CORRESPONDENCE.

GRADUATION FORMULAS.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—In his *Prize Essay on Friendly Societies*, which appears in the current number of the *Journal*, Mr. George F. Hardy makes kindly reference to some contributions of mine on the subject of graduation, and then proceeds to add a formula for the purpose which he describes as new, and superior both in smoothness and simplicity to any yet given (*J.I.A.*, xxvii, 277). If Mr. Hardy had stated the manner in which he arrived at this formula, I think it would appear that he has paid me the further compliment of adopting, with a trifling alteration, a formula of mine which will be found in *J.I.A.*, xxv, 23.

The stages of procedure, stated in each case in the same order, are as follows:

HIGHAM.	HARDY.
Divide by 12·5 (which is done by multiplying by ·08).	Divide by 12.
Sum three middle terms of five.	Identical with mine.
Sum two outer terms of same.	Ditto.
Deduct the two from the three.	Ditto.
Sum in fives.	Sum in fours.
Sum in fives, and divide by 10.	Identical with mine.
Sum in fives.	Sum in sixes.

The only alteration which Mr. Hardy makes is, that he substitutes a summation in fours and one in sixes for two summations in fives. The change in the divisor is not an independent alteration but results from the other. Working with fours and sixes we bring out the coefficients of the terms of u as Mr. Hardy prints them; and the sum of these being 120, that number becomes his divisor, as 125 is mine and Mr. Woolhouse's.

In regard to simplicity, the formula as originally written has somewhat the advantage. It is a little easier to multiply by 8 than to divide by 12; and it is a little easier to work continuously with fives than to shift from fours to fives and sixes—to say nothing of the inconvenience of inter-spacing the fours and sixes for half-terms, as printed in the Table on page 278.

That Table is more compact than mine (already referred to) for two reasons.

First.—Mr. Hardy does not show his initial division by 12.

Second.—He is content with a final column to the same number of decimals as the first, and this enables him to keep down the figures by making the division his first operation; whereas I had to make the division the last operation in order to give correctly an additional figure corresponding with an extension of the radix from 10,000 to 100,000.

But for the same requirement, Mr. Ackland's columnar arrangement of Mr. Woolhouse's formula (*J.I.A.*, xxiii, 354) would look much less formidable than it does.

To measure the improvement in smoothness it is necessary to obtain the original unadjusted data by multiplying Mr. Hardy's first column by 12; after which I multiply by ·08, in order that the succeeding table may start from the same point as his.

Age	$\frac{1}{12}u_x$	u_x (Unadjusted Material)	$\times$ ·08, giving $\frac{1}{12\cdot5}u_x$	Say
19	·068	·816	·06528	·065
20	·064	·768	·06144	·061
21	·070	·840	·06720	·067
22	·064	·768	·06144	·061
23	·064	·768	·06144	·061
24	·068	·816	·06528	·065
25	·071	·852	·06816	·068
26	·066	·792	·06336	·063
27	·065	·780	·06240	·062
28	·056	·672	·05376	·054
29	·066	·792	·06336	·063
30	·065	·780	·06240	·062
31	·069	·828	·06624	·066
32	·067	·804	·06432	·064
33	·076	·912	·07296	·073
34	·075	·900	·07200	·072
35	·081	·972	·07776	·078
	1·155	13·860	1·10880	1·105

It will be observed that the last column, reduced to two significant figures, adds up ·004 less than $\frac{1}{12\cdot5}$ of the sum of the terms of u_x in consequence of the preponderance of terms in which the two rejected figures are below 50. An arbitrary amendment of this would be unfair to one side or the other in a comparison limited to the question of orderly progression.

Graduation by Formula in J.I.A., xxv, 23.

Age	$\frac{1}{12\cdot5}u_x$	Three Middle Terms of Five	Two Outer Terms	(3)−(4)	Sum in Fives	$\frac{1}{10}$ Sum in Fives	Sum in Fives (Adjusted Values)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
10	·065	...	...	...	...	...	...
11	·065	...	...	...	...	...	...
12	·065	195	130	65	...	...	...
13	·065	195	130	65	...	...	...
14	·065	195	130	65	325	...	...
15	·065	195	130	65	325	...	...
16	·065	195	130	65	329	163	...
17	·065	195	130	65	323	163	...
18	·065	195	126	69	325	162	·807
19	·065	191	132	59	323	160	·802
20	·061	193	126	67	321	159	·798
21	·067	189	126	63	304	158	·795
22	·061	189	126	63	315	159	·795
23	·061	187	135	52	321	159	·795
24	·065	194	124	70	332	160	·794
25	·068	196	123	73	317	159	·787
26	·063	193	119	74	319	157	·778
27	·062	179	131	48	300	152	·768
28	·054	179	125	54	300	150	...
29	·063	179	128	51	282	150	...
30	·062	191	118	73	303	...	...
31	·066	192	136	56	314	...	...
32	·064	203	134	69	...	...	...
33	·073	209	144	65	...	...	...
34	·072	...	...	...	...	...	...
35	·078	...	...	...	...	...	...

The number of figures in the two graduations is precisely the same. My results are smaller than Mr. Hardy's for a reason already stated. One has to look closely for the improved smoothness of the latter; nevertheless it is there.

His differences are 4, 3, 1, 0, 0, 3, 6, 9, 9.

Mine are 5, 4, 3, 0, 0, 1, 7, 9, 10.

If a summation be made twice in fives the resulting coefficients become

1, 2, 3, 4, 5, 4, 3, 2, 1; divisor 25.

If in fours and sixes the coefficients are

1, 2, 3, 4, 4, 4, 3, 2, 1; divisor 24.

The flattening in the middle tends to soften asperities; but when it is combined with the other processes, the formula ceases to be correct to third differences.* The error, however, only amounts to

* Mr. Hardy notices this defect, and suggests in a foot note a correction which is absolutely a change back to my formula without alteration.

$\frac{1}{12}$ of the second difference $+\frac{7}{12}$ of the third difference, and this seems practically as unimportant as the shades of modification previously noticed.

To sum the matter up: the formula which Mr. Hardy produces can hardly be regarded as new. It is rather to be described as a previously known formula in which an alteration has been made with the result of rendering it slightly more smooth, slightly less simple, slightly less accurate, than it was.

I am, Sir,
Your obedient servant,

Royal Exchange,
October 1888.

J. A. HIGHAM.

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