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A UNIFORM L^∞ ESTIMATE OF THE SMOOTHING OPERATORS RELATED TO PLANE CURVES

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ABSTRACT. In dealing with the spectral synthesis property for a plane curve with nonzero curvature, a key step is to have a uniform L^∞ estimate for some smoothing operators related to the curve. In this paper, we will show that the same L^∞ estimate holds true for a plane curve that may have zero curvature.

1. **Introduction.** Let $S(R^n)$ be the space of Schwartz class functions and $S'(R^n)$ be the dual space of $S(R^n)$. It is obvious that for $1 \le p \le \infty$, we have $L^p(R^n) \subset S'(R^n)$. For $f \in S(R^n)$, we define the Fourier transform of $f(x)$ by $\hat{f}(\xi) = \int_{R^n} e^{i\xi\cdot x} f(x)\,dx$. Also for $T \in S'(R^n)$, define $\hat{T}$ by the formula $\langle \hat{T}, f\rangle = \langle T, \hat{f}\rangle$. For $1 \le p \le \infty$, let $FL^p(R^n) = \{T \in S'(R^n)\ ;\ \hat{T} \in L^p(R^n)\}$. For a compact subset E of R^n, denote

$$I(E) = \{f \in FL^1(R^n)\ ;\ f(E) = 0\}$$
$$J(E) = \{f \in S(R^n)\ ;\ f(E) = 0\}$$
$$K(E) = \{f \in S(R^n)\ ;\ \operatorname{supp} f \cap E = \emptyset\}$$

Obviously $\overline{K(E)} \subset \overline{J(E)} \subset I(E)$ in FL^1 norm. We call E a set of spectral synthesis if $\overline{K(E)} = I(E)$ and a set of weak spectral synthesis if $\overline{J(E)} = I(E)$.

It is easy to see that the unit ball of R^n is a set of spectral synthesis. For $n \ge 3$, L. Schwartz [11] discovered that the unit sphere S^{n-1} in R^n is not a set of spectral synthesis. The first surprising result is due to C. Herz [9], who proved that the unit circle S^1 of R^2 is a set of spectral synthesis. Then N. Varopoulos [12] obtained that S^{n-1} is a set of weak spectral synthesis. Y. Domar ([1], [2]) used a totally different approach to generalize Herz's result to compact smooth plane curves with non-vanishing curvature and generalize Varopoulos's result to compact smooth $(n-1)$-dimensional manifolds in R^n with non-vanishing Gaussian curvature. Domar's idea was followed by D. Muller [10] and the author [7], slightly weakening the curvature and the smoothness assumptions on the manifolds.

The basic idea in [1] is to prove a uniform L^∞ estimate for some smoothing operators related to a curve with nonzero curvature. Motivated from Domar's work in [3], in this paper we will show that one could get the same uniform L^∞ estimate for a plane curve that may have zero curvature, namely, we will prove

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THEOREM A. *Let $k \geq 2$ be an integer and $M = \Big\{\big(x, \psi(x)\big)\ ;\ x \in [a,b]\Big\}$, where $\psi(x) \in C^{k+1}([a,b])$ such that $\psi^{(k)}(x) > 0$ for all $x \in [a,b]$. Let $T \in FL^\infty(R^2)$ with* $\mathrm{supp}(T) \subset M$. *Then for all small positive h, there exists a family of smooth measures T_h on M such that*

$$\lim_{h\to 0} \hat{T}_h(\eta,\xi) = \hat{T}(\eta,\xi) \tag{1}$$

$$\|\hat{T}_h\|_{L^\infty(R^2)} \leq C\|\hat{T}\|_{L^\infty(R^2)} \tag{2}$$

where C is independent of all small h.

The structure of this paper is as follows. In Section 2, some preliminary results will be given. Section 3 will state and prove Theorem B, a local result, from which Theorem A will be derived in Section 4.

In the rest of this paper, the same letter C will stand for different uniform constants, but the involvement of parameters in each occurrence of C will be stated explicitly.

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2. **Preliminaries.** The following lemma can be proved by the Beurling-Pollard technique (see the proof of Lemma 1 in [6]).

LEMMA 1.1. *Let E be a compact C^1 curve in $R^n (n \geq 2)$. Let $T \in FL^\infty(R^n)$ with* $\mathrm{supp}(T) \subset E$ *and $f \in FL^1(R^n)$ with $f(E) = 0$. If f is Lipschitz continuous on a neighborhood of E, then we have $\langle T, f\rangle = 0$.*

LEMMA 1.2. *Let $k \geq 2$, $x_0 \in [a,b]$ and $f(x) \in C^k([a,b])$ with $f(x_0) = 0$. Let $g(x) = \frac{f(x)}{x-x_0}$. Then*

(i) $g(x) = \int_0^1 f'\big((1-t)x_0 + tx\big)\, dt$, $x \in [a,b]$

(ii) $\|g\|_{C^m([a,b])} \leq \|f\|_{C^{m+1}([a,b])}$, *for all* $0 \leq m \leq k-1$

(iii) $\inf_{x\in[a,b]} |g(x)| \geq \inf_{x\in[a,b]} |f'(x)| - \|f\|_{C^2([a,b])}(b-a)$

PROOF. (i) is obvious if we let $u = (1-t)x_0 + tx$ so that $dt = \frac{1}{x-x_0}\, du$. (ii) follows immediately from (i), while (iii) follows from (ii) with $m = 1$ and the identity $f'(x) = g'(x)(x-x_0) + g(x)$.

REMARK. One corollary of (iii) is that if $f'(x) \geq 1$ for all $x \in [a,b]$ and $(b-a) \leq \frac{1}{2\|f\|_{C^2([a,b])}}$, then we have $g(x) \geq \frac{1}{2}$ for all $x \in [a,b]$. The author thanks Dr. Yibiao Pan for suggesting the above simple proof of (ii) in Lemma 1.2.

LEMMA 1.3. *Given $a < x_1 < x_2 < \cdots < x_{m-1} < x_m < b$, there exist $\phi_j(x) \in C(R)$, $1 \leq j \leq m$, such that*

(i) $0 \leq \phi_j(x) \leq 1$, *for all* $x \in R$

(ii) $\sum_{j=1}^m \phi_j(x) = 1$, *for all* $x \in R$

(iii) $\|\phi_1'(x)\|_{L^\infty} \leq \frac{2}{x_2-x_1}$, *and* $\|\phi_m'(x)\|_{L^\infty} \leq \frac{2}{x_m-x_{m-1}}$

$\|\phi_j'(x)\|_{L^\infty} \leq \max\{\frac{2}{x_j-x_{j-1}}, \frac{2}{x_{j+1}-x_j}\}$, *for* $2 \leq j \leq m-1$

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(iv) $\operatorname{supp}(\phi_1)\cap[a,b]\subset[a,x_1+\frac{3}{4}(x_2-x_1)]$
$\operatorname{supp}(\phi_j)\subset[x_{j-1}+\frac{1}{4}(x_j-x_{j-1}),x_j+\frac{3}{4}(x_{j+1}-x_j)]$, *for* $2\le j\le m-1$
$\operatorname{supp}(\phi_m)\cap[a,b]\subset[x_{m-1}+\frac{1}{4}(x_m-x_{m-1}),b]$

PROOF. We define the functions $\phi_1(x)$, $\phi_m(x)$ and $\phi_j(x)$, $2\le j\le m-1$ as follows.

$$\phi_1(x)=\begin{cases}1 & \text{if } x<x_1+\frac{1}{4}(x_2-x_1)\\ -\frac{2}{x_2-x_1}\Big(x-\big(x_1+\frac{3}{4}(x_2-x_1)\big)\Big) & \text{if } x_1+\frac{1}{4}(x_2-x_1)\\ & \quad\le x<x_1+\frac{3}{4}(x_2-x_1)\\ 0 & \text{if } x\ge x_1+\frac{3}{4}(x_2-x_1)\end{cases}$$

$$\phi_j(x)=\begin{cases}0 & \text{if } x<x_{j-1}+\frac{1}{4}(x_j-x_{j-1})\\ \frac{2}{x_j-x_{j-1}}\Big(x-\big(x_{j-1}+\frac{1}{4}(x_j-x_{j-1})\big)\Big) & \text{if } x_{j-1}+\frac{1}{4}(x_j-x_{j-1})\\ & \quad\le x<x_{j-1}+\frac{3}{4}(x_j-x_{j-1})\\ 1 & \text{if } x_{j-1}+\frac{3}{4}(x_j-x_{j-1})\\ & \quad\le x<x_j+\frac{1}{4}(x_{j+1}-x_j)\\ -\frac{2}{x_{j+1}-x_j}\Big(x-\big(x_j+\frac{1}{4}(x_{j+1}-x_j)\big)\Big) & \text{if } x_j+\frac{1}{4}(x_{j+1}-x_j)\\ & \quad\le x<x_j+\frac{3}{4}(x_{j+1}-x_j)\\ 0 & \text{if } x\ge x_j+\frac{3}{4}(x_{j+1}-x_j)\end{cases}$$

$$\phi_m(x)=\begin{cases}0 & \text{if } x<x_{m-1}+\frac{1}{4}(x_m-x_{m-1})\\ \frac{2}{x_m-x_{m-1}}\Big(x-\big(x_{m-1}+\frac{1}{4}(x_m-x_{m-1})\big)\Big) & \text{if } x_{m-1}+\frac{1}{4}(x_m-x_{m-1})\\ & \quad\le x<x_{m-1}+\frac{3}{4}(x_m-x_{m-1})\\ 1 & \text{if } x\ge x_{m-1}+\frac{3}{4}(x_m-x_{m-1})\end{cases}$$

The above definitions give (i), (ii) and (iv) directly. It remains to verify (iii). It is easy to check that in the distributional sense, $\phi_j'(x)(2\le j\le m-1)$ is a step function, taking the values 0, $\frac{2}{x_j-x_{j-1}}$, 0, $-\frac{2}{x_{j+1}-x_j}$, 0 on the blocks in the definition of $\phi_j(x)$, while $\phi_1'(x)$ takes the values 0, $-\frac{2}{x_2-x_1}$, 0 and $\phi_m'(x)$ takes the values 0, $\frac{2}{x_m-x_{m-1}}$, 0 respectively. This verifies (iii).

LEMMA 1.4 (CARLSON). *If f, $f'\in L^2(R)$, then f can be changed on a set of measure zero such that $\hat{f}\in L^1(R)$, and*

$$\|\hat{f}\|_{L^1}\le C(\|f\|_{L^2}\|f'\|_{L^2})^{\frac{1}{2}}$$

The following lemma is a corollary of Lemma 1.4.

LEMMA 1.5. *Let I be an interval and denote $|I|$ the length of I. Let $\tau(x)\in C(R)$ such that* $\operatorname{supp}(\tau)\subset I$ *and $\tau'(x)\in L^\infty(R)$. If $\|\tau\|_{L^\infty}\le C$ and $\|\tau'\|_{L^\infty}\le C|I|^{-1}$, then for any function $f\in C^1(R)$, we have*

$$\|\tau f\|_{FL^1(R)}\le C\{\|f\|_{L^\infty(I)}+(|I|\,\|f\|_{L^\infty(I)}\|f'\|_{L^\infty(I)})^{\frac{1}{2}}\}$$

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Now let the interval I in Lemma 1.5 be contained in $(-1,1)$ and let $\psi(x)$ be a function on $[-1,1]$ to be specified later. For real η and ξ ($\xi \neq 0$), let $g(x) = \frac{\eta}{\xi}x + \psi(x)$. For $\phi(x)$, $\theta(x) \in C_0^\infty(-1,1)$ and small positive h such that $\mathrm{supp}(\theta) + [-h,h] \subset [-1,1]$, define

$$K(x) = \theta(x)\int_{-1}^{1} e^{i\xi g(x-hy)}\phi(y)\,dy$$
$$L(x) = \theta(x)\int_{-1}^{1} e^{i\xi(g(x-hy)-g(x))}\phi(y)\,dy$$

The proof of the following two technical lemmas follows easily from Lemma 1.5 and some standard calculations such as changing variables and integration by parts. The detail computation could be found in [3], where the reader will see that the constant C in the lemmas does not depend on η, ξ, h and I.

LEMMA 1.6.

$$\|\tau K\|_{FL^1(R)} \leq C\Big(1 + (|I|h^{-1})^{\frac{1}{2}}\Big) \tag{3}$$

$$\|\tau L\|_{FL^1(R)} \leq C\Big(1 + (|I||\xi h|\|g''\|_{L^\infty(I+[-h,h])})^{\frac{1}{2}}\Big) \tag{4}$$

LEMMA 1.7. *If* $\|g'\|_{L^\infty(I+[-h,h])}\|\frac{1}{g'}\|_{L^\infty(I+[-h,h])} \leq C$, *then we have*

$$\begin{aligned}\|\tau L\|_{FL^1(R)} \leq C\Big\{|\xi h|^{-1}\Big\|\frac{1}{g'}\Big\|_{L^\infty(I+[-h,h])}\Big(1 + \Big\|\frac{hg''}{g'}\Big\|_{L^\infty(I+[-h,h])}\Big)\Big\}^{\frac{1}{2}} \\ \cdot \Big\{1 + \Big(|I|\Big\|\frac{g''}{g'}\Big\|_{L^\infty(I+[-h,h])}\Big)^{\frac{1}{2}}\Big\}\end{aligned} \tag{5}$$

3**. The local result.** Let $k \geq 2$ such that $\psi(x) \in C^{k+1}[-1,1]$ and $\Psi^{(k)}(x) \geq 1$ for all $x \in [-1,1]$. Let $\Gamma = \big\{\big(x,\psi(x)\big)\ ;\ x \in (-1,1)\big\}$. Let $T \in FL^\infty(\mathbf{R}^2)$ with $\mathrm{supp}(T) \subset \Gamma$. Following Domar, we construct a family of smooth measures $\{T_h\}$ on Γ for all small positive h as follows. Let

$$\alpha\colon R^2 \longrightarrow R \quad \text{given by} \quad (x,y) \longrightarrow x,$$
$$\beta\colon (-1,1) \longrightarrow \Gamma \quad \text{given by} \quad x \longrightarrow \big(x,\psi(x)\big).$$

We first define a distribution $\Sigma \in S'(R)$ by

$$\langle \Sigma, g\rangle = \langle T, g\circ\alpha\rangle \quad \text{for } g \in S(R).$$

This makes sense since $\mathrm{supp}(T)$ is compact. From the construction of Σ, it is obvious that $\mathrm{supp}(\Sigma) \subset (-1,1)$. It follows that one can find $\theta(x) \in C_0^\infty(-1,1)$ such that $\Sigma = \theta\Sigma$. Let $\phi(x) \in C_0^\infty(-1,1)$ with $\int_R \phi(x)\,dx = 1$. Denote $\phi_h(x) = \frac{1}{h}\phi(\frac{x}{h})$ and $\check{\phi}_h(x) = \phi_h(-x)$. Let $U = (-1,1)$, then for $0 < h < \frac{1}{2}\,\mathrm{dist}\big(\partial U, \mathrm{supp}(\sigma)\big)$ (we shall call such h small), we see that $\mathrm{supp}(\Sigma * \check{\phi}_h) \subset (-1,1)$. Now we define $T_h \in S'(R^2)$ by

$$\langle T_h, f\rangle = \langle \Sigma * \check{\phi}_h, f\circ\beta\rangle \quad \text{for } f \in S(R^2).$$

It is easy to check that for all small h, T_h are mass measures on Γ. Our local result is the following estimate.

THEOREM B. *Let (η, ξ) be any point in R^2 and let T and T_h be as above. Then one has*

$$\lim_{h\to 0} \hat{T}_h(\eta, \xi) = \hat{T}(\eta, \xi) \tag{6}$$

$$\|\hat{T}_h\|_\infty \leq C\|\hat{T}\|_\infty, \tag{7}$$

where C is independent of all small h.

For $(\eta, \xi) \in R \times R \setminus \{0\}$ and $(x, y) \in R \times R$, let $X(x, y) = e^{i(\eta x+\xi y)}$. Then from the construction of T_h, we have

$$\begin{aligned} \hat{T}_h(\eta, \xi) &= \langle T_h, X\rangle = \langle \theta\Sigma, (X \circ \beta) * \phi_h\rangle \\ &= \left\langle T, \theta(x) \int_R e^{i[\eta(x-\sigma)+\xi\psi(x-\sigma)]}\phi_h(\sigma)\, d\sigma \right\rangle \end{aligned} \tag{8}$$

Let $g(x)$, $K(x)$ and $L(x)$ be as in Lemma 1.6 and Lemma 1.7. Then (8) implies

$$|\hat{T}_h(\eta, \xi)| \leq C\|K\|_{FL^1(R)}\|\hat{T}\|_{L^\infty(R^2)} \tag{9}$$

And Lemma 1.1 and (8) yield

$$|\hat{T}_h(\eta, \xi)| \leq C\|L\|_{FL^1(R)}\|\hat{T}\|_{L^\infty(R^2)} \tag{10}$$

We notice that in (9) and (10), $K(x)$, $L(x)$ depend on η, ξ and h, but the constant C is independent of η, ξ and h.

REMARK. If the curve Γ has nonzero curvature, then one can use (10) alone to get (7) (see [1], or [7]), that is, one can control $\|L\|_{FL^1(R)}$ uniformly for all η, ξ and small h. When a curve has zero curvature at some points, Gustavsson [8] gave an example, showing that in this case $\|L\|_{FL^1(R)}$ is not uniformly bounded for all η, ξ and small h. Following the idea in [3], in this paper we divide the set of η, ξ, h into two subsets S_1 and S_2 so that a uniform estimate of $\|K\|_{FL^1(R)}$ on S_1 and a uniform estimate of $\|L\|_{FL^1(R)}$ on S_2 could be obtained. The inequality (7) follows from these two estimates.

PROOF OF THEOREM B. The identity (6) follows from the construction of T_h, so it remains to verify (7). Let $M = \max_{x\in[-1,1]} |\psi''(x)|$. We divide our discussion into two cases.

CASE 1. $|\eta| \geq 2M|\xi|$.

From the definition of $L(x)$, we have $L(x) = e^{i\eta x}\theta(x) \int_R e^{i(-h\eta\sigma)+h\xi\frac{\psi(x-h\sigma)-\psi(x)}{h})}\phi(\sigma)\, d\sigma = e^{i\eta x}L_1(x)$, where $L_1(x) = \theta(x) \int_R e^{i(-h\eta\sigma)+h\xi\frac{\psi(x-h\sigma)-\psi(x)}{h})}\phi(\sigma)\, d\sigma$. So it is enough to control $\|L_1\|_{FL^1(R)}$ since $\|L\|_{FL^1(R)} = \|L_1\|_{FL^1(R)}$.

Integrating by parts for $L_1(x)$ yields that $\|L_1\|_{L^2} \leq C(|h\eta|)^{-1}$ with C independent of η, ξ and h. Also it is trivial to see that $\|L_1'\|_{L^2} \leq C|h\xi| \leq \frac{C}{2M}|h\eta|$ with C independent of η, ξ and h. Thus (7) follows from Lemma 1.4 and (10).

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CASE 2. $|\eta| \leq 2M|\xi|$.

In this case, when $|\xi h| \leq 1$, (7) follows from the argument given in Case 1, so from now on we assume that $|\xi h| \geq 1$. Let $P_{\eta,\xi}(x) = \frac{\eta}{\xi} + \psi'(x)$. Then $P_{\eta,\xi}(x) \in C^k[-1,1]$ and $P_{\eta,\xi}^{(k-1)}(x) = \psi^{(k)}(x)$.

Let u be a nonnegative integer and let $a_1, a_2, \ldots, a_u$ be the zeros in $[a,b]$ of $P_{\eta,\xi}(x)$ with multiplicity $l_1, l_2, \ldots, l_u$ respectively. From Rolle's theorem, we see that $k_1 = \sum_{j=1}^{u} l_i \leq k-1$. Thus we have

$$P_{\eta,\xi}(x) = \underline{+}(x-a_1)^{l_1}(x-a_2)^{l_2}\cdots(x-a_u)^{l_u} Q_{\eta,\xi}(x), \tag{11}$$

where $Q_{\eta,\xi}(x)$ is a C^{k-k_1} function on $[a,b]$ such that $Q_{\eta,\xi}(x) > 0$ for all $x \in [a,b]$. We emphasize that what makes the argument complicated is that the roots a_j are depending on η and ξ. If $Q_{\eta,\xi}(x) \geq C_1 > 0$ with C_1 independent of η, ξ and $x \in [a,b]$ (we keep in mind that $|\frac{\eta}{\xi}| \leq 2M$), then we say that $P_{\eta,\xi}(x)$ has a level 1 structure (11). If there is no such C_1, let $c = \inf_{x\in[a,b]} Q_{\eta,\xi}(x)$ (we know that $c > 0$) and let $a_{u+1}, a_{u+2}, \ldots, a_{u+v}$ be the zeros of $Q_{\eta,\xi}(x) - c$ with even multiplicity $l_{u+1}, l_{u+2}, \ldots, l_{u+v}$ respectively (it is possible that $a_j = a_i$ for some j, i with $1 \leq j \leq u$, $u+1 \leq i \leq u+v$). Then $k_2 = \sum_{j=u+1}^{u+v} l_j \leq (k-1) - k_1$ and we have

$$\begin{aligned} P_{\eta,\xi}(x) = \underline{+}&(x-a_1)^{l_1}(x-a_2)^{l_2}\cdots(x-a_u)^{l_u}[(x-a_{u+1})^{l_{u+1}} \\ &\cdot (x-a_{u+2})^{l_{u+2}}\cdots(x-a_{u+v})^{l_{u+v}} S_{\eta,\xi}(x) + c], \end{aligned} \tag{12}$$

where $S_{\eta,\xi}(x)$ is a $C^{k-(k_1+k_2)}$ function on $[a,b]$ such that $S_{\eta,\xi}(x) > 0$ for all $x \in [a,b]$. Again if $S_{\eta,\xi}(x) \geq C_2 > 0$ with C_2 independent of η, ξ and $x \in [a,b]$, then we say that $P_{\eta,\xi}(x)$ has a level 2 structure (12).

Similarly one can define a structure of level 3, level 4 and so on. Combining Lemma 1.2 and an induction argument (if necessary one can divide the interval $[-1,1]$ into finite many subintervals), one can follow the remark after Lemma 1.2 to see that there are at most $k-1$ levels. To simplify the notation, we only give the proof of Case 2 when $P_{\eta,\xi}(x)$ has a structure of level 2 since the proof for other levels follows the same line.

Now assume that $P_{\eta,\xi}$ has the expression (12). We remark that $|a_j| \leq C$, $1 \leq j \leq u+v$ with C independent of η, ξ. From the choice of $\theta(x)$, one can find a small $\epsilon > 0$ such that $\operatorname{supp}(\theta) \subset [-1+\epsilon, 1-\epsilon]$. Based on whether the points a_j are all contained in $[-1+\frac{1}{2}\epsilon, 1-\frac{1}{2}\epsilon]$ or not, we have the following two subcases.

CASE 2.1. All a_j are contained in the interval $[-1+\frac{1}{2}\epsilon, 1-\frac{1}{2}\epsilon]$.

PROOF OF CASE 2.1. From the argument below we will see that we may assume that $a_i \neq a_j$ if $i \neq j$. Also we will see that the order of a_j is not important, so let us assume that $-1 < a_1 < a_2 < \cdots < a_u < a_{u+1} < \cdots < a_{u+v} < 1$.

Applying Lemma 1.3 for $m = u+v$, $a = -1$, $x_j = a_j$, $b = 1$, one can find $\phi_j(x)$ as in Lemma 1.3 so that (if some a_j are the same, then only the distinct a_j will be used in the partition of unity)

$$\begin{aligned}\hat{T}_h(\eta,\xi) &= \Big\langle T, \theta(x)\int_R e^{i[\eta(x-\sigma)+\xi\psi(x-\sigma)]}\phi_h(\sigma)\,d\sigma\Big\rangle\\ &= \Big\langle T, \sum_{j=1}^m \phi_j(x)\theta(x)\int_R e^{i[\eta(x-h\sigma)+\xi\psi(x-h\sigma)]}\phi(\sigma)\,d\sigma\Big\rangle\\ &= \sum_{j=1}^m\Big\langle T, \phi_j(x)\theta(x)\int_R e^{i[\eta(x-h\sigma)+\xi\psi(x-h\sigma)]}\phi(\sigma)\,d\sigma\Big\rangle\\ &= \sum_{j=1}^m I_j(\eta,\xi)\end{aligned}$$

As in Lemma 1.5, we let $g(x) = \frac{\eta}{\xi}x + \psi(x)$ so that $g'(x) = P_{\eta,\xi}(x)$. Then using the product rule, we see that $g''(x)$ has $u+v+1$ terms (if $S(x)$ is a constant, then $g''(x)$ has only $u+v$ terms). For each term, we take the absolute value and then sum all the $u+v+1$ terms together to get a new function denoted by $\overline{g}(x)$, which contains all factors as shown in (13). For $\delta > 0$ and each j, $1 \le j \le u+v$, let $\overline{G_j}(\delta)$ be the function of δ obtained from $\overline{g}(x)$, replacing $S(x)$, $S'(x)$ by 1, $(x-a_j)$ by δ, and $(x-a_i)$ by $|a_i - a_j| + \delta$ if $i \ne j$. Similarly let $\overline{F_j}(\delta)$ be the function of δ obtained from $g'(x)$, replacing $S(x)$ by 1, $(x-a_j)$ by δ, and $(x-a_i)$ by $|a_i - a_j| + \delta$ if $i \ne j$. It is easy to see that $\overline{G_j}(\delta)$ is an increasing function of δ and that $\delta\overline{G_j}(\delta) \le C\overline{F_j}(\delta)$ for $0 < \delta \le 1$ with C independent of η, ξ and h.

For $I_j(\eta,\xi)$, we define $0 < \delta_j \le 1$ (since $|\xi h| \ge 1$) by the equation

$$|\xi h|\delta_j\overline{G_j}(\delta_j) = 1 \tag{13}$$

Set $d_j = \max\{h, \delta_j\}$. Since h is small, we see that $0 < d_j \le 1$. It should keep in mind that d_j does depend on η and ξ since δ_j does. Thanks to the similarity of the argument for each I_j, we will only show

$$\|I_1\|_{L^\infty} \le C\|\hat{T}\|_{L^\infty}, \tag{14}$$

where C is independent of η, ξ and h.

Find L, N such that $a_1 + 1 = 2^L d_1$ and $\frac{3}{4}(a_2 - a_1) = 2^N d_1$. Denote the integer part of L, N by $[L], [N]$ respectively. When $L > 0$ and $N > 0$, we cut the interval $[-1, a_1 + \frac{3}{4}(a_2 - a_1)]$ by the points $\{a_1 - 2^l d_1\ ;\ 1 \le l \le [L]\}$ and $\{a_1 + 2^n d_1\ ;\ 1 \le n \le [N]\}$. When $N > 0$ and $L \le 0$, we cut the same interval by the points $\{a_1 + 2^n d_1\ ;\ 1 \le n \le [N]\}$. When $N \le 0$ and $L > 0$, we cut the interval by the points $\{a_1 - 2^l d_1\ ;\ 1 \le l \le [L]\}$. Finally if $L \le 0$ and $N \le 0$, we leave the interval alone. To simplify the notation and show the idea, we restrict ourselves to the case when $N > 0$ and $L \le 0$ (if N is an integer, then we use $\{a_1 + 2^n d_1\ ;\ 1 \le n \le N-1\}$ to cut the interval). The treatment for other cases is similar. In the rest of this section, the letter C will stand for the constants independent of η, ξ and h.

From Lemma 1.3, there exist functions $z_n(x) \in C(R)$, $1 \le n \le [N]$ such that

(i) $0 \le z_n(x) \le 1$ for all $x \in R$
(ii) $\sum_{n=1}^{[N]} z_n(x) = 1$, for all $x \in R$
(iii) $\|z_1'(x)\|_{L^\infty} \le \frac{1}{d_1}$, and $\|z_n'(x)\|_{L^\infty} \le \frac{1}{2^{n-2}d_1}$, for $2 \le n \le [N]$

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(iv) $\operatorname{supp}(z_1)\cap[-1,a_1+\frac{3}{4}(a_2-a_1)]\subset[-1,a_1+\frac{7}{2}d_1]$
$\operatorname{supp}(z_n)\subset[a_1+\frac{5}{4}2^{n-1}d_1,a_1+\frac{7}{4}2^nd_1]$, for $2\le n\le[N]-1$
$\operatorname{supp}(z_{[N]})\cap[-1,a_1+\frac{3}{4}(a_2-a_1)]\subset[a_1+\frac{5}{4}2^{[N]-1}d_1,a_1+\frac{3}{4}(a_2-a_1)]$

From the construction of $z_n(x)$, we have

$$\begin{aligned}I_1(\eta,\xi) &= \sum_{n=1}^{[N]}\Big\langle T, z_n(x)\phi_1(x)\theta(x)\int_R e^{i\xi g(x-h\sigma)]}\phi(\sigma)\,d\sigma\Big\rangle\\ &= \sum_{n=1}^{[N]} J_n(\eta,\xi)\end{aligned}$$

To prove (14), it is sufficient to show that

$$\|J_n\|_{L^\infty}\le C2^{-n}\|\hat{T}\|_{L^\infty} \tag{15}$$

Let $\tau_n(x)=z_n(x)\phi_1(x)$, then $|\operatorname{supp}(\tau_n)|\le C2^nd_1$ and $\|\tau_n'(x)\|_{L^\infty}\le C(2^nd_1)^{-1}$. Let $K(x)$, $L(x)$ be as in Lemma 1.6. It is easy to see that

$$\|J_n\|_{L^\infty}\le C\|\tau_nK\|_{FL^1(R)}\|\hat{T}\|_{L^\infty(R^2)} \tag{16}$$

Let $\beta(x,y)\in C_0^\infty(R^2)$ such that $\beta=1$ on a neighborhood of $\overline{\Gamma}$. For fixed η,ξ and h, define

$$f(x,y)=\beta(x,y)(e^{i(\eta x+\xi y)}e^{-i\xi g(x)}-1)\,\tau_n(x)\,\theta(x)\int_R e^{i\xi g(x-h\sigma)]}\phi(\sigma)\,d\sigma$$

We observe that $f(x,y)\in C(R^2)$ such that f is Lipschitz continuous on a neighborhood of $\overline{\Gamma}$ and $f(\overline{\Gamma})=0$. Also using Lemma 3.1 in [10], one can verify that $f\in FL^1(R^2)$. Evoking Lemma 1.1, one has

$$\|J_n\|_{L^\infty}\le C\|\tau_nL\|_{FL^1(R)}\|\hat{T}\|_{L^\infty(R^2)} \tag{17}$$

First we control $J_1(\eta,\xi)$. When $d_1=h$, (15) ($n=1$) follows from (3) of Lemma 1.6 and (16). So assume that $d_1=\delta_1$. In this case it is easy to see that $\|g''\|_{L^\infty(\operatorname{supp}(\tau_1)+[-h,h])}\le C\overline{G}_1(\delta_1)$. Thus from the definition of δ_1, one obtains (15) ($n=1$) by using (4) of Lemma 1.6 and (17).

Now we prove (15) for $n\ge 2$. The formula $\Big|\frac{\left(\prod_1^m f_j(x)\right)'}{\prod_1^m f_j(x)}\Big|\le\sum_1^m\Big|\frac{f_j'(x)}{f_j(x)}\Big|$ leads us to the inequality

$$\left|\frac{g''(x)}{g'(x)}\right|\le C\left(\left|\frac{S'(x)}{S(x)}\right|+\sum_1^{u+v}\left|\frac{1}{x-a_j}\right|\right) \tag{18}$$

From (18), one easily has (since $h\le d_1$)

$$\left\|\frac{hg''}{g'}\right\|_{L^\infty(\operatorname{supp}(\tau_n)+[-h,h])}\le C \tag{19}$$

Since $2^{[N]}d_1\le\frac{3}{4}(a_j-a_1)$ for all $2\le j\le u+v$, it is easy to see that

$$\|g'\|_{L^\infty(\operatorname{supp}(\tau_n)+[-h,h])}\le C\overline{F}_1(2^nd_1) \tag{20}$$

$$\left\|\frac{1}{g'}\right\|_{L^\infty(\operatorname{supp}(\tau_n)+[-h,h])}\le C\big(\overline{F}_1(2^nd_1)\big)^{-1} \tag{21}$$

Moreover $\overline{F}_1(2^nd_1)\ge C2^nd_1\overline{G}_1(2^nd_1)\ge C2^n\delta_1\overline{G}_1(\delta_1)$, so (15) follows from the definition of δ_1, (17), (19), (20), (21) and Lemma 1.7.

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CASE 2.2. Some a_j are not contained in the interval $[-1+\frac{1}{2}\epsilon, 1-\frac{1}{2}\epsilon]$.

PROOF OF CASE 2.2. Without loss of generality, let us assume that only a_1 is not contained in the interval $[-1+\frac{1}{2}\epsilon, 1-\frac{1}{2}\epsilon]$. In this case, we only use $a_2, \ldots, a_{u+v}$ to cut the interval $[-1, 1]$ and obtain $\{\phi_j(x), 2 \le j \le u+v\}$ in the partition of unity. Since $|x - a_1| \ge \frac{1}{2}\epsilon$ if $x \in \operatorname{supp}(\theta)$, a minor modification of the argument for Case 2.1 yields the proof for this case.

The proof of Theorem B is now complete.

4. **Proof of Theorem A.** Let the compact curve M and the distribution T be as in Theorem A. For any open ball U, denote by U^x the projection of U onto the x-axis. There exist three open balls U_j with $U_j^x = (a_j, b_j)$, $1 \le j \le 3$, functions $\alpha_j(x) \in C_0^\infty(R^2)$, $1 \le j \le 3$ such that

(i) $M \subset \cup_1^3 U_j$
(ii) $\psi^k(x) \ge c > 0$ on $[a_j, b_j]$
(iii) $\operatorname{supp}(\alpha_j) \subset U_j$
(iv) $\sum_1^3 \alpha_j(x) = 1$ in a neighborhood of M

Since $\operatorname{supp}(T) \subset M$, we see that $T = \sum_1^3 (\alpha_j T) = \sum_1^3 T_j$. We may assume $(a_2, b_2) \subset (a, b)$, (a_1, b_1) contains the point a and (a_3, b_3) contains the point b. For T_2, one can apply Theorem B directly, so it remains to control T_1 and T_3.

For T_1, following the proof of Theorem B, we first define Σ_1 and let $\theta(x) \in C_0^\infty(a_1, b_1)$ so that $\Sigma_1 = \theta\Sigma_1$. Then let $\phi(x) \in C_0^\infty(-1, 0)$ ($\int_R \phi(x)\,dx = 1$) so that $\operatorname{supp}(\check{\phi}_h) \subset (0, h)$. This implies that for all small h, $\operatorname{supp}(\Sigma_1 * \check{\phi}_h) \subset [a, b_1)$ since $\operatorname{supp}(\Sigma_1) \subset [a, b_1)$. Now we define $T_{1h} \in S'(R^2)$ by

$$\langle T_{1h}, f\rangle = \langle \Sigma_1 * \check{\phi}_h, f \circ \beta\rangle \quad \text{for } f \in S(R^2).$$

This construction guarantees (1) when T, T_h are replaced by T_1, T_{1h} respectively. The verification of (2) is the same as the proof of Theorem B.

The treatment for T_3 is similar. This finishes the proof of Theorem A.

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