REAL INTERPOLATION FOR DIVISIBLE CONES

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We give necessary and sufficient conditions on a general cone of positive functions to satisfy the Decomposition Property (DP) introduced in [5] and connect the results with the theory of interpolation of cones introduced by Sagher [9]. One of our main result states that if Q satisfies DP or equivalently is divisible, then for the quasi-normed spaces E_0 and E_1,

$$(Q \cap E_0, Q \cap E_1)_{\theta,q} = Q \cap (E_0^Q, E_1^Q)_{\theta,q},$$

where $E^Q = \{f; \tilde{Q}f \in E\}$ with $\tilde{Q}f = \inf\{g \in Q; |f| \le g\}$.

According to this formula, it yields that the interpolation theory for divisible cones can be easily obtained from the classical theory.

1. Introduction and preliminaries

Given a vector space V, a set $Q \subset V$ is called a cone if $Q + Q \subset Q$ and $\lambda Q \subset Q$, for every $\lambda > 0$.

In [9], the author gives the definition of the interpolation spaces $(Q_0, Q_1)_{\theta,q}$ where Q_i is a cone instead of a quasi-normed space E_i, as in the classical case (see [2]).

In particular, if Q is a cone contained in the sum space $E_0 + E_1$ and we consider $Q_i = Q \cap E_i$, it yields that

$$(Q_0, Q_1)_{\theta,q} = \left\{ f \in Q; \left(\int_0^\infty \left(\frac{K^Q(t, f; \overline{E})}{t^\theta} \right)^q \frac{dt}{t} \right)^{1/q} < \infty \right\},$$

where

$$K^Q(t, f; \overline{E}) = \inf\{\|f_0\|_{E_0} + t\|f_1\|_{E_1} : f = f_0 + f_1, f_i \in Q\}.$$

From this fact, it follows immediately that

$$(Q_0, Q_1)_{\theta,q} \subset Q \cap (E_0, E_1)_{\theta,q},$$

and a cone $Q \subset E_0 + E_1$ is called a *Marcinkiewicz cone* (MC) (see [9]) if, for every $0 < \theta < 1$ and every $0 < q \leq \infty$,

$$(Q_0, Q_1)_{\theta,q} = Q \cap (E_0, E_1)_{\theta,q}.$$

To get such an equality is not an easy task, but it follows immediately if we can show that, for every $f \in Q$,

$$K^Q(t, f; \overline{E}) \approx K(t, f; \overline{E}). \tag{1}$$

For the case when $Q = D_c$ is the cone of positive and decreasing right-continuous functions on $(0, \infty)$, this has been studied in [5] where the authors define Marcinkiewicz pairs as those pairs $\overline{E}$ satisfying (1).

For this purpose, it turns out to be fundamental what they called the Decomposition Property for cones:

Definition (Decomposition Property). We say that a cone Q satisfies the Decomposition Property or simply Q satisfies DP, if the following condition holds: given $f, f_0, f_1 \in Q$ so that $f \leq f_0 + f_1$, there exists $\phi_0, \phi_1 \in Q$ such that $\phi_0 \leq f_0$, $\phi_1 \leq f_1$ and $f = \phi_0 + \phi_1$.

In [5] the Decomposition Property is proved for $Q = D_c$.

Another example of a cone satisfying DP is given by the cone $Q = C$ of positive concave functions in $(0, \infty)$ (see [3]). Moreover, the Decomposition Property for this cone is fundamental in the study of the K-divisibility property for compatible couples ([1, 3]).

The results of [9] and [5] have had interesting applications in the setting of Fourier series and in the theory of boundedness of operators acting on decreasing functions, respectively.

The paper is organized as follows. In Section 2, we give necessary and sufficient conditions on a general cone of positive functions to satisfy DP and in Section 3, we establish the connection of this property with the theory of interpolation of cones, we have mentioned above. Section 4 is devoted to present some formulae for the K-functional for different divisible cones.

Important remark. In Section 2, we deal with a cone Q of pointwise defined functions on an arbitrary set Ω and hence $f = g$ means $f(x) = g(x)$ for every $x \in \Omega$, while in Section 3, we work with a cone $[Q]$ of equivalence classes of measurable functions defined on a σ-finite measure space Ω and consequently $f = g$ means $f(x) = g(x)$ almost everywhere. (We use the same convention for the symbols $\leq$ and $\geq$).

Also, we shall write $f \lesssim g$ or $g \gtrsim f$ if there exists a strictly positive constant M such

that, $f \leq Mg$. If $f \lesssim g$ and $g \lesssim f$, f and g are said to be equivalent and we write $f \approx g$.

2. Some general results concerning the Decomposition Property

Let $\mathcal{F}(\Omega)$ be the set of all real valued functions defined on an arbitrary set Ω and let $\overline{\mathcal{F}}(\Omega)$ be $\mathcal{F}(\Omega)$ together with the constant function $+\infty$.

Let us consider an arbitrary nonempty cone Q of $\mathcal{F}(\Omega)$. Throughout the paper the functions in Q are assumed to take nonnegative values. To this cone, we associate the sublinear function $\tilde{Q} : \mathcal{F} \to \overline{\mathcal{F}}$ defined as

$$\tilde{Q}f := \inf\{g \in Q; |f| \leq g\},$$

where $\tilde{Q}f \equiv +\infty$ when no $g \in Q$ majorizes $|f|$.

Let us define, for $x, y \in \Omega$,

$$h^Q(x, y) := \inf\left\{\frac{f(y)}{f(x)}; f \in Q, f(x) \neq 0\right\}.$$

In what follows this function will be fundamental.

We also need to deal with an explicit formula for $\tilde{Q}f$ and this will be given in the next proposition. We observe that if $\inf_i f_i \in Q$ whenever $f_i \in Q$ for every i, then one can easily see that

$$\tilde{Q}f(y) = \inf_{h \in Q}\left[\sup_x\left(\frac{|f(x)|}{h(x)}\right)h(y)\right].$$

However, for our purpose, it will be very much useful to interchange the inf and sup in the above expression.

Proposition 2.1. *Let Q be a cone and let us define, for $y \in \Omega$,*

$$F_f^Q(y) := \sup_x(|f(x)|h^Q(x, y)).$$

(i) *If $F_f^Q \in Q$, then $F_f^Q = \tilde{Q}f$.*

(ii) *If there exists $y \in \Omega$ such that $F_f^Q(y) = +\infty$, then $\tilde{Q}f \equiv +\infty$.*

Proof.   (i) First we note that

$$F_f^Q(y) \geq |f(y)|h^Q(y, y) = |f(y)|,$$

and hence F_f^Q is a majorant of $|f|$ and thus, $\tilde{Q}f \leq F_f^Q$. On the other hand, if $g \in Q$ is a majorant of $|f|$, then

$$F_f^Q(y) \leq \sup_x \left(g(x) \inf_{r \in Q} \frac{r(y)}{r(x)} \right) \leq \sup_x \left(g(x) \frac{g(y)}{g(x)} \right) = g(y),$$

and since $F_f^Q \in Q$, it follows that $F_f^Q \leq \tilde{Q}f$ and the proof is complete.

(ii) By hypothesis, given $\lambda > 0$, there exists an x such that $\lambda < |f(x)|h^Q(x, y)$. Assume that $\tilde{Q}f$ is finite; that is, there exists $g \in Q$ which is a majorant of $|f|$. Then $\lambda < |f(x)|h^Q(x, y) \leq g(x)h^Q(x, y) \leq g(y)$. Hence $g \notin Q$ and we are done.   □

Let now Ψ_Q be the collection of sets of the type $\{x \in \Omega; f(x) \geq g(x)\}$ for $f, g \in Q$. Then, our first main result is the following:

Theorem 2.2.   *Assume that the cone $Q \subset \mathcal{F}(\Omega)$ satisfies the following conditions:*

(a) *$F_f^Q \in Q$ for all $f \in \mathcal{F}$ such that F_f^Q is finite.*

(b) *If $g_0, g_1 \in Q$ and $E \in \Psi_Q$ are such that $\tilde{Q}(g_i \chi_E) = g_i$, for $i = 0, 1$, then $\tilde{Q}((g_0 + g_1)\chi_E) = g_0 + g_1$.*

Then, Q satisfies DP.

Proof.   First, we observe that (a) implies that if $\{f_\alpha\}$ is a linearly ordered subset of Q, then $\inf f_\alpha \in Q$. In fact, since $f_\alpha \in Q$ and (a) holds, we have that $f_\alpha = \sup_x f_\alpha(x)h^Q(x, \cdot)$ and, by taking infimum on both sides, we find that

$$\inf_\alpha f_\alpha \leq \sup_x \left[\inf_\alpha f_\alpha(x)h^Q(x, \cdot) \right] \leq \inf_\alpha f_\alpha,$$

which again is in Q by (a).

Take f, f_0, f_1 in Q such that $f \leq f_0 + f_1$. We begin by following the proof of Theorem 3.2.2 from [3]. Consider the set

$$A := \{(g, h) \in Q \times Q; f \leq g + h, g \leq f_0 \text{ and } h \leq f_1\}.$$

On A we define the order

$$(g, h) \leq (g', h') \Leftrightarrow g \leq g' \text{ and } h \leq h'.$$

We will now show that a minimal element of A gives the decomposition of f that we are looking for. Let (g_0, g_1) be a minimal element in A, which exists by Zorn's Lemma

and (a), and let us prove that $f = g_0 + g_1$. Assume that, for some $\alpha_0 < 1$, and for every $\alpha \in [\alpha_0, 1)$, the set

$$E_\alpha := \{x; f(x) < \alpha(g_0(x) + g_1(x))\},$$

is nonempty. Hence since $\tilde{Q}f = f$ and

$$f = \frac{fg_0}{g_0 + g_1} + \frac{fg_1}{g_0 + g_1},$$

we get

$$f \le \tilde{Q}\left(\frac{fg_0}{g_0 + g_1}\right) + \tilde{Q}\left(\frac{fg_1}{g_0 + g_1}\right) \le g_0 + g_1.$$

Thus, by the minimality of (g_0, g_1) we have that, for $i = 0, 1$,

$$\tilde{Q}\left(\frac{fg_i}{g_0 + g_1}\right) = g_i,$$

which, by Proposition 2.1, implies that

$$g_i(y) = \sup_x \frac{f(x)g_i(x)}{g_0(x) + g_1(x)} h^Q(x, y).$$

We now observe that it is sufficient to take supremum over E_α^c, since,

$$\sup_{x \in E_\alpha} \frac{f(x)g_i(x)}{g_0(x) + g_1(x)} h^Q(x, y) \le \alpha \sup_{x \in E_\alpha} g_i(x)h^Q(x, y) < g_i(y).$$

Hence,

$$g_i(y) = \sup_{x \in E_\alpha^c} \frac{f(x)g_i(x)}{g_0(x) + g_1(x)} h^Q(x, y) \le \sup_{x \in E_\alpha^c} g_i(x)h^Q(x, y) \le g_i(y),$$

and we have that $\tilde{Q}(g_i \chi_{E_\alpha^c}) = g_i$. Hence, it follows from (b) that

$$(g_0 + g_1)\chi_{E_1} = \left(\tilde{Q}(g_0 \chi_{E_\alpha^c}) + \tilde{Q}(g_1 \chi_{E_\alpha^c})\right)\chi_{E_1} = \tilde{Q}((g_0 + g_1)\chi_{E_\alpha^c})\chi_{E_1}$$

$$\le \frac{1}{\alpha}\tilde{Q}(f\chi_{E_\alpha^c})\chi_{E_1} \le \frac{1}{\alpha} f\chi_{E_1},$$

and letting α go to 1, we find that

$$(g_0 + g_1)\chi_{E_1} \le f\chi_{E_1} < (g_0 + g_1)\chi_{E_1},$$

which is a contradiction. Therefore we conclude that $E_\alpha = \emptyset$ for every $0 < \alpha < 1$ and hence $f = g_0 + g_1$ which finishes the proof. $\qquad\square$

The next result shows that, under a fairly mild restriction, condition (b) above is also necessary.

Theorem 2.3. *If a cone $Q \subset \mathcal{F}(\Omega)$ satisfies DP and for every $g \in Q$ and every set E, $\tilde{Q}(g\chi_E) \in Q$, then (b) holds for every set $E \subset \Omega$.*

Proof.   Let us assume that (b) does not hold. Then there exist g_0 and g_1 in Q and E so that $\tilde{Q}(g_i\chi_E) = g_i$ and

$$\tilde{Q}((g_0 + g_1)\chi_E) \neq g_0 + g_1. \tag{2}$$

Since the Decomposition Property holds, we have that $\tilde{Q}((g_0 + g_1)\chi_E) = h_0 + h_1$ for some $h_i \leq g_i$, $h_i \in Q$. Obviously, $h_i\chi_E = g_i\chi_E$ and hence $\tilde{Q}(h_i\chi_E) = g_i$. Therefore,

$$\tilde{Q}((g_0 + g_1)\chi_E) = h_0 + h_1 \geq \tilde{Q}(h_0\chi_E) + \tilde{Q}(h_1\chi_E) = g_0 + g_1 \geq \tilde{Q}((g_0 + g_1)\chi_E),$$

which contradicts (2) and we are done. $\qquad\square$

In the sequel, we say that Q is a divisible cone if it satisfies DP and a non-divisible cone if it does not.

2.1. Divisible cones

We already know that the cone C of concave and the cone D_c of decreasing right continuous functions on $\mathbb{R}^+$ are divisible cones.

We should point out that it is very easy to check that the cone D of decreasing functions on $\mathbb{R}^+$ satisfies the conditions in Theorem 2.2 and hence, we obtain a proof of Cerdà and Martin's result [5, Lemma 1] without assuming that the functions are right continuous. On the contrary, this is not the case for C, since although this cone satisfies (b), it fails to satisfy condition (a).

Also, if a cone satisfies that given f_0 and f_1 in the cone, the functions $f_0 f_1$ and f_0/f_1 belong to Q (whenever it is well defined), then the Decomposition Property holds trivially. Hence we have, for example, that the cone of positive and radial functions or the cone of positive and periodic functions on $\mathbb{R}^n$ are in the set of divisible cones.

Let us give a list of other divisible cones:

1. Decreasing sequences in $\mathbb{Z}$.

2. Quasi-monotone sequences (see [9]); that is, those sequences $(a_n)_n$ so that a_n/n^β is decreasing for some $\beta > 0$.

3. Sequences $(a_n)_n$ so that a_n/b_n is decreasing for some fixed sequence (b_n).

4. The cone of all functions f so that fh is decreasing for a fixed function h.

5. Increasing functions (and all the above examples with the increasing property instead of decreasing).

However, although one can easily see that all these cones satisfy the conditions (a) and (b) of our Theorem 2.2, the Decomposition Property can be proved as an easy consequence of the decreasing case D.

Let us consider now the following cone: let A be any set in $\mathbb{R}^+$ and let us say that a function f is decreasing in A if, for every $x, y \in A, x \geq y$ we have $f(x) \leq f(y)$. If

$$D^A = \{f : \mathbb{R}^+ \to \mathbb{R}^+; f \text{ is decreasing in } A\},$$

then D^A is a divisible cone.

To see this, we first observe that

$$h^{D^A}(x, y) = \begin{cases} 1 & \text{if } x = y, \\ 1 & \text{if } x \geq y \text{ and } x, y \in A, \\ 0 & \text{elsewhere.} \end{cases}$$

Hence,

$$F_f^{D^A}(y) = \begin{cases} |f(y)| & \text{if } y \notin A, \\ \sup_{x \geq y} |f(x)\chi_A(x)| & \text{if } y \in A, \end{cases}$$

and condition (a) holds,

Also, for a function $f \in D^A$, we can deduce from the previous expression that $\widetilde{D^A}(f\chi_E) = f$ if and only if

$$f(y) = \begin{cases} 0 & \text{if } y \notin E \cup A, \\ \lim_{x \to x_y, x \in A \cap E} f(x) & \text{if } y \in A \text{ and } x_y \notin A \cap E \cap [y, \infty), \\ f(x_y) & \text{if } y \in A \text{ and } x_y \in A \cap E \cap [y, \infty), \end{cases}$$

where $x_y = \inf A \cap E \cap [y, \infty)$ and the above limit is zero whenever $A \cap E \cap [y, \infty) = \emptyset$. From this, also condition (b) is easily checked.

Using the same technique one can easily see that, for a collection of sets $\{A_i\}_i$ in an ordered measure set Ω (say, $\mathbb{R}^+$ or $\mathbb{Z}$), also

$$Q = \{f : \Omega \to \mathbb{R}^+; f \text{ is decreasing in } A_i, \forall i\},$$

is a divisible cone.

2.2. Non-divisible cones

Using Theorem 2.3 we can also give some examples of non-divisible cones.

Example 1. Let $Q = QC$ be the cone of quasi-concave functions; that is, increasing functions so that $f(x)/x$ is decreasing. If we consider

$$g_0(y) = \begin{cases} 1 & \text{if } y \leq 1, \\ y & \text{if } y > 1, \end{cases} \qquad g_1(y) = \begin{cases} 1 & \text{if } y \leq 1/2, \\ 2y & \text{if } y > 1/2, \end{cases}$$

and $E = (0, 1/3] \cup [8/3, \infty)$, then one can easily see that $\tilde{Q}(g_i \chi_E) = g_i (i = 0, 1)$ but

$$\tilde{Q}((g_0 + g_1)\chi_E) = \begin{cases} 2 & \text{if } y \leq 2/3, \\ 3y & \text{if } y > 2/3, \end{cases}$$

and hence $\tilde{Q}((g_0 + g_1)\chi_E) \neq g_0 + g_1$. Therefore, QC is a non-divisible cone.

Similarly, one can easily see that, if h_1 is an increasing function, h_2 is decreasing and h_1/h_2 is a bijection, then

$$Q_{h_1}^{h_2} = \{f : \mathbb{R}^+ \to \mathbb{R}^+; \quad fh_1 \text{ increases and } fh_2 \text{ decreases}\}$$

is a non-divisible cone. To see this, we only have to observe that $f \in Q_{h_1}^{h_2}$ if and only if $g_0(f \circ g_1) \in QC$ where

$$g_0 = h_1 \circ \left(\frac{h_1}{h_2}\right)^{-1} \qquad \text{and} \qquad g_1 = \left(\frac{h_1}{h_2}\right)^{-1}. \tag{3}$$

Hence, we get that $Q_{h_1}^{h_2}$ is a non-divisible cone since this holds for QC and g_j $(j = 0, 1)$ are increasing functions.

In particular, given $\alpha > \beta$, the cone

$$Q_\alpha^\beta = \{f : \mathbb{R}^+ \to \mathbb{R}^+; \quad f(x)x^\alpha \text{ increases and } f(x)x^\beta \text{ decreases}\}$$

is a non-divisible cone.

Example 2. Let us now consider the cone $Q = D_n$ of positive functions on $\mathbb{R}^n_+$ so that they are decreasing in each variable. For simplicity, we shall restrict ourselves to the case $n = 2$.

If $\bar{z} = (z_1, z_2) \in \mathbb{R}^2_+$ and by $\bar{x} \leq \bar{y}$ we mean $x_i \leq y_i$, $i = 1, 2$, we find that

$$h^Q(\bar{x}, \bar{y}) = \begin{cases} 1 & \text{if } \bar{y} \leq \bar{x}, \\ 0 & \text{elsewhere}, \end{cases}$$

and, thus,

$$F_f(\bar{y}) = \sup_{\bar{x} \geq \bar{y}} |f(\bar{x})|.$$

Therefore, if we take $E = \{(1, 2), (2, 1)\}$, then

$$g_0(x, y) = \begin{cases} 3 & \text{if } 0 < x \leq 1 \text{ and } 0 < y \leq 2, \\ 2 & \text{if } 1 < x \leq 2 \text{ and } 0 < y \leq 1, \\ 0 & \text{elsewhere,} \end{cases}$$

and

$$g_1(x, y) = \begin{cases} 7 & \text{if } 0 < x \leq 2 \text{ and } 0 < y \leq 1, \\ 3 & \text{if } 0 < x \leq 1 \text{ and } 1 < y \leq 2, \\ 0 & \text{elsewhere,} \end{cases}$$

so that $\tilde{Q}(g_i \chi_E) = g_i$ but $\tilde{Q}((g_0 + g_1)\chi_E) \neq g_0 + g_1$ and, hence, D_2 and similarly D_n are non-divisible cones.

3. Connection between the Decomposition Property and interpolation theory

Let now $L_0(\Omega)$ be the set of all equivalence class $[f]$ of real valued measurable functions defined on a σ-finite measure space (Ω, Σ, μ).

A function lattice E is understood to be a quasi-normed space in $L_0(\Omega)$ which satisfies that if $|f| \leq |g|$ a.e. and $g \in E$, then $f \in E$ and $\|f\| \leq \|g\|$.

Throughout, the convention that $\|f\|_X = +\infty$ for $f \notin X$ will be used and $X = Y$ will mean that X and Y are equal as sets and their quasi-norms are equivalent.

Also the letter Q will indicate through the rest of the paper a cone of measurable functions pointwise defined on Ω while we shall write

$$[Q] = \{[f]; \ f \in Q\}.$$

From now on, we shall write as usual $f, g, \ldots$ instead of $[f], [g], \ldots$ when it is clear from the context but we shall try to be precise with the notation Q and $[Q]$. However, when working with a couple of lattices $\overline{E}$, we shall write $K^Q(t, f; \overline{E})$ to indicate $K^{[Q]}(t, [f]; \overline{E})$.

Following the notation in the previous section, let $\overline{L}_0(\Omega)$ be $L_0(\Omega)$ together with the constant function $+\infty$ and set $\tilde{Q} : L_0 \to \overline{L}_0$ defined by

$$\tilde{Q}f := \inf\{g \in [Q]; \ |f| \leq g\},$$

in the sense that either $\tilde{Q}f \equiv \infty$ or $\tilde{Q}f$ is a measurable function so that $|f| \leq \tilde{Q}f$ and for every $g \in [Q]$ so that $|f| \leq g$ we have $\tilde{Q}f \leq g$.

For a function lattice E, we define

$$E^Q := \{f \in L_0; \ \tilde{Q}f \in E\},$$

which will be a function lattice under the quasi-norm

$$\|f\|_{E^Q} := \|\tilde{Q}f\|_E.$$

Clearly, we have $E^Q \hookrightarrow E$, and if $\tilde{Q}$ is bounded from E into E, then $E = E^Q$. We will also consider F^Q, with the obvious definition, when F is only a subset of a lattice. It trivially holds that

$$[Q] \cap E \hookrightarrow E^Q \hookrightarrow E.$$

Throughout the rest of the paper, we shall assume that $\tilde{Q}f$ is measurable, for every $f \in L_0(\Omega)$ and when working with $\overline{E}^Q = (E_0^Q, E_1^Q)$ we shall also assume that

$$\tilde{Q}f \in [Q] \qquad \text{for all} \quad f \in (E_0 + E_1)^Q.$$

Moreover,

Theorem 3.1. *If $[Q]$ satisfies DP, then*

$$K(t, f; \overline{E}^Q) = K^Q(t, \tilde{Q}f; \overline{E}).$$

Proof.   Let $\tilde{Q}f = f_0 + f_1$, with $f_i \in [Q]$. Then, according to the lattice property, we find that

$$K(t, f; \overline{E}^Q) \leq K(t, \tilde{Q}f; \overline{E}^Q) \leq \|f_0\|_{E_0} + t\|f_1\|_{E_1}.$$

For the converse, we take $f = f_0 + f_1$. According to the sublinearity property of $\tilde{Q}$, we have that $\tilde{Q}f \leq \tilde{Q}f_0 + \tilde{Q}f_1$ and, by the Decomposition Property, there exist $\phi_0, \phi_1 \in [Q]$ such that $\tilde{Q}f = \phi_0 + \phi_1$ and $\phi_i \leq \tilde{Q}f_i$, $i = 0, 1$. Hence

$$K^Q(t, \tilde{Q}f; \overline{E}) \leq \|\phi_0\|_{E_0} + t\|\phi_1\|_{E_1} \leq \left\|\tilde{Q}f_0\right\|_{E_0} + \left\|\tilde{Q}f_1\right\|_{E_1} = \|f_0\|_{E_0^Q} + t\|f_1\|_{E_1^Q},$$

and the proof follows by taking the infimum. □

In particular, we have that if $[Q]$ is divisible,

$$K^Q(t, f; \overline{E}) = K(t, f; \overline{E}^Q), \tag{4}$$

for every $f \in [Q]$ and then, the following useful corollary is easily obtained.

Corollary 3.2. *If $[Q]$ satisfies DP and $Q_i = E_i \cap [Q]$, then*

$$(Q_0, Q_1)_{\theta,q} = [Q] \cap (E_0^Q, E_1^Q)_{\theta,q},$$

for all $0 < \theta < 1$ and $0 < q \leq \infty$.

Remark 3.3.   (a) Since $[Q] \cap E_i = [Q] \cap E_i^Q$, it follows from Corollary 3.2, that if $[Q]$ satisfies DP, then $[Q]$ is a MC with respect to the couple (E_0^Q, E_1^Q), (this fact also follows from Lemma 2 in [6]).

(b) Since $\tilde{Q} : E_i^Q \to E_i^Q$ is bounded, it follows by interpolation that

$$(E_0^Q, E_1^Q)_{\theta, q} = (E_0^Q, E_1^Q)_{\theta, q}^Q. \tag{5}$$

(c) Following the notation in [4], we have that $Q_i = Q_{E_i; \tilde{Q}}$ and hence formula (4) could also be obtained from Remark 2.4 above and Theorem 2 in the mentioned paper. However, with this technique we can only obtain $K^Q(t, f; \overline{E}) \approx K(t, f; \overline{E}^Q)$ but not the equality.

(d) We should point out that the Decomposition Property for cones is connected with the S-divisibility property introduced in [4], in the following sense: if a cone $[Q]$ satisfies DP, then $[Q]$ is $\tilde{Q}$-divisible with respect to $[Q]$ and every couple of lattices $\overline{E}$.

Now, using Corollary 3.2 and expression (5), we can obtain the following reiteration formula which, in the case $Q = D_c$, was proved in [5]:

Theorem 3.4.   *If $[Q]$ satisfies DP and $Q_i = E_i \cap [Q]$, then*

$$\left((Q_0, Q_1)_{\theta_0, q_0}, (Q_0, Q_1)_{\theta_1, q_1}\right)_{\lambda, q} = (Q_0, Q_1)_{\theta, q},$$

where $\theta_0 \neq \theta_1$ and $\theta = (1 - \lambda)\theta_0 + \lambda\theta_1$.

Proof.   By Corollary 3.2, (5) and the classical reiteration theorem, we obtain that

$$
\begin{aligned}
\left((Q_0, Q_1)_{\theta_0, q_0}, (Q_0, Q_1)_{\theta_1, q_1}\right)_{\lambda, q} &= \left([Q] \cap (E_0^Q, E_1^Q)_{\theta_0, q_0}, [Q] \cap (E_0^Q, E_1^Q)_{\theta_1, q_1}\right)_{\lambda, q} \\
&= [Q] \cap \left((E_0^Q, E_1^Q)_{\theta_0, q_0}^Q, (E_0^Q, E_1^Q)_{\theta_1, q_1}^Q\right)_{\lambda, q} \\
&= [Q] \cap \left((E_0^Q, E_1^Q)_{\theta_0, q_0}, (E_0^Q, E_1^Q)_{\theta_1, q_1}\right)_{\lambda, q} \\
&= [Q] \cap (E_0^Q, E_1^Q)_{\theta, q} = (Q_0, Q_1)_{\theta, q}. \qquad \square
\end{aligned}
$$

Similar technique can be used to prove the following extension of the Holmstedt formula (see also [5] for the decreasing case):

Theorem 3.5.   *If $[Q]$ satisfies DP and $Q_i = E_i \cap [Q]$, then*

$$K(t, f; (Q_0, Q_1)_{\theta_0, q_0}, (Q_0, Q_1)_{\theta_1, q_1}) \approx \left[\int_0^{t^{1/\lambda}} \left(\frac{K^Q(t, f; \overline{E})}{t^{\theta_0}}\right)^{q_0} \frac{dt}{t}\right]^{1/q_0} + \left[\int_{t^{1/\lambda}}^\infty \left(\frac{K^Q(t, f; \overline{E})}{t^{\theta_1}}\right)^{q_1} \frac{dt}{t}\right]^{1/q_1},$$

where $0 \leq \theta_0 < \theta_1 \leq 1$ and $\lambda = \theta_1 - \theta_0$.

Proof.   Let $f \in [Q]$. Then

$$
\begin{aligned}
K(t, f; (Q_0, Q_1)_{\theta_0, q_0}, (Q_0, Q_1)_{\theta_1, q_1}) &= K(t, f; [Q] \cap (E_0^Q, E_1^Q)_{\theta_0, q_0}, [Q] \cap (E_0^Q, E_1^Q)_{\theta_1, q_1}) \\
&= K^Q(t, f; (E_0^Q, E_1^Q)_{\theta_0, q_0}, (E_0^Q, E_1^Q)_{\theta_1, q_1}) \\
&= K(t, f; (E_0^Q, E_1^Q)_{\theta_0, q_0}^Q, (E_0^Q, E_1^Q)_{\theta_1, q_1}^Q) \\
&= K(t, f; (E_0^Q, E_1^Q)_{\theta_0, q_0}, (E_0^Q, E_1^Q)_{\theta_1, q_1}),
\end{aligned}
$$

and the result now follows from the classical Holmstedt formula and (4). $\square$

Remark 3.6.   Let $\overline{F} = (F_0, F_1)$ be a compatible couple of lattices. Using again (4) and the general reiteration formula (see [1, 8, 4]) we get that if

$$
\overline{E}_{F; K^Q} := \{f \in (E_0 + E_1) \cap [Q]; K^Q(\cdot, f; \overline{E}) \in F\},
$$

then

$$
K(t, f; \overline{E}_{F_0; K^Q}, \overline{E}_{F_1; K^Q}) \approx K(t, K^Q(\cdot, f; \overline{E}); F_0^C, F_1^C),
$$

and, consequently, for every lattice G,

$$
\big((E_0, E_1)_{F_0; K^Q}, (E_0, E_1)_{F_1; K^Q}\big)_{G; K} = (E_0, E_1)_{(F_0^C, F_1^C)_{G; K}; K^Q}.
$$

Moreover, if we deal with Marcinkiewicz cones (see [9]), we have the following result:

Corollary 3.7.   *If $[Q] \subset E_0 + E_1$ is a Marcinkiewicz cone satisfying DP, then*

$$
(E_0^Q, E_1^Q)_{\theta, q} = (E_0, E_1)_{\theta, q}^Q, \tag{6}
$$

for all $0 < \theta < 1$ and $0 < q \leq \infty$.

Proof.   By Theorem 3.1, we have that, if $Q_i = E_i \cap [Q]$, then

$$
(E_0^Q, E_1^Q)_{\theta, q} = \{f \in E_0^Q + E_1^Q; \tilde{Q}f \in (Q_0, Q_1)_{\theta, q}\},
$$

and, hence,

$$
\begin{aligned}
(E_0^Q, E_1^Q)_{\theta, q} &= \{f \in E_0^Q + E_1^Q; \tilde{Q}f \in [(E_0, E_1)_{\theta, q} \cap [Q]]\} \\
&= \{f \in E_0^Q + E_1^Q; \tilde{Q}f \in (E_0, E_1)_{\theta, q}\} = (E_0, E_1)_{\theta, q}^Q. \quad \square
\end{aligned}
$$

Our next step is to extend (6) to cones which are not divisible. For this purpose we have to introduce the notion of equivalent cones.

Definition 3.8. We say that the cones Q_0 and Q_1 are c-equivalent, and we write $Q_0 \approx Q_1$, if the following condition is satisfied: for every $f \in Q_0$ or $f \in Q_1$, there exists $g \in Q_1$ or $g \in Q_0$, respectively, so that $\frac{1}{c}g \leq f \leq cg$ and if so, we write $f \overset{c}{\approx} g$.

The next result establishes the connection between two cones Q_i which are equivalent and the corresponding equivalence for the operators $\tilde{Q}_i$.

Proposition 3.9. *Let $c > 0$ be fixed.*

(a) *If $Q_0 \overset{c}{\approx} Q_1$, then $\tilde{Q}_0 \overset{c^2}{\approx} \tilde{Q}_1$.*

(b) *Assume that $\tilde{Q}_i f \in Q_i$, for every $f \in Q_{1-i}$ ($i = 0, 1$). If $\tilde{Q}_0 \overset{c}{\approx} \tilde{Q}_1$, then $Q_0 \overset{c}{\approx} Q_1$.*

Proof. (a) Let $f : \Omega \to \mathbb{R}^+$ and let $g \in Q_0$ so that $f \leq g$. Let $h \in Q_1$ so that $f \leq g \leq h \leq c^2 g$. Then,

$$\tilde{Q}_1 f \leq h \leq c^2 g,$$

and therefore $\tilde{Q}_1 f \leq c^2 \tilde{Q}_0 f$.

(b) Let $f \in Q_0$. Since $\tilde{Q}_0 \overset{c}{\approx} \tilde{Q}_1$, we find that $\frac{1}{c}\tilde{Q}_1 f \leq \tilde{Q}_0 f = f \leq c\tilde{Q}_1 f$ and since $\tilde{Q}_1 f \in Q_1$ there exists an equivalent function to f in Q_1. Using the same argument for $f \in Q_1$, we get the result. □

We say that a cone is quasi-divisible if it is equivalent to a divisible cone.

It is known that $C \overset{2}{\approx} QC$ (see [3]) and, from this, we obtain that, for every h_1 and h_2 as before, the cone

$$C_{h_1}^{h_2} = \{f : \mathbb{R}^+ \to \mathbb{R}^+ ; g_0(f \circ g_1) \in C\},$$

with g_j as in (3), is equivalent to the cone $Q_{h_1}^{h_2}$. Moreover, using that C is a divisible cone, one can easily see that same holds for $C_{h_1}^{h_2}$ and, therefore, $Q_{h_1}^{h_2}$ is a quasi-divisible cone.

Observe that if Q_0 and Q_1 are two cones in $\mathcal{F}(\Omega)$ so that $Q_0 \approx Q_1$, then $[Q_0] \approx [Q_1]$ and hence $[C_{h_1}^{h_2}] \approx [Q_{h_1}^{h_2}]$.

Remark 3.10. (1) Obviously, if $\tilde{Q}_0 \overset{c}{\approx} \tilde{Q}_1$, then $E^{Q_0} = E^{Q_1}$ and, hence, Corollary 3.7 can be extended, in the obvious way, to quasi-divisible cones.

(2) Also, it is important to observe that, for $Q_{h_1}^{h_2}$, formula (6) is a reiteration result in the following sense: We have that

$$h^{Q_{h_1}^{h_2}}(x, y) = \min\left(\frac{h_1(x)}{h_1(y)}, \frac{h_2(x)}{h_2(y)}\right),$$

and

$$(\tilde{Q}_{h_1}^{h_2} f)(y) = \text{ess} \sup_x \left(|f(x)| h^{Q_{h_1}^{h_2}}(x, y) \right)$$

$$= \frac{1}{h_1(y)} \text{ess} \sup_x \left[|f(x)| \min \left(h_1(x), \frac{h_1(y)}{h_2(y)} h_2(x) \right) \right]$$

$$\approx \frac{1}{h_1(y)} K \left(\frac{h_1(y)}{h_2(y)}, f; L_\infty(h_1), L_\infty(h_2) \right).$$

Hence, if, for a lattice E, we define $\hat{E}$ by $\|f\|_{\hat{E}} = \|(f \circ (h_1/h_2))/h_1\|_E$, with g_j as in (3), then, for $Q = Q_{h_1}^{h_2}$,

$$\|f\|_{E^Q} = \|\tilde{Q}f\|_E = \|f\|_{(L_\infty(h_1), L_\infty(h_2))_{\hat{E};K}},$$

and hence $E^Q = (L_\infty(h_1), L_\infty(h_2))_{\hat{E};K}$. Therefore, if $\overline{A} = (L_\infty(h_1), L_\infty(h_2))$, formula (6) reads

$$(\overline{A}_{\hat{E}_0;K}, \overline{A}_{\hat{E}_1;K})_{0,q} = \overline{A}_{(E_0, E_1)\hat{}_{0,q};K},$$

which also follows from the more general reiteration formula for K-spaces in [8] and the fact that $(E_0, E_1)\hat{}_{0,q} = (\hat{E}_0, \hat{E}_1)_{0,q}$.

4. The K-functional for divisible cones

The purpose of this section is to compute some K-functionals for several divisible cones and for the couples $(L^{p_0}(w_0), L^{p_1}(w_1))$ and (X, L^∞) where X is a lattice. We shall use formula (4) from Section 3 and the fact that the K-functional for the couple $\overline{E}^Q$, can sometimes be explicitly computed as follows (see [4]):

Theorem 4.1. *Let $\overline{E}$ be a compatible couple of lattices and let us assume that the function $h^Q(x, \cdot)$ is measurable for a.e. x and that the functions $g_j(x) = \|h^Q(x, \cdot)\|_{E_j^Q}$ are also measurable. If, for each $t > 0$, there exist measurable sets $B_0(t)$ and $B_1(t)$ such that $B_0(t) \cup B_1(t) = \Omega$ and*

$$\sup_{t>0} \max \left(\frac{1}{t} \|\chi_{B_0(t)}/g_1\|_{E_0^Q}, t\|\chi_{B_1(t)}/g_0\|_{E_1^Q} \right) < \infty, \tag{7}$$

then

$$K(t, f; \overline{E}^Q) \approx \|f\chi_{B_0(t)}\|_{E_0^Q} + t\|f\chi_{B_1(t)}\|_{E_1^Q}, \tag{8}$$

for all $f \in [Q]$.

Lemma 4.2.   (a) *The cone $[D]$ is divisible.*

(b) *Let h_1 and h_2 be as in Section 2. Then $[C_{h_1}^{h_2}]$ is a divisible cone.*

Proof. (a) Let $f, f_0, f_1 \in [D]$ be such that $f \leq f_0 + f_1$. Let E be so that $|E| = 0$ and

$$(f\chi_{E^c})(x) \leq (f_0\chi_{E^c})(x) + (f_1\chi_{E^c})(x),$$

for every x. Then, $(f\chi_{E^c})$, $(f_0\chi_{E^c})$ and $(f_1\chi_{E^c})$ belong to the cone $D^{E^c} = \{g; g \text{ is decreasing in } E^c\}$. Since, we have shown in Section 2 that this cone is divisible, we get that there exists $g_i \in D^{E^c}$ $(i = 0, 1)$ so that $(f\chi_{E^c}) = g_0 + g_1$ and $g_i \leq (f_i\chi_{E^c})$. Then $[f] = [g_0] + [g_1]$ with $[g_i] \leq f_i$. Since clearly $[g_i] \in [D]$, we are done. (An alternative proof can be given using the fact that the cone D_c is divisible and if $f \in [D]$ there exists a representative in the class that it is in D_c.)

(b) Let now $f, f_0, f_1 \in [C_{h_1}^{h_2}]$ be such that $f \leq f_0 + f_1$ and set E as in (a). Then,

$$g_0(x)(f \circ g_1)(x) \leq g_0(x)(f_0 \circ g_1)(x) + g_0(x)(f_1 \circ g_1)(x),$$

for every $x \in E^c$, where g_i are as in (3). Now, since the above functions are continuous (they are concave), we get that the above inequality holds for every $x \in \Omega$ and hence, using that C is a divisible cone we get that there exist F_0 and F_1 so that $g_0(x)(f \circ g_1)(x) = F_0(x) + F_1(x)$ and $F_i(x) \leq g_0(x)(f_i \circ g_1)(x)$ for every $x \in \Omega$. From this the results follows immediately. $\square$

Let us now consider the couple $\overline{E} = (L^{p_0}(w_0), L^{p_1}(w_1))$ and let us write $W_i(t) = \int_0^t w_i(s)\, ds$.

Corollary 4.3. (see [4]) *If w_0 and w_1 satisfy that, for every $t > 0$, there exists $a_t \in [0, \infty]$ such that*

$$\left(\int_0^{a_t} \frac{w_0(s)}{W_1(s)^{p_0/p_1}}\, ds\right)^{\frac{1}{p_0}} \lesssim t \tag{9}$$

and

$$\left(\frac{W_1(a_t)}{W_0(a_t)^{p_1/p_0}} + \int_{a_t}^{\infty} \frac{w_1(s)}{W_0(s)^{p_1/p_0}}\, ds\right)^{\frac{1}{p_1}} \lesssim \frac{1}{t}, \tag{10}$$

then, for every decreasing function f,

$$K^D(t, f; L^{p_0}(w_0), L^{p_1}(w_1)) \approx \left(\int_0^{a_t} f(s)^{p_0} w_0(s)\, ds\right)^{\frac{1}{p_0}} + t\left(\int_{a_t}^{\infty} f(s)^{p_1} w_1(s)\, ds\right)^{\frac{1}{p_1}}.$$

In particular, the conditions (9) and (10) are satisfied in the following cases:

(i) If $w_i(x) \leq W_i(x)/x$, $i = 0, 1$, for almost every x and $W_0(t)^{1/p_0} W_1(t)^{-1/p_1} t^{-c}$ is increasing for some $c > 0$.

(ii) If there exist $\alpha > 0$ and $\beta > 0$ such that $\alpha p_0 < \beta p_1$ and W_0^α / W_1^β increases.

As a corollary of the previous result, one can easily get conditions on the weights w_0 and w_1 so that K^{Q_h} can be computed for the cone Q_h of positive functions f such that fh is decreasing.

If we now want to deal with the cone $Q = C_{h_1}^{h_2}$, we first need to observe that $h^Q = h^{Q_{h_1}^{h_2}}$ and

$$g_j(x) = \|h^Q(x, \cdot)\|_{(L^{p_j}(w_j))^Q} = \|h^Q(x, \cdot)\|_{L^{p_j}(w_j)}$$
$$= \left[\int_0^x \left(\frac{h_2(x)}{h_2(y)}\right)^{p_j} w_j(y)\, dy + \int_x^\infty \left(\frac{h_1(x)}{h_1(y)}\right)^{p_j} w_j(y)\, dy\right]^{1/p_j}.$$

Using this fact together with Theorem 4.1, we can obtain the following result:

Proposition 4.4.   *If w_0 and w_1 satisfy that*

$$\int_x^\infty \left(\frac{h_1(x)}{h_1(y)}\right)^{p_0} w_0(y)\, dy \lesssim \int_0^x \left(\frac{h_2(x)}{h_2(y)}\right)^{p_0} w_0(y)\, dy, \tag{11}$$

or

$$\int_0^x \left(\frac{h_2(x)}{h_2(y)}\right)^{p_1} w_1(y)\, dy \lesssim \int_x^\infty \left(\frac{h_1(x)}{h_1(y)}\right)^{p_1} w_1(y)\, dy, \tag{12}$$

and, for every $t > 0$, there exists $a_t \in [0, \infty]$ such that

$$\left(\int_0^{a_t} \frac{w_0(s)}{g_1(s)^{p_0}}\, ds\right)^{\frac{1}{p_0}} + \frac{h_1(a_t)}{g_1(a_t)} \left(\int_{a_t}^\infty \frac{w_0(s)}{h_1(s)^{p_0}}\, ds\right)^{\frac{1}{p_0}} \lesssim t, \tag{13}$$

and

$$\frac{h_2(a_t)}{g_0(a_t)} \left(\int_0^{a_t} \frac{w_1(s)}{h_2(s)^{p_1}}\, ds\right)^{\frac{1}{p_1}} + \left(\int_{a_t}^\infty \frac{w_1(s)}{g_0(s)^{p_1}}\, ds\right)^{\frac{1}{p_1}} \lesssim \frac{1}{t}, \tag{14}$$

then, for every $f \in C_{h_1}^{h_2}$,

$$K^{C_{h_1}^{h_2}}(t, f; L^{p_0}(w_0), L^{p_1}(w_1)) \approx \left(\int_0^{a_t} f(s)^{p_0} w_0(s)\, ds\right)^{\frac{1}{p_0}} + t\left(\int_{a_t}^\infty f(s)^{p_1} w_1(s)\, ds\right)^{\frac{1}{p_1}}. \tag{15}$$

Proof.   Let us assume that (11) holds. The proof follows similarly if instead (12) holds.

Conditions (13) and (14) imply that (7) holds and, thus, it only remains to prove that, for $f \in C_{h_1}^{h_2}$,

$$\|f\chi_{(0,a_t)}\|_{(L^{p_0}(w_0))^\varrho} + t\|f\chi_{(a_t,\infty)}\|_{(L^{p_1}(w_1))^\varrho} \approx \|f\chi_{(0,a_t)}\|_{L^{p_0}(w_0)} + t\|f\chi_{(a_t,\infty)}\|_{L^{p_1}(w_1)}.$$

In fact,

$$\|f\chi_{(0,a_t)}\|_{(L^{p_0}(w_0))^\varrho} \approx \|f\chi_{(0,a_t)}\|_{L^{p_0}(w_0)} + \left\| f(a_t)\chi_{(a_t,\infty)}\frac{h_1(a_t)}{h_1} \right\|_{L^{p_0}(w_0)} = I + II,$$

and to estimate II, we have to use (11) and (13) as follows:

$$\begin{aligned}
II &= f(a_t)h_1(a_t)\left(\int_{a_t}^\infty \frac{w_0}{h_1^{p_0}}\right)^{1/p_0} \lesssim f(a_t)g_1(a_t)t \\
&\approx t\left(f(a_t)h_1(a_t)\left(\int_{a_t}^\infty \frac{w_1}{h_1^{p_1}}\right)^{1/p_1} + f(a_t)h_2(a_t)\left(\int_0^{a_t} \frac{w_1}{h_2^{p_1}}\right)^{1/p_1}\right) \\
&\lesssim t\left(\int_{a_t}^\infty f^{p_1}w_1\right)^{1/p_1} + f(a_t)g_0(a_t) = t\|f\chi_{(a_t,\infty)}\|_{L^{p_1}(w_1)} + f(a_t)g_0(a_t) \\
&= III + IV.
\end{aligned}$$

Now, to estimate IV, we observe that (11) is equivalent to saying that

$$g_0(x) \approx \int_0^x \left(\frac{h_2(x)}{h_2(y)}\right)^{p_0} w_0(y)\, dy$$

and, hence,

$$IV \approx f(a_t)h_2(a_t)\left(\int_0^{a_t} \frac{w_0}{h_2^{p_0}}\right)^{1/p_0} \leq \|f\chi_{(0,a_t)}\|_{L^{p_0}(w_0)}.$$

Similarly,

$$\begin{aligned}
t\|f\chi_{(a_t,\infty)}\|_{(L^{p_1}(w_1))^\varrho} &\approx t\left\|\chi_{(0,a_t)}f(a_t)\frac{h_2(a_t)}{h_2}\right\|_{L^{p_1}(w_1)} + t\|f\chi_{(a_t,\infty)}\|_{L^{p_1}(w_1)} \\
&= V + VI.
\end{aligned}$$

Moreover, we use (14) and the above observation on g_0, to obtain the following estimate of V:

$$\begin{aligned}
V &= tf(a_t)h_2(a_t)\left(\int_0^{a_t} \frac{w_1}{h_2^{p_1}}\right)^{1/p_1} \lesssim f(a_t)g_0(a_t) \\
&\approx f(a_t)h_2(a_t)\left(\int_0^{a_t} \frac{w_0}{h_2^{p_0}}\right)^{1/p_0} \lesssim \left(\int_0^{a_t} f^{p_0}w_0\right)^{1/p_0} = \|f\chi_{(0,a_t)}\|_{L^{p_0}(w_0)},
\end{aligned}$$

from which the result follows easily. $\square$

Our next aim is to present a concrete example of how Proposition 4.4 can be applied and, for this purpose, let us assume that the following "B_p type" conditions hold, for $i = 0, 1$:

$$h_1(x)^{p_i} \int_x^\infty \frac{w_i(y)}{h_1(y)^{p_i}} \, dy \lesssim h_2(x)^{p_i} \int_0^x \frac{w_i(y)}{h_2(y)^{p_i}} \, dy. \tag{16}$$

Let us write $U_i(x) = \int_0^x \frac{w_i(y)}{h_2(y)^{p_i}} \, dy$. Then, one can easily see that $g_i(x) \approx h_2(x) U_i(x)^{1/p_i}$ and the following results follow:

Corollary 4.5. *Assume that* (16) *holds.*

(a) *Let* $U_1(\infty) = \infty$ *and there exist* $\alpha > 0$ *and* $\beta > 0$ *so that* $U_0^\alpha U_1^{-\beta}$ *increases and* $\beta p_1 > \alpha p_0$, *then* (15) *holds with* a_t *so that* $t = U_0^{1/p_0}(a_t) U_1^{-1/p_1}(a_t)$.

(b) *If*

$$\frac{w_i(y)}{h_2^{p_i}(y)} \lesssim \frac{U_i(y)}{y},$$

and there exists $c > 0$ *so that* $U_0(t)^{1/p_0} U_1(t)^{-1/p_1} t^{-c}$ *is increasing, then* (15) *holds for* a_t *so that* $t = U_0^{1/p_0}(a_t) U_1^{-1/p_1}(a_t)$.

Proof. Some straightforward calculations show that conditions (13) and (14) are satisfied. □

Remark 4.6. If $h_1(x) = x^\alpha$ and $h_2(x) = x^{-\beta}$, then the condition (16) reads $w_i(x)x^{\beta p_i} \in B_{(\alpha+\beta)p_i}$ which in the concave case reduces to $w_i(x)x^{p_i} \in B_{p_i}$.

Concerning the couple (X, L^∞) where X is a lattice, we have the following result:

Theorem 4.7. *Let* $[Q]$ *be a divisible cone such that*

(i) *For every* $f \in [Q]$, $\min(f, 1) \in [Q]$,

(ii) *For every* $f \in [Q]$, *there exists* $g \in [Q]$ *so that* $(f - 1)_+ \leq g$ *and* $\|g\|_X \lesssim \|(f - 1)_+\|_X$.

Then, $[Q]$ *is a Marcinkiewicz cone with respect the couple* (X, L^∞).

Proof. It is known (see [7]) that

$$K(t, f; X, L^\infty) = \inf_{\lambda > 0} \{ \|(f - \lambda)_+\|_X + t\| \min(f, \lambda)\|_\infty \}.$$

Let $f \in [Q]$. Since, $f = (f - \lambda)_+ + \min(f, \lambda)$, we have that $f \leq g + \min(f, \lambda)$ and by (i) and the Decomposition Property of $[Q]$, we have that there exist $h_j \in [Q]$ so that

$f = h_0 + h_1, h_0 \leq g$ and $h_1 \leq \min(f, \lambda)$. Hence,

$$K^Q(t, f; X, L^\infty) \leq \|h_0\|_X + t\|h_1\|_\infty \leq \|g\|_X + t\|\min(f, \lambda)\|_\infty$$
$$\lesssim \|(f - \lambda)_+\|_X + t\|\min(f, \lambda)\|_\infty,$$

and since λ is arbitrary, we find that, for every $f \in [Q]$,

$$K^Q(t, f; X, L^\infty) \lesssim K(t, f; X, L^\infty) \leq K^Q(t, f; X, L^\infty),$$

and the proof is complete. $\square$

Corollary 4.8. (a) $[D]$ *is a Marcinkiewicz cone with respect to the couple* (X, L^∞).

(b) *If* X *satisfies that the left translation operator* $\tau_a : X \to X$ *is uniformly bounded in* a*, then* $[C]$ *is a Marcinkiewicz cone with respect to the couple* (X, L^∞).

Proof. (a) The cone $[D]$ trivially satisfies (i) and (ii) with $g = (f - 1)_+$. Since by Lemma 4.2 $[D]$ is divisible we are done.

(b) Since f is increasing, we can take $g = \tau_a[(f - 1)_+]$ for some a and, hence, the conditions (i) and (ii) are satisfied. $\square$

A different proof of (a) was given in [5].

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