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A WILD BOOTSTRAP FOR DEPENDENT DATA

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This paper introduces a novel wild bootstrap for dependent data (WBDD) as a means of calculating standard errors of estimators and constructing confidence regions for parameters based on dependent heterogeneous data. The consistency of the bootstrap variance estimator for smooth function of the sample mean is shown to be robust against heteroskedasticity and dependence of unknown form. The first-order asymptotic validity of the WBDD in distribution approximation is established when data are assumed to satisfy a near epoch dependent condition and under the framework of the smooth function model. The WBDD offers a viable alternative to the existing non parametric bootstrap methods for dependent data. It preserves the second-order correctness property of blockwise bootstrap (provided we choose the external random variables appropriately), for stationary time series and smooth functions of the mean. This desirable property of any bootstrap method is not known for extant wild-based bootstrap methods for dependent data. Simulation studies illustrate the finite-sample performance of the WBDD.

1. INTRODUCTION

The bootstrap of Efron (1979) is a very popular and powerful nonparametric method to approximate the sampling distribution and the variance of complicated statistics based on independent and identically distributed (i.i.d.) observations. When dealing with independent but heterogeneously distributed observations, the so-called wild bootstrap, introduced by Wu (1986) and further studied by Liu (1988) and Mammen (1993), is commonly used and is known to perform well for approximatively linear statistics. As extensions of Efron's i.i.d. bootstrap and Wu's wild bootstrap to dependent observations, blocking-based and wild-based bootstrap methods for dependent data can be used, respectively, to approximate the sampling distributions and variances of statistics in time series. To form a bootstrap sample, blockwise bootstrap methods involve resampling blocks of observations, whereas wild-based bootstrap methods for dependent data require the use of

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auxiliary variables (which weights the data), in a manner appropriate to capture temporal dependence nonparametrically.

Künsch (1989) and Liu and Singh (1992) proposed the moving block bootstrap (MBB) method, which samples the overlapping blocks with replacement and then pastes the resampled blocks together to form a bootstrap sample. Based on the idea of resampling blocks, a few variants of the MBB have been developed, such as the nonoverlapping block bootstrap (NBB) (Carlstein, 1986), the stationary bootstrap (SB) (Politis and Romano, 1994), and the tapered block bootstrap (TBB) (Paparoditis and Politis, 2001, Paparoditis and Politis, 2002), among others. The idea of modifying and extending Wu's method (Wu, 1986) to dependent data appears in Yeh (1998), Inoue (2001), and Shao (Shao, 2010; Shao, 2011), among others. More recently, in the context of noisy diffusion models, Hounyo, Gonçalves, and Meddahi (2017), Hounyo et al. (2017), and Christensen, Hounyo, and Podolskij (2018) have proposed and studied a wild-based bootstrap method for dependent data, called the wild blocks of blocks bootstrap (WBBB). The WBBB method allows some time series heterogeneity, which are present in financial high-frequency data.

It is well known that block-based bootstrap methods (e.g., MBB and SB) can accommodate a large class of heterogeneous weakly dependent time series, see for example, Gonçalves and White ((Gonçalves and White, 2002), (Gonçalves and White, 2004)), see also Gonçalves et al. (2019). Moreover, these methods (e.g., MBB, NBB, and SB) are known to provide approximations to the distribution of a statistic that are (theoretically) superior to those obtained from using a Gaussian asymptotic approximation that is, asymptotic refinement. This is one of the key properties of any bootstrap method, in order to get better inferences. To the best of our knowledge, there is no result on a potential higher-order correctness for extant wild-based bootstrap methods for dependent data.

In this paper, we introduce a new wild-based bootstrap method for dependent data, that is generally applicable for dependent heterogeneous arrays. We name this novel approach the WBDD. As in Gonçalves and White (2002), the data are assumed to satisfy a near epoch dependent (NED) condition, which includes the more restrictive mixing assumption as a special case. NED processes also allow for considerable heterogeneity. For the smooth function model, we found that the WBDD is robust against heteroskedasticity and dependence of unknown form. The validity of wild-based bootstrap methods for dependent data has not yet been studied in heterogeneous context, and with the degree of dependence considered here. Our results broaden considerably the scope for application of the new WBDD in economics and finance, where the homogeneity of data and the mixing assumption are often a concern. Although it is convenient to implement the WBDD on dependent heterogeneous data, we also should note that (like the extant wild-based bootstrap methods,) the WBDD is not as widely applicable as blocking-based bootstrap methods.

We show that the external random variable u_t used to generate the WBDD observations, can be chosen such that the WBDD variance estimator of the asymptotic variance of the statistic of interest is *exactly* equal to the TBB variance estimator. Therefore, the WBDD preserves the favorable bias and mean squared error properties of the TBB, which is the state-of-the-art block-based method in terms of asymptotic accuracy of variance estimation and distribution approximation. Finally, our results show that similarly to block-based bootstrap methods, the WBDD can provide an asymptotic refinement over the first-order asymptotic distribution. We study the second-order correctness properties of the WBDD procedure under the restrictive stationarity (not heterogeneous) assumption, as for the equivalent settings in Götze and Künsch (1996). Specifically, we build upon the work of Götze and Künsch (1996), and show the second-order accuracy of the WBDD in the smooth function model, for stationary time series data. In this framework, conditional on the observed data, we show that the WBDD statistic of interest can be written as an average of independent random vectors. Hence, the WBDD approach naturally lends itself to the asymptotic expansions, which may be derived by simply using the Edgeworth expansion theory for independent random vectors. As for the plain wild bootstrap, we find that the main reason for the second-order correctness of the WBDD procedure is the asymptotically correct skewness of the bootstrap distribution. More specifically, if the external random variable u_t in addition to having mean 0, second moment satisfying $\ell\mathbb{E}(u_t)^2 \to 1$, where ℓ denotes the block size, and/or a bandwidth parameter, and where its third central moment satisfying $\ell\mathbb{E}(u_t)^3 \to 1$, then the WBDD shares with the standard wild bootstrap and block-based bootstrap methods the property of second-order accuracy after studentization. This desirable feature of asymptotic refinement of the WBDD is not known for extant dependent wild-based bootstrap methods. Therefore, the WBDD method constitutes a viable alternative to the existing methods.

The WBDD applies the standard wild bootstrap to overlapping tapered or nontapered blocks. However, the WBDD is not a block-based bootstrap in the sense that it does not involve any block resampling or some random block selection as all other existing block bootstraps (e.g., MBB, NBB, TBB, and SB). Instead, an implied data block structure is used in the WBDD approach only to obtain a scheme for a "multiplier-type" bootstrap. Our WBDD can be related to Paparoditis and Politis's (Paparoditis and Politis, 2001) TBB in the same way that Wu's (Wu, 1986) wild bootstrap is related to Efron's (Efron, 1979) bootstrap.

There are at least two different valid interpretations of the WBDD method. One is that the WBDD can be viewed as a simple variant of the traditional wild bootstrap. The main difference from the traditional wild bootstrap is that the data are first transform (with or without tapering) before applying the traditional wild bootstrap on the transformed data. As Wu's (Wu, 1986) wild bootstrap, the WBDD can also handle elegantly heteroskedasticity in the data. The other interpretation is that the WBDD method is akin to the DWB of Shao (2010). As the DWB, the WBDD extends the traditional wild bootstrap of Wu (1986) to the time series setting by allowing a transformation of the auxiliary variables involved in the wild bootstrap to be dependent, hence, the WBDD is capable of mimicking the dependence in the original series nonparametrically. Our new approach can also be viewed as a kind of generalized DWB. In particular, in contrast to the DWB, the WBDD method does not require the choice of the external random variable to be from a stationary process.

The remainder of the paper proceeds as follows. In Section 2, after outlining some preliminaries, we introduce the WBDD, describe the assumptions, and establish the consistency of our bootstrap for both variance estimation and distribution approximation of the smooth function of sample mean when data are assumed to satisfy an NED condition. In Section 2, we also provide examples of weights for the WBDD, which can be used in practice. Section 3 studies the second-order correctness property of the WBDD. The results from simulation studies are reported in Section 4. Section 5 concludes. Additional assumptions for the second-order validity of the WBDD and its proof are collected in Appendices A and B, respectively. An online supplementary appendix provides technical lemmas and the proofs of the WBDD variance and distribution consistency results.

2. THE WILD BOOTSTRAP FOR DEPENDENT DATA

This section introduces the framework, notations, the WBDD method, provides the assumptions and establishes the asymptotic validity of the bootstrap. Moreover, we illustrate the choice of auxiliary variables in our context.

2.1. Some Preliminaries

The set-up follows that of Gonçalves and White (2002) in which general dependence conditions and also heterogeneity in data are allowed. Suppose $\{X_{Nt}, N, t = 1,2\ldots\}$ is a double array of not necessarily stationary (can be heterogeneous) random vectors defined on a given probability space $(\Omega, \mathcal{F}, P)$ and NED on a mixing process $\{V_t\}$. We define $\{X_{Nt}\}$ to be NED on a mixing process $\{V_t\}$ if $\|X_{Nt}\|_2 < \infty$ and $\upsilon_k \equiv \sup_{N,t} \left\|X_{Nt} - \mathbb{E}_{t-k}^{t+k}(X_{Nt})\right\|_2 \to 0$ as $k \to \infty$. Here and in what follows, $\|X_{Nt}\|_p \equiv \left(\mathbb{E}|X_{Nt}|^p\right)^{1/p}$ is the L_p norm and $\mathbb{E}_{t-k}^{t+k}(\cdot) \equiv \mathbb{E}\left(\cdot|\mathcal{F}_{t-k}^{t+k}\right)$, where $\mathcal{F}_{t-k}^{t+k} \equiv \sigma(V_{t-k}, \ldots, V_{t+k})$ is the σ-field generated by $V_{t-k}, \ldots, V_{t+k}$. If $\upsilon_k = O\left(k^{-a-\delta}\right)$ for some $\delta > 0$, we say $\{X_{Nt}\}$ is NED of size $-a$. We assume $\{V_t\}$ is strong mixing. The strong mixing coefficients are $\alpha_k \equiv \sup_m \sup_{\{A\in\mathcal{F}_{-\infty}^m, B\in\mathcal{F}_{m+k}^\infty\}} |P(A\cap B) - P(A)P(B)|$, and we require $\alpha_k \to 0$ as $k \to \infty$ at an appropriate rate.

Let $\mu_{Nt} \equiv \mathbb{E}(X_{Nt})$ for $t = 1,2,\ldots,N$, $\bar{\mu}_N \equiv N^{-1}\sum_{t=1}^N \mu_{Nt}$, and $\theta \equiv H(\bar{\mu}_N)$, for some (smooth) function $H : \mathbb{R}^d \to \mathbb{R}$. Given a realization of $\{X_{Nt}\}$ denoted by $\mathcal{X}_N$, the goal is to make inferences about θ based on $\hat{\theta}_N = H\left(\bar{X}_N\right)$, where $\bar{X}_N = N^{-1}\sum_{t=1}^N X_{Nt}$ is the sample mean. In particular, we are interested in constructing a confidence region for θ or constructing an estimate of the variance $Var(\sqrt{N}\hat{\theta}_N)$, or its asymptotic limit $\sigma_\infty^2 = \lim_{N\to\infty} Var(\sqrt{N}\hat{\theta}_N)$. Typically, an estimate of the sampling distribution of $\hat{\theta}_N$ is required, and the WBDD method proposed here is developed for this purpose.

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Notice that this framework (in addition to allowing general dependence conditions and heterogeneity) is sufficiently general to include many statistics of practical interest, such as autocovariance, autocorrelation, the Yule-Walker estimators of autoregressive parameters in an auto-regression model, and other interesting statistics in time series (see e.g., (Bhattacharya and Ghosh, 1978; Hall, 1992; Lahiri, 2003)).

Let $h(x) = \{\partial H(x)/\partial x_1, \partial H(x)/\partial x_2, \ldots, \partial H(x)/\partial x_d\}'$ be the vector of first-order partial derivatives of H at x. Consider now the new series

$$Y_{Nt} \equiv h\left(\bar{X}_N\right)' \left(X_{Nt} - \bar{X}_N\right) \text{ for } t = 1, 2, \ldots, N. \tag{1}$$

The WBDD involves perturbing appropriately a weighted and centered version of Y_{Nt}. In practice, the WBDD method can be implemented with or without tapering. Following Paparoditis and Politis (Paparoditis and Politis, 2001; Paparoditis and Politis, 2002) for block tapering, we introduce a sequence of data-tapering windows $w_n(\cdot)$ for $n = 1, 2, \ldots$; the weights $w_n(t)$ are values in $[0,1]$, with $w_n(t) = 0$ for $t \notin \{1, 2, \ldots, n\}$. From the above, it is immediate that $\|w_n\|_1 \leq n$ and $\|w_n\|_2 \leq n^{1/2}$, where $\|w_n\|_1 = \sum_{t=1}^{n} |w_n(t)|$ and $\|w_n\|_2 = \left(\sum_{t=1}^{n} w_n^2(t)\right)^{1/2}$. The idea behind the (multiplicative) application of a tapering window to data is to give reduced weight to data near the end-points of the window. The notion of tapering for time series especially in connection to spectral estimation is well-studied; see, for example, Priestley (1981) and Künsch (1989).

As usual in the bootstrap literature, P^* ($\mathbb{E}^*$ and Var^*) denotes the probability measure (expected value and variance) induced by the bootstrap resampling, conditional on a realization of the original time series. In addition, let "$\to^d$" and "$\to^P$" denote convergence in distribution and in probability, respectively, and let $O_p(1)$ and $o_p(1)$ denote being bounded in probability and convergence to zero in probability, respectively. Finally, for $\alpha = (\alpha_1, \ldots, \alpha_d)' \in \mathbb{N}^d$, let D^α denote the differential operator $D^\alpha = \frac{\partial^{\alpha_1 + \ldots + \alpha_d}}{\partial x_1^{\alpha_1}, \ldots, \partial x_d^{\alpha_d}}$ on $\mathbb{R}^d$.

2.2. The Bootstrap Method

To present the WBDD method, we let $\ell = \ell_N \in \mathbb{N}$ denote the block size, and/or a bandwidth parameter such that $1 \leq \ell < N$, and

$$\bar{Y}_{\ell,w} = \frac{1}{Q} \sum_{j=1}^{Q} \sum_{i=1}^{\ell} \frac{w_\ell(i)}{\|w_\ell\|_1} Y_{N,i+j-1} = \sum_{t=1}^{N} \underbrace{\sum_{j=1}^{Q} \frac{w_\ell(t-j+1)}{Q\|w_\ell\|_1}}_{\equiv a_N(t)} Y_{N,t} = \sum_{t=1}^{N} a_N(t) Y_{Nt},$$

be the tapered moving (overlapping) block sample mean, where $Q \equiv N - \ell + 1$. Similarly, we let $\bar{X}_{\ell,w} = \sum_{t=1}^{N} a_N(t) X_{Nt}$. Note that $\sum_{t=1}^{N} a_N(t) = 1$. For $t = 1, 2, \ldots, N$, we define the WBDD pseudo-observations as

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$$Y_{Nt}^{*}-\bar{Y}_{N}=\sum_{j=1}^{Q}\left(\sum_{i=1}^{\ell}\left(\frac{w_{\ell}(i)}{\|w_{\ell}\|_{2}}\left(Y_{N,i+j-1}-\bar{Y}_{\ell,w}\right)\right)1_{\{t\}}(i+j-1)\right)\sqrt{\ell}u_{j}, \tag{2}$$

where $\bar{Y}_N = N^{-1}\sum_{t=1}^{N} Y_{Nt}$, $1_{\{\cdot\}}$ is the indicator function, $u_j, j = 1,\ldots,Q$, is a sequence of i.i.d. external random variables subject to mild regularity conditions, including mean 0 and second moment satisfies $\ell\mathbb{E}(u)^2 \to 1$, which are formalized in Assumption WBDD (cf. Section 2.3). Y_{Nt}^* can be equivalently rewritten as

$$Y_{Nt}^{*}-\bar{Y}_{N}=\left(Y_{Nt}-\bar{Y}_{\ell,w}\right)\eta_{t}, \tag{3}$$

where $\eta_t \equiv \sum_{j=1}^{Q}\frac{w_\ell(t-j+1)}{\|w_\ell\|_2}\sqrt{\ell}u_j$.

We now offer some remarks on the WBDD. The decomposition in (2) highlights that for $t = 1,\ldots,N,$ the tapered multiplicative weights $\left\{\frac{w_\ell(i)}{\|w_\ell\|_2}\right\}_{i=1}^{\ell}$ used at time point t depend on the position that Y_{Nt} occupies within a block of ℓ consecutive observations. An implied data block structure is used in the WBDD approach only to obtain a scheme for a "multiplier-type" bootstrap.

Remark 1. In the case of the sample mean, without tapering, our WBDD can be viewed as the *overlapping* version of the *nonoverlapping* wild-based bootstrap method for dependent data studied by Shao (2011), in the context of approximation of the sampling distribution of the Cramér-von Mises test statistic; see also the related work of Yeh (1998) and Inoue (2001). It is also related to the overlapping WBBB method of Christensen et al. (2018) and the nonoverlapping WBBB method of Hounyo et al. (Hounyo et al., 2017) and Hounyo (2017). The latter is a modified version of Shao's (Shao, 2011) approach, where one replaces the global sample mean by sample means computed locally over blocks of consecutive ℓ observations.

The reformulation (3) emphasises the way the WBDD method is related to the DWB method, which is proposed by Shao (2010) for stationary time series. As the DWB, the auxiliary variables $\{\eta_t\}_{t=1}^{N}$ involved in the wild bootstrap are dependent. However, notice that unlike the DWB, the random variables $\{\eta_t\}_{t=1}^{N}$ are not stationary even in the case of no tapering, and observations are centered around $\bar{Y}_{\ell,w}$, but not around $\bar{Y}_N$, (see the RHS of (3)). The centering and the non-stationarity of $\{\eta_t\}_{t=1}^{N}$ are all important for higher-order refinement. The DWB pseudo-time series are generated such that one can mainly capture the second-order moment structure of the observed time series. The first-order asymptotic validity of the DWB follows essentially from the convergence of the DWB variance estimator of σ_∞^2 towards σ_∞^2; see Shao (2010). As explained in Shao (2010) (cf. page 223), for the DWB, "it is a *hard task* to design the joint distribution for [the DWB auxiliary variables] to match the higher-order cumulants of the unknown data-generating process." In contrast, for the WBDD, it is very easy to choose auxiliary variables to match higher-order cumulants. Specifically, given (2),

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the WBDD pseudo-observations are obtained by generating simple i.i.d $\{u_t\}_{t=1}^{Q}$ random variables, as for the plain wild bootstrap. Motivated by the practical flexibility of choosing the weights $\{u_t\}_{t=1}^{Q}$, we study in Section 3 the second-order correctness of the WBDD method.

2.3. Assumptions

It is customary to obtain the sequence of data-tapering windows $w_n(\cdot)$ by means of dilations of a single function $w: \mathbb{R} \to [0,1]$, so that

$$w_n(t) = w\left(\frac{t-0.5}{n}\right). \tag{4}$$

We follow Paparoditis and Politis (Paparoditis and Politis, 2001, Paparoditis and Politis, 2002) and assume that the function $w(\cdot)$ satisfies the following assumptions.

Assumption 1. We have $w(t) \in [0,1]$ for all $t \in \mathbb{R}$, $w(t) = 0$ if $t \notin [0,1]$, and $w(t) > 0$ for t in a neighbourhood of $\frac{1}{2}$.

Assumption 2. The function $w(t)$ is symmetric about $t = \frac{1}{2}$ and nondecreasing for $t \in \left[0, \frac{1}{2}\right]$.

Assumption 3. The self-convolution is twice continuously differentiable at the point $t = 0$, where $w * w(t) = \int_{-1}^{1} w(x)\, w(x + |t|)\, dx$.

To state our results, we need a smoothness assumption on the function H. We make the following assumption.

Assumption 4. The function H is differentiable in a neighborhood of $\bar{\mu}_N$ that is, $N_H = \left\{x \in \mathbb{R}^d : \|x - \bar{\mu}_N\|_2 \leq \epsilon\right\}$ for some $\epsilon > 0$, $\sum_{|\alpha|=1} |D^{\alpha} H(\bar{\mu}_N)| \neq 0$, and the first partial derivatives of H satisfy a Lipschitz condition of order $s > 0$ on N_H.

We also follow Gonçalves and White (2002) and make the following assumption on $\{X_{Nt}\}$.

Assumption 5.

(a) For some $r > 0$, $\|X_{Nt}\|_{3r} \leq \Delta < \infty$ for all $N, t = 1, 2, \ldots$
(b) $\{X_{Nt}\}$ is near epoch dependent (NED) on $\{V_t\}$ with NED coefficients α_k of size $-\frac{2(r-1)}{(r-2)}$; $\{V_t\}$ is an α-mixing sequence with α_k of size $-\frac{2r}{r-2}$.

Finally, we impose the following general condition on the external random variable u_j.

Assumption WBDD. The sequence of random variables $u_j, j = 1, \ldots, Q = N - \ell + 1$, is independent of the original observed sample $\mathcal{X}_N$. Moreover, it is independent and identically distributed and satisfies the following regularity

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conditions: $\mathbb{E}(u)=0$, $Var(\sqrt{\ell}u)=\ell\mathbb{E}(u)^2\to 1$ and for some $\delta>0, \ell\mathbb{E}\left(|u|^{2+\delta}\right)\to C_\delta<\infty$, as $N\to\infty, \ell\to\infty$ such that $\ell/N=o(1)$, where C_δ is a nonrandom constant.

It is worth noting that the process u_j should be written as u_{Nj}, where the dependence on N is suppressed for notational convenience. When $\ell=1$ the requirements on the weights $\{u_j\}_{j=1}^{Q}$ in Assumption WBDD become $\mathbb{E}(u)=0$, $Var(u)=1$ and for some $\delta>0, \mathbb{E}\left(|u|^{2+\delta}\right)<\infty$, which are standard in the wild bootstrap literature, see for example, Kline and Santos (2012).

2.4. Bootstrap Validity

We here establish the validity of the WBDD. Let us denote $S_N\equiv N^{1/2}\left(\bar{X}_N-\bar{\mu}_N\right)$. Recall that under some suitable conditions (cf. Assumption 4), W_N is asymptotically equivalent to $h(\bar{\mu}_N)'S_N$. Additionally, we have

$$W_N\equiv\sqrt{N}\left(H\left(\bar{X}_N\right)-H\left(\bar{\mu}_N\right)\right)\to^d N\left(0,\sigma_\infty^2\right). \quad (5)$$

The asymptotic variance of W_N can be written as

$$\sigma_\infty^2=\lim_{N\to\infty}\sigma_N^2,\ \text{with}\ \sigma_N^2\equiv h(\bar{\mu}_N)'\,Var(S_N)\,h(\bar{\mu}_N). \quad (6)$$

We define the WBDD variance estimator of σ_N^2 as $\sigma_N^{*2}\equiv Var^*\left(W_N^*\right)$, where $W_N^*\equiv N^{1/2}\left(\bar{Y}_N^*-\bar{Y}_N\right)$.

Remark 2. Note that a straightforward analytical calculation (see part (b) of Lemma C1.1. in the supplementary appendix for further details) shows that

$$\sigma_N^{*2}=\underbrace{\frac{Q}{N}}_{\to 1}\underbrace{(\ell Var(u))}_{\to 1}\sigma_N^{*2(\mathrm{TBB})}, \quad (7)$$

where $\sigma_N^{*2(\mathrm{TBB})}=\frac{1}{Q}\frac{1}{\|w_l\|_2^2}\sum_{j=1}^{Q}\left(\sum_{i=1}^{\ell}w_\ell(i)X_{i+j-1}-\|w_\ell\|_1\bar{X}_{\ell,w}\right)^2$ denote the TBB estimator of the asymptotic variance σ_∞^2 based on block size ℓ. This implies that the WBDD method preserves the favorable bias and MSE properties of the TBB. Furthermore, it is useful to note that if in (2), we used $\left(\frac{Q}{N}(\ell Var(u))\right)^{-1/2}u_j, j=1,\ldots,Q$, as external random variable, (instead of u_j) to obtain the WBDD pseudo-observations, then $\sigma_N^{*2}=\sigma_N^{*2(\mathrm{TBB})}$, implying that $\mathrm{MSE}\left(\sigma_N^{*2}\right)=\mathrm{MSE}\left(\sigma_N^{*2(\mathrm{TBB})}\right)$.

As Gonçalves and White (2002) pointed out (for the MBB and the SB), we also found for the WBDD, (see Theorem 2.1) that under arbitrary heterogeneity in $\{X_{Nt}\}$ the WBDD variance estimator σ_N^{*2} is not consistent for σ_N^2, but for $h(\bar{\mu}_N)'(Var(S_N)+U_N)h(\bar{\mu}_N)$. The bias term $h(\bar{\mu}_N)'U_Nh(\bar{\mu}_N)$ is related to the heterogeneity in the means $\{\mu_{Nt}\}$ and can be interpreted as the WBDD variance estimate of the scaled sample mean $h(\bar{\mu}_N)'\sqrt{N}\bar{\mu}_N^*=h(\bar{\mu}_N)'\left(N^{-1/2}\sum_{t=1}^{N}\mu_{Nt}^*\right)$

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that would result if we could resample the time series $\{\mu_{Nt}\}$. We follow Gonçalves and White (2002) and call $\{\mu_{Nt}^*\}$ the "resampled version" of $\{\mu_{Nt}\}$. σ_N^2 can be easily obtained using the WBDD variance σ_N^{*2} under some homogeneity condition. The following theorem and its corollary provide the theoretical justification.

THEOREM 2.1. *Suppose that equation (4), Assumptions 1–5 and Assumption WBDD hold. If* $\ell_N \to \infty$ *as* $N \to \infty$ *such that* $\ell_N = o\left(N^{1/2}\right)$*, then,*

(a) $\sigma_N^{*2} - h(\bar{\mu}_N)'(Var(S_N) + U_N)h(\bar{\mu}_N) \xrightarrow{P} 0$, *where* $U_N \equiv Var^*\left(N^{-1/2}\sum_{t=1}^{N}\mu_{Nt}^*\right)$.

(b) $U_N = \frac{Q}{N}\ell Var(u)\sum_{\tau=-\ell+1}^{\ell-1}\frac{v_\ell(\tau)}{v_\ell(0)}\sum_{t=1}^{N-|\tau|}\beta_{N,t,\tau}\left(\mu_{Nt}-\bar{\mu}_{\ell,w}\right)\left(\mu_{N,t+|\tau|}-\bar{\mu}_{\ell,w}\right)'$,
where $v_\ell(\tau) = \sum_{i=1}^{\ell-|\tau|} w_\ell(i)\,w_\ell(i+|\tau|)$, $\bar{\mu}_{\ell,w} = \sum_{t=1}^{N} a_N(t)\,\mu_{Nt}$, *and* $\beta_{N,t,\tau} = \frac{1}{v_\ell(\tau)}\frac{1}{Q}\sum_{j=1}^{Q} w_\ell(t-j+1)\,w_\ell(t-j+1+|\tau|)$ *with* $\tau < j$.

(c) $\sigma_N^{*2} - h(\bar{\mu}_N)'\,Var(S_N)\,h(\bar{\mu}_N) \xrightarrow{P} 0$, *as* $\lim_{N\to\infty} U_N = 0$.

Thus, the condition $\lim_{N\to\infty} U_N = 0$ is the homogeneity condition on the mean, analogous conditions are given by Liu (1988) for the plain wild bootstrap, by Gonçalves and White (2002) for the MBB, and recently by Hounyo et al. (2017) and Hounyo ((Hounyo, 2017), cf. equation (13)) for the wild blocks of blocks bootstrap method. In our setting, for the WBDD approach, to ensure this condition, one can for example suppose that

Assumption 6. $N^{-1}\sum_{t=1}^{N}(\mu_{Nt}-\bar{\mu}_N)(\mu_{Nt}-\bar{\mu}_N)' = o\left(\ell_N^{-1}\right)$ where $\ell_N = o\left(N^{1/2}\right)$.

Assumption 6 amounts to Assumption 2.2 in Gonçalves and White (2002). As they explain, this assumption is rather general allowing for breaks in mean. See Gonçalves and White (2002) for particular examples of processes that satisfy Assumption 6. The following consistency result is an immediate consequence of the previous Theorem 2.1.

COROLLARY 2.1. *Suppose that equation (4), Assumptions 1–6 and Assumption WBDD hold. If* $\ell_N \to \infty$ *as* $N \to \infty$ *such that* $\ell_N = o\left(N^{1/2}\right)$*, then* $\sigma_N^{*2} - h(\bar{\mu}_N)'\,(Var(S_N))\,h(\bar{\mu}_N) \xrightarrow{P} 0$.

In the following theorem, we provide a theoretical justification for using the WBDD distribution of W_N^* to estimate the distribution of W_N under general dependence conditions. As in Gonçalves and White (2002), we require a slightly stronger dependence condition than Assumption 5.(b). Specifically, we impose:

Assumption 5.(b'). For some small $\delta > 0$, $\{X_{Nt}\}$ is $L_{2+\delta}$-NED on $\{V_t\}$ with NED coefficients υ_k of size $-\frac{2(r-1)}{r-2}$; $\{V_t\}$ is an α-mixing sequence with α_k of size $-\frac{(2+\delta)r}{r-2}$.

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THEOREM 2.2. *Suppose that equation (4), Assumptions 1–6 and Assumption WBDD hold, strengthened by Assumption 5.(b'). If* $\ell_N \to \infty$ *as* $N \to \infty$ *such that* $\ell_N = o\left(N^{1/2}\right)$, *then* $\sup_{x\in\mathbb{R}} \left|P^*\left(W_N^* \leq x\right) - P\left(W_N \leq x\right)\right| = o_p(1)$.

Theorem 2.2 justifies using the WBDD to build asymptotically valid confidence intervals for (or test hypotheses about) $\bar{\mu}_N$, even though there may be considerable heterogeneity.

2.5. Examples of Weights for the Wild Bootstrap for Dependent Data

This section contains some examples of external random variables which satisfy Assumption WBDD. Our intention in including these examples is to show the scope of random variables that are covered by Assumption WBDD (and can be used in practice). In each case, we can make connection with well known weights used in practice with the standard wild bootstrap based on independent data.

Example 2.3 (WBDD1). $u_j = \frac{1}{\ell}\sum_{i=(j-1)l+1}^{j\ell} \tilde{v}_i, j = 1,\dots,Q,$ where $\tilde{v}_i = v_i - \mathbb{E}(v_i)$ such that $v_i \sim$ i.i.d. $N(0,1)$, implying that $\mathbb{E}(u_j) = 0$, $\ell\mathbb{E}(u_j^2) = 1$ and $\ell\mathbb{E}(u_j^3) = 0$.

Example 2.4 (WBDD2). $u_j = \frac{1}{\ell}\sum_{i=(j-1)l+1}^{j\ell} \tilde{v}_i, j = 1,\dots,Q,$ where $\tilde{v}_i = v_i - \mathbb{E}(v_i)$ with $v_i \sim$ i.i.d. two-point distribution such that:

$$v_i = \begin{cases} 1, & \text{with prob} \quad p = \frac{1}{2}, \\ -1, & \text{with prob} \quad 1-p, \end{cases}$$

for which we have $\mathbb{E}(u_j) = 0$, $\ell\mathbb{E}(u_j^2) = 1$ and $\ell\mathbb{E}(u_j^3) = 0$.

Example 2.5 (WBDD3). $u_j = \frac{1}{\ell}\sum_{i=(j-1)l+1}^{j\ell} \tilde{v}_i, j = 1,\dots,Q,$ where $\tilde{v}_i = v_i - \mathbb{E}(v_i)$ with $v_i \sim$ i.i.d. two-point distribution such that:

$$v_i = \begin{cases} \frac{\ell+\sqrt{\ell^2+4}}{2}, & \text{with prob} \quad p = \frac{\sqrt{\ell^2+4}-\ell}{2\sqrt{\ell^2+4}}, \\ \frac{\ell-\sqrt{\ell^2+4}}{2}, & \text{with prob} \quad 1-p, \end{cases}$$

for which $\mathbb{E}(u_j) = 0$, $\ell\mathbb{E}(u_j^2) = 1$ and $\ell\mathbb{E}(u_j^3) = 1$.

Example 2.6 (WBDD4). $u_j = \frac{1}{\ell}\sum_{i=(j-1)l+1}^{j\ell} \tilde{v}_i, j = 1,\dots,Q,$ where $\tilde{v}_i = v_i - \mathbb{E}(v_i)$ such that $v_i \sim$ i.i.d.$\Gamma(\alpha,\beta)$, that is, $v_i \sim$ i.i.d. with gamma distribution having density

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x)\, 1_{\{x>0\}},$$

where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}\exp(-x)\,dx$. Implying that $\mathbb{E}(u_j) = 0$, $\ell\mathbb{E}(u_j^2) = 1$ and $\ell\mathbb{E}(u_j^3) = 1$.

Example 2.7 (WBDD5). $u_j = \left(v_j - \mathbb{E}(v_j)\right), j = 1, \ldots, Q,$ where $v_j \sim$ i.i.d. such that

$$\left(v_1, \ldots, v_Q\right)' \sim \text{Multinomial}\left(\frac{N}{\ell}, \left(Q^{-1}, \ldots, Q^{-1}\right)\right),$$

with $\frac{N}{\ell}$ (assume here for simplicity to be) an integer. It follows that $\mathbb{E}(u_j) = 0$, $\ell\mathbb{E}(u_j^2) \to 1,$ and $\ell\mathbb{E}(u_j^3) \to 1,$ as $N \to \infty, \ell \to \infty$ such that $\ell/N = o(1)$ and $Q = N - \ell + 1$.

For further details on $\mathbb{E}(u_j), \ell\mathbb{E}(u_j^2)$ and $\ell\mathbb{E}(u_j^3)$, see Appendix C2, in the Supplementary Appendix. It is worth mentioning that in Examples 2.3–2.6, the external random variables $\sqrt{\ell}u_j,\ j = 1, \ldots, Q$ are independent block-variables defined on nonoverlapping blocks of length ℓ. Specifically, $\sqrt{\ell}u_j$ are scaled sum of ℓ (nonoverlapping) mean-zero i.i.d. random variables. The effect due to *block averaging* is important in these cases to ensure the moment conditions on $\sqrt{\ell}u_j$. When $\ell = 1$, WBDD2 boils down to the so-called Rademacher distribution originally proposed by Liu (1988). WBDD3 generalizes the two-point distribution suggested by Mammen (1993) for independent data in our context of dependent data. Similarly, WBDD4 generalizes the example of external random variable which ensures the second-order correctness of the wild bootstrap originally proposed by Liu (1988) (cf. Example 4) in our setting of dependent data. WBDD5 is related to the weight used in Gonçalves et al. (2019) to rewrite the MBB sample mean as a weighted average sum, in the context of estimating standard errors of parameters estimated via multi-stage QMLE estimators.

A natural question is whether the WBDD distribution can provide second-order accuracy that improves on the normal approximation. We will show in Section 3 that the further condition $\ell\mathbb{E}(u_j^3) \to 1$ (satisfied by WBDD3, WBDD4, and WBDD5) is a necessary condition for asymptotic refinement of the WBDD method.

3. SECOND-ORDER CORRECTNESS OF THE WBDD

In this section, we access the second-order accuracy of the WBDD method for the studentized statistic. To do so, we follow Götze and Künsch (1996) and impose stationarity in our framework. Under stationarity, σ_N^2 can be written as

$$\sigma_N^2 = h(\mu)' \left[\sum_{j=-N}^{N} (1 - |j|/N)\,\Gamma(j)\right] h(\mu), \tag{8}$$

where $\mu = \mu_{Nt}$ for all N and t, $\Gamma(j) = Cov\left(X_0, X_j\right) = \mathbb{E}\left(X_0 - \mu\right)\left(X_j - \mu\right)'$. Note that X_j should be written as X_{Nj}, where the dependence on N is suppressed for notational convenience (under stationarity). A class of consistent estimators of the

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asymptotic variance σ_∞^2 is given by

$$\hat{\sigma}_N^2 = h\left(\bar{X}_N\right)' \left[\sum_{j=0}^{\ell-1} \tilde{k}_{jN}\hat{\Gamma}(j)\right] h\left(\bar{X}_N\right),$$

where $\hat{\Gamma}(j) = N^{-1}\sum_{i=1}^{N-\ell}\left(X_i - \bar{X}_N\right)\left(X_{i+j} - \bar{X}_N\right)'$. The lag weights $\tilde{k}_{jN} = 2\tilde{k}(j/\ell)$, $1 \le j \le \ell - 1$ for some continuous function $\tilde{k} : [0,1) \to [0,1]$ with $\tilde{k}(0) = 1$. We consider the following studentized statistic:

$$T_{N,stud} = W_N \hat{\sigma}_N^{-1}, \tag{9}$$

which has a standard normal distribution, asymptotically. Our aim is to use the WBDD and propose a better than normal approximation of the distribution of $T_{N,stud}$.

Note that we can equivalently rewrite W_N^* as follows:

$$W_N^* = N^{-1/2}\sum_{j=1}^{Q} \underbrace{B_j \cdot \left(\sqrt{\ell} u_j\right)}_{=B_j^*}, \tag{10}$$

where $B_j = \tilde{B}_j - \frac{1}{Q}\sum_{j=1}^{Q}\tilde{B}_j$, with $\tilde{B}_j = h\left(\bar{X}_N\right)' \frac{1}{\|w_\ell\|_2}\sum_{i=1}^{\ell} w_\ell(i) X_{N,i+j-1}$, see Lemma C1.1 in the supplementary appendix for further details.

Remark 3. The representation of W_N^* given in (10) would drive the second-order correctness property of the WBDD. The reason is that conditional on $\mathcal{X}_N$, W_N^* is a simple scaled sum of a collection of Q independent random vectors. Hence, an expansion for W_N^* may be derived by simply using the Edgeworth expansion theory for independent random vectors.

In the following, and throughout this section, we let $w(t) = 1_{[0,1]}$ (i.e., no tapering), implying that $\|w_\ell\|_2 = \ell^{1/2}$ and $B_j = h(\bar{X}_N)' A_j$, with $A_j \equiv \ell^{-1/2}\sum_{i=1}^{\ell}(X_{i+j-1} - \tilde{X}_N)$, where $\tilde{X}_N = \ell^{-1}Q^{-1}\sum_{j=1}^{Q}\sum_{i=1}^{\ell} X_{i+j-1}$. Under Assumption WBDD, $u_j \sim$ i.i.d., and given (10), it follows that

$$\sigma_N^{*2} = \underbrace{\frac{Q}{N}}_{\to 1} \underbrace{(\ell Var(u))}_{\to 1} \cdot \sigma_N^{*2(\mathrm{MBB})}, \tag{11}$$

where $\sigma_N^{*2(\mathrm{MBB})} \equiv Q^{-1}\sum_{j=1}^{Q} B_j^2$ is the MBB variance estimate of σ_∞^2 (i.e., analog of σ_N^{*2} based on MBB resampling approach). As bootstrap variance estimator of σ_N^{*2}, we propose

$$\hat{\sigma}_N^{*2} \equiv Q^{-1}\sum_{j=1}^{Q}\hat{B}_j^{*2},$$

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where $\hat{B}_j^* = \hat{B}_j\sqrt{\ell}u_j$, with $\hat{B}_j \equiv h(\bar{X}_N)'\ell^{-1/2}\sum_{i=1}^{\ell}(X_{i+j-1} - \bar{X}_N) = B_j + h(\bar{X}_N)'\ell^{1/2}(\tilde{X}_N - \bar{X}_N)$.

The studentized WBDD statistic is then

$$T_{N,stud}^* = W_N^*\hat{\sigma}_N^{*-1}.$$

Note that when $\check{Y}_i \equiv h(\mu)'(X_i - \mu)$ are correlated, $\hat{\sigma}_N^{*2} - \sigma_N^2$ is always at least of order $O_p\left(N^{-1/3}\right)$. Therefore, in using $\hat{B}_j$ instead of B_j, we can remove the error due to the wrong variance estimation. In particular, we have $\mathbb{E}^*\left(\hat{\sigma}_N^{*2}\right) = \sigma_N^{*2} + \underbrace{h\left(\bar{X}_N\right)'\ell^{1/2}\left(\tilde{X}_N - \bar{X}_N\right)}_{=O_p(\ell/N)}$. We can prove the following theorem.

THEOREM 3.1. *Let conditions (C.1)-(C.7) (in Appendix A) hold. Assume that ℓ satisfies*

$$(\log(N))^C \ll \ell \le N^{1/3},$$

with C large enough and $\tilde{k} \equiv 1$. Furthermore, suppose $\{R_j^* \equiv (B_j, A_j')' \cdot \sqrt{\ell}u_j, j = 1,\ldots,Q\}$, where $u_j \sim$ i.i.d. satisfy Assumption WBDD with

$$\ell\mathbb{E}\left(u_j\right)^3 \to 1, \qquad \textbf{(12)}$$

$$\mathbb{E}|\sqrt{\ell}u_j|^{qs+\delta} \le U_{N,qs+\delta} \to C_{qs+\delta} < \infty, \qquad \textbf{(13)}$$

for some $\delta > 0$, as $N \to \infty, \ell \to \infty$ such that $\ell/N = o(1)$ and for some integers $q \ge 3, s \ge 8$. Suppose, in addition that the $\sqrt{\ell}u_j$ satisfy Cramér's condition, that is, for all $r > 0$, there exists $M_r \in (0,1)$ such that

$$\left|\phi_{Q,j}(t)\right| \le M_r \text{ for all } \|t\| \ge r \text{ and } Q \ge 1, 1 \le j \le Q, \qquad \textbf{(14)}$$

where $\phi_{Q,j}$ is the characteristic function of R_j^* under P^*. Then

$$\sup_{x\in\mathbb{R}}\left|P^*\left\{T_{N,stud}^* \le x\right\} - P\left\{T_{N,stud} \le x\right\}\right| = O_p\left(\ell N^{-1+2/s} + \ell^{-1}N^{-1/2}\right).$$

Theorem 3.1 shows that the WBDD approximation to the distribution of the studentized statistic $T_{N,stud}$ is more accurate than the limiting normal approximation. Thus, like the standard wild bootstrap of Wu (1986) and Liu (1988), the WBDD also outperforms the asymptotic normal approximation under dependence. This feature is not known for extant wild-based bootstrap methods for dependent data. Notice that in Theorem 3.1, results are derived for the WBDD with no tapering. We conjecture that the tapering case would follow using the proof strategies as in Götze and Künsch (1996), under some additional regularity conditions, although a rigorous proof is well beyond the scope of this paper.

Remark 4. A sufficient condition for (14) is that the probability distribution of R_j^* has an absolutely continuous component with respect to the Lebesgue

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measure on $\mathbb{R}^{d+1}$. Hence, the external random variable WBDD4 (see Example 2.6) with a distribution that has a density satisfies (14). Unfortunately, the external random variables WBDD2 and WBDD3 (cf. Examples 2.4 and 2.4) do not satisfy condition (14) and hence it is unlikely that the second-order Edgeworth expansions of the WBDD studentized statistic $T^*_{N,stud}$ exist for both choices. As the plain wild bootstrap, the WBDD provides the usual skewness correction whenever the requirement $\ell\mathbb{E}(u_j^3) \to 1$ holds. See for example, WBDD3, WBDD4, and WBDD5 in Examples 2.5–2.7. The finite sample properties of all these external random variables are examined in the simulation study.

4. SIMULATION STUDIES

In this section, we study via simulations the finite-sample performance of the WBDD. We focus on the inference of the population mean μ of a time series. Performance is measured in terms of empirical coverage probability of two-sided 95% level intervals, length of confidence intervals and empirical MSE. In the simulation studies, we considered three different models generating the observations, namely:

Model 1. Nonlinear autoregressive model, NAR,

$$X_t = \rho \sin(X_{t-1}) + \upsilon_t,$$

for $t \in \mathbb{Z}$, where $\{\upsilon_t\}$ i.i.d. $N(0,1)$, with $\rho \in \{0.2, 0.6\}$.

Model 2. Heteroskedastic AR(1), with periodic innovation variance

$$X_t = \rho X_{t-1} + \upsilon_t, \quad \text{and} \quad \upsilon_t = s_t \tilde{\upsilon}_t,$$

for $t \in \mathbb{Z}$, where $\{\tilde{\upsilon}_t\}$ i.i.d. $N(0,1)$, with $\rho \in \{0.2, 0.6\}$. Here, $\{s_t\}$ denotes a sequence of real numbers that might be regarded as seasonal effects. Throughout, we choose $\{s_t\}$ to be the infinite repetition of the sequence $\{1,1,1,1,2,3,1,1,1,1,2,4,6\}$.

Model 3. Heteroskedastic AR(1), with permanent shifts in the innovation variance

$$X_t = \rho X_{t-1} + \upsilon_t, \quad \text{and} \quad \upsilon_t = s_t \tilde{\upsilon}_t,$$

for $t \in \mathbb{Z}$, where $\{\tilde{\upsilon}_t\}$ i.i.d. $N(0,1)$, with $\rho \in \{0.2, 0.6\}$. We follow Cavaliere (2004), see also Phillips and Xu (2006), Xu and Phillips (2008), Cavaliere and Taylor (Cavaliere and Taylor (2007), Cavaliere and Taylor (2009)), and assume that $s_t = g(t/n)$, where

$$g(r)^2 = s_0^2 + \left(s_1^2 - s_0^2\right) 1_{\{r \geq \tau\}}, r \in [0,1].$$

The steepness of the variance change is characterized by the post-break and pre-break variance ratio $\delta^2 = s_1^2/s_0^2$. We set $\tau = 0.2, \delta = 0.2$ and $s_0^2 = 1$.

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Note that, among the block-based bootstrap methods, the theoretical advantage of the TBB over the MBB has been confirmed for model 1, (in particular, with $\rho = 0.6$) through simulation studies by Paparoditis and Politis (2001). Moreover, using model 1, Shao (2010) (cf. Section 6), assessed the finite sample properties of the DWB and block-based bootstrap methods. He found that the TBB and the DWB have comparable empirical performance at a range of block sizes. However, for large value of ℓ, the DWB outperforms the TBB. For these reasons, it seems natural to study our new WBDD method in this case and compare its finite-sample performance directly with the DWB.

In order to investigate the performance of the WBDD when there are dependent "strongly" heterogeneous data, we also consider model 2. The latter is used by Politis, Romano, and Wolf (1997) in another context for heteroskedastic times series. Note that in model 2, the innovations are independent but heteroskedastic. Then model 2 generates a weakly dependent, heteroskedastic time series. Moreover, we consider model 3, which allows for time varying unconditional variance with permanent changes in volatility.

We generate repeated trials of length $N \in \{100, 200\}$ from these processes. In order to generate the WBDD observations we need a data-tapering window function $w(\cdot)$. We define the following family of trapezoidal functions as

$$w_c^{trap}(t) = \begin{cases} \frac{t}{c}, & \text{if} \quad t \in [0, c], \\ 1, & \text{if} \quad t \in [c, 1-c], \\ \frac{1-t}{c}, & \text{if} \quad t \in [1-c, 1], \\ 0, & \text{if} \quad t \notin [0, 1], \end{cases} \tag{15}$$

where c is some fixed constant in $(0, 1/2]$. To make the comparison fair, in our simulation, we took $c = 0.43$, since it was found in Paparoditis and Politis (2001) that $w(t) = w_{0.43}^{trap}(t)$ offers the optimal (theoretical) MSE provided we fix the covariance structure of a time series. We also use $\gamma(t) = w_{0.43}^{trap} * w_{0.43}^{trap}(t) / w_{0.43}^{trap} * w_{0.43}^{trap}(0)$, where $\gamma(\cdot)$ is the covariance function of the external random variable used to generate the DWB observations. With this choice of the kernel function for the DWB, the favorable bias and MSE properties of the TBB variance estimator over other block-based counterparts in the mean case automatically carries over to the DWB. In addition, in view of the discussion in Remark 2, (showing the connection between the TBB and WBDD variance estimators) and from Paparoditis and Politis (2002), we can deduce that (in our setting) for the all three methods TBB, DWB, and WBDD, the (large-sample) MSE is minimized at the same value of block size and/or bandwidth, which should be picked proportional to $N^{1/5}$. As a consequence, in practice, we advocate using the same practical block size choice suggested by Paparoditis and Politis (2002), Politis and White (2004) see also Patton, Politis and White (2009) to implement the DWB and the WBDD.

For the choice of external random variables of the DWB, we use multivariate normal as in Shao (2010), whereas to generate the WBDD data we use five different external random variables. See Examples 2.3–2.7. Note that all five choices of u_j are asymptotically valid when used to construct the unstudentized bootstrap intervals, since they satisfy Assumption WBDD. The further condition $\ell\mathbb{E}(u_j^3) \to 1$ (satisfied by WBDD3, WBDD4, and WBDD5) is a necessary condition for refinements. WBDD2 also satisfies the necessary conditions for refinements in the case of unskewed disturbances. Davidson and Flachaire (2007) advocated the use of the Rademacher distribution (i.e., the WBDD2 with $\ell = 1$).

For each time series, we generated 499 DWB and WBDD pseudo-series to obtain the bootstrap-based critical values. Then we repeated this procedure 1000 times and reported in Table 1, the empirical coverage of nominal 95% symmetric bootstrap percentile[1] confidence intervals for μ, the average lengths of confidence intervals, and the empirical MSEs of the variance estimators of σ_N^2. In Figure 1, we plotted the empirical coverage rate as a function of block size, for the WBDD, DWB, TBB, and MBB methods. For the WBDD method, in Figure 1, we focus on WBDD3, WBDD4, and WBDD5 (for a matter of readability).

In Table 1, we compare the DWB directly to the WBDD. As a first observation, for all three models, it is striking how close all bootstrap methods analyzed here are in terms of empirical MSE. In line with our theoretical results, all methods are globally equivalent when we consider the empirical MSEs of the variance estimators of σ_N^2. However, as results in Table 1 suggest (see also Figure 1), there are notable differences among the different bootstrap methods when considering their empirical coverage probabilities. For all bootstrap methods, finite sample performance is far from perfect (especially for models 2 and 3) and gets worse as the degree of dependence in the data increases. Models 2 and 3 exhibit overall larger coverage distortions than model 1.

For the WBDD method, none of the five choices of the external random variables used (i.e., WBDD1, WBDD2, WBDD3, WBDD4, and WBDD5) clearly dominates the others. One exception is in the model 1, with $\rho = 0.2,$ where the performance of WBDD4 is comparable to that of WBDD5, and both tend to outperform DWB, WBDD1, WBDD2, and WBDD3. Specifically, in this model, when $n = 100, \rho = 0.2,$ the WBDD4 and WBDD5-based intervals have a coverage probability equal to 93.40% and 93.50%, respectively; whereas, the coverage rates are equal to 88.70%, 88.10%, 88.10%, and 88.30%, for the DWB, WBDD1, WBDD2, and WBDD3, respectively.

In models 2 and 3, the DWB seems to perform very poorly compared to the WBDD schemes. Results based on models 2 and 3 in Table 1, clearly show that the intervals based on the WBDD have better coverage rates than intervals based on the DWB. For instance when $\rho = 0.6,$ and $n = 100,$ in model 3, the DWB-based intervals have a coverage probability equal to 63.70% only, instead of the desired nominal 95%. In contrast, this rate is equal to 83.50%, 83.90%, 84.50%, 84.30%, and 84.70% for the WBDD1, WBDD2, WBDD3, WBDD4, and WBDD5, respectively. In Table 2, we report the bootstrap block size/ bandwidth choices made. As expected, the average chosen block size/bandwidth is larger for larger

[1]We have performed a similar exercise using bootstrap percentile-t confidence intervals, applying the same set of tuning parameters. The results are qualitatively identical to those reported in Table 1, the latter is excluded for brevity.

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Table 1. Comparison of nominal 95% confidence intervals for μ and variance estimators of σ_N^2

	$\rho = 0.2$						$\rho = 0.6$					
	DWB	WBDD1	WBDD2	WBDD3	WBDD4	WBDD5	DWB	WBDD1	WBDD2	WBDD3	WBDD4	WBDD5
$N = 100$												
Model 1												
Coverage (%)	88.70	88.10	88.10	88.30	93.40	93.50	82.70	89.30	89.90	89.80	90.40	90.40
Length	0.39	0.39	0.39	0.39	0.41	0.41	0.42	0.52	0.52	0.52	0.53	0.53
MSE·10	0.11	0.11	0.11	0.11	0.11	0.11	0.18	0.18	0.18	0.18	0.18	0.19
Model 2												
Coverage (%)	88.80	89.10	88.80	88.40	92.60	92.60	65.70	84.40	84.70	84.70	85.10	85.00
Length	0.97	1.06	1.06	1.07	1.11	1.11	1.18	1.92	1.91	1.94	1.94	1.96
MSE·10	0.79	0.78	0.79	0.79	0.81	0.82	2.50	2.53	2.52	2.52	2.52	2.52
Model 3												
Coverage (%)	87.60	87.70	87.20	87.70	89.40	89.70	63.70	83.50	83.90	84.50	84.30	84.70
Length	0.23	0.25	0.25	0.25	0.26	0.26	0.28	0.44	0.44	0.44	0.44	0.45
MSE·10	0.04	0.04	0.04	0.04	0.04	0.04	0.12	0.13	0.13	0.13	0.13	0.14
$N = 200$												
Model 1												
Coverage (%)	90.90	91.00	91.00	90.70	93.40	93.50	83.00	91.20	91.10	91.20	91.90	92.80
Length	0.28	0.29	0.29	0.29	0.30	0.30	0.30	0.38	0.39	0.39	0.39	0.39
MSE·10	0.05	0.06	0.06	0.06	0.06	0.06	0.09	0.10	0.10	0.10	0.10	0.10

(Continued)

TABLE 1. (Continued)

	$\rho = 0.2$						$\rho = 0.6$					
	DWB	WBDD1	WBDD2	WBDD3	WBDD4	WBDD5	DWB	WBDD1	WBDD2	WBDD3	WBDD4	WBDD5
Model 2												
Coverage (%)	86.90	91.90	91.60	91.60	92.30	91.90	67.90	89.80	89.50	89.80	90.10	90.20
Length	0.70	0.80	0.80	0.80	0.81	0.81	0.20	0.34	0.34	0.34	0.34	0.35
MSE·10	0.42	0.43	0.43	0.43	0.43	0.43	1.47	1.49	1.49	1.49	1.50	1.51
Model 3												
Coverage (%)	89.50	91.80	91.90	91.70	92.60	93.10	68.30	88.90	89.00	89.10	89.50	89.40
Length	0.17	0.19	0.19	0.19	0.19	0.19	0.20	0.34	0.34	0.34	0.34	0.35
MSE·10	0.02	0.02	0.02	0.02	0.02	0.02	0.08	0.08	0.08	0.08	0.08	0.08

Notes: DWB denotes the DWB method. WBDD1, WBDD2, WBDD3, WBDD4, and WBDD5 refer to the proposed WBDD based on the external random variables WBDD1, WBDD2, WBDD3, WBDD4, and WBDD5, respectively. Coverage is the estimated coverage probability of confidence intervals; Length gives the average lengths of confidence intervals. The bootstrap percentile is used for all bootstrap-based methods. MSE refers to the empirical MSEs of the variance estimators of σ_N^2. 1,000 Monte Carlo trials with 499 bootstrap replications each.

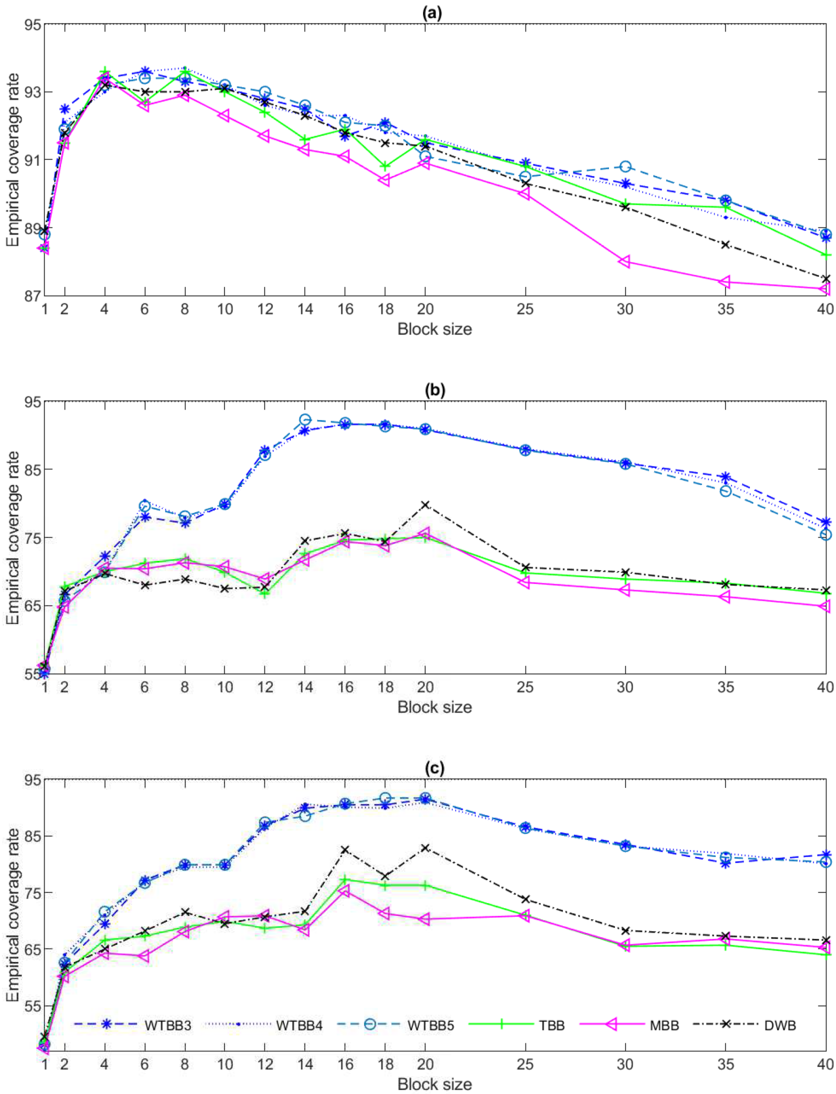

FIGURE 1. Empirical coverage, as a function of the block size ℓ, of 95% symmetric confidence intervals of the mean, obtained for the nonlinear AR model, Model 1, the heteroskedastic AR(1) model, Model 2, and the heteroskedastic AR(1), with permanent shifts in the innovation variance, Model 3, for a sample size $N = 200$ and $\rho = 0.6$. (a) Model 1. (b) Model 2. (c) Model 3.

sample sizes, and also increases with ρ that is, the degree of dependence in the data.

Overall, the results suggest that in a context of "strongly" heteroskedastic times series, the WBDD is a viable alternative to the DWB.

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TABLE 2. Block size/ bandwidth choices and SE (in parentheses)

	$N = 100$		$N = 200$	
	$\rho = 0.2$	$\rho = 0.6$	$\rho = 0.2$	$\rho = 0.6$
Model 1	3.74 (0.11)	6.26 (0.14)	3.97 (0.08)	7.05 (0.10)
Model 2	3.99 (0.08)	10.21 (0.15)	4.72 (0.07)	12.71 (0.17)
Model 3	5.65 (0.18)	11.32 (0.21)	6.88 (0.21)	14.38 (0.28)

Notes: This table provides the average block size/ bandwidth selected and their standard errors. Simulations where done with 1,000 Monte Carlo trials.

5. CONCLUDING REMARKS

This paper proposes a new bootstrap method for time series, the WBDD, that is generally applicable to variance estimation and sampling distribution approximation for the smooth function model. We show the consistency of the WBDD for both variance estimation and distribution approximation of the smooth function of sample mean when data are assumed to satisfy an NED condition. Computationally, it is very convenient to implement the new WBDD method. In particular, the choice of the external random variable is very flexible, as for the plain wild bootstrap. Furthermore, we show that the WBDD can provide asymptotic refinements over the limiting normal approximation by choosing the external random variables appropriately, a result that does not seem to be available for extant wild-based bootstrap methods for dependent data. Finally, simulation studies demonstrate that the WBDD performs well even for moderate sample sizes and in most cases outperforms the DWB. On the downside, as the DWB (and other wild-based bootstrap methods) the WBDD is not as widely applicable as the block-based bootstrap (in its many variations).

APPENDIX

In this appendix, we first list some additional assumptions (which are useful for the result of the higher-order correctness of the WBDD derived in Section 3) then prove Theorem 3.1.

A. Additional Regularity Conditions and Notations

We follow Götze and Künsch (1996) and assume that the sequence of random vectors $R_j \equiv (X_j, \check{Y}_j) \in \mathbb{R}^d \times \mathbb{R}$ satisfies the following conditions:

(C.1) $\mathbb{E}(X_j) = 0, j = 1, 2, \ldots$
(C.2) $\beta_s \equiv \mathbb{E}\|X_j\|^{s+r} < \infty,$ for some integer $s \geq 8,$ and $r > 0$ arbitrary small.

(C.3) There exists a sequence $\mathcal{D}_k, k \in \mathbb{Z}$ of sub-σ fields of $\mathcal{A}$ and a constant $\delta \in (0,1)$ such that for $j, m = 1, 2, \ldots,$ with $m > \delta^{-1}$, the random variable X_j can be approximated by a $\mathcal{D}_{j-m,j+m} \equiv \sigma\left(\mathcal{D}_p : |p-j| \leq m\right)$-measurable random vector $\bar{X}_{j,m}$ with

$$\mathbb{E}\left\|X_j - \bar{X}_{j,m}\right\| \leq \delta^{-1}\exp\left(-\delta m\right).$$

(C.4) There exists a $\delta \in (0,1)$ such that for all $m, j = 1, 2, \ldots, A \in \mathcal{D}_{-\infty,j}, B \in \mathcal{D}_{j+m,\infty}$,

$$|P\{A \cap B\} - P\{A\}P\{B\}| \leq \delta^{-1}\exp\left(-\delta m\right).$$

(C.5) There exists a $\delta \in (0,1)$ such that for all $m, j = 1, 2, \ldots, \delta^{-1} < m < j$ and $\|t\| \geq \delta$,

$$\mathbb{E}\left|\mathbb{E}\left\{\exp\left(\iota t\left[\check{Y}_{j-m} + \ldots + \check{Y}_{j+m}\right]\right)|\mathcal{D}_j : j \neq m\right\}\right| \leq \exp\left(-\delta\right),$$

and

$$\liminf_N N^{-1} Var\left(\sum_{i=1}^{N}\check{Y}_i\right) > 0.$$

(C.6) There exists a $\delta \in (0,1)$ such that for all $m, j, p = 1, 2, \ldots, A \in \mathcal{D}_{j-p,j+p}$

$$\mathbb{E}\left|P\{A|\mathcal{D}_l : l \neq j\} - P\{A|\mathcal{D}_l : 0 < |l-j| \leq m+p\}\right| \leq \delta^{-1}\exp\left(-\delta m\right).$$

(C.7) The function $H : \mathbb{R}^d \to \mathbb{R}$ is 3-times continuously differentiable, $DH(\mu) \neq 0$ and there are constant $c_0, a_0 > 0$, such that

$$\|D^3 H(x)\| \leq c_0(1 + \|x\|^{a_0})$$

for every $x \in \mathbb{R}^d$ for some integer $a_0 \geq 1$.

Following Götze and Künsch (1996) (cf. page 1918), by approximating $T_{N,stud}$ and $T^*_{N,stud}$ by quadratic statistics (and using a Taylor expansion), we have

$$T_{N,stud} = M_N + O_p\left(\frac{\ell}{N}\right),$$

where the leading term is

$$\begin{aligned} M_N &= h(\mu)' S_N \sigma_N^{-1} \\ &+ N^{-1/2}\left[\frac{1}{2}S_N' D^2 H(\mu) S_N \sigma_N^{-1} - \frac{1}{2}h(\mu)' S_N S_N' \zeta \sigma_N^{-3} - \frac{1}{2}h(\mu)' S_N V_N \sigma_N^{-3}\right] \\ &- \frac{1}{2}h(\mu)' S_N\left(\tau_N^2 - \sigma_N^2\right)\sigma_N^{-3}, \end{aligned}$$

such that

$$V_N \equiv \sum_{j=0}^{\ell}\tilde{k}_{jN} N^{-1/2}\sum_{i=1}^{N-\ell}\left(\check{Y}_i\check{Y}_{i+j} - \mathbb{E}\left(\check{Y}_i\check{Y}_{i+j}\right)\right),$$

$$\zeta \equiv 2D^2 H(\mu)\left[\sum_{j=-\infty}^{\infty}\Gamma(j)\right] DH(\mu) \text{ and } \tau_N^2 \equiv \sum_{j=0}^{\ell}\tilde{k}_{jN}\mathbb{E}\left(\check{Y}_0\check{Y}_j\right).$$

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Similarly, for the WBDD, we have

$$T^*_{N,stud} = M^*_N + O_{p^*}\left(\frac{\ell}{N}\right),$$

where the leading term is

$$M^*_N = h\left(\bar{X}_N\right)' S^*_N \sigma^{*-1}_N + N^{-1/2} \times \left[\frac{1}{2} S^{*\prime}_N D^2 H\left(\bar{X}_N\right) S^*_N \sigma^{*-1}_N - \frac{1}{2} h\left(\bar{X}_N\right)' S^*_N S^{*\prime}_N \zeta^* \sigma^{*-3}_N - \frac{1}{2}\ell^{1/2} h\left(\bar{X}_N\right)' S^*_N V^*_N \sigma^{*-3}_N\right],$$

such that

$$S^*_N \equiv N^{1/2}\left(\bar{X}^*_N - \bar{X}_N\right), V^*_N \equiv N^{-1/2}\sum_{j=1}^{Q}\left(B^{*2}_j - \mathbb{E}^*\left(B^{*2}_j\right)\right) = N^{-1/2}\sum_{j=1}^{Q} B^2_j\left(\ell u^2_j - \ell\mathbb{E}^*\left(u^2_j\right)\right),$$

$$\zeta^* \equiv 2D^2 H\left(\bar{X}_N\right)\mathbb{E}^*\left[Q^{-1}\sum_{j=1}^{Q} A^*_j A^{*\prime}_j\right] DH\left(\bar{X}_N\right), \text{ with } A^*_j = A_j\sqrt{\ell}u_j.$$

Next, we let

$$\mu_{3,N} = N^{1/2}\mathbb{E}\left(N^{-1/2}\sum_{i=1}^{N}\check{Y}_i\right)^3 = N^{1/2}\mathbb{E}\left(N^{-1/2}\sum_{i=1}^{N} h(\mu)'(X_i - \mu)\right)^3,$$

$$\pi_N \equiv \mathbb{E}\left[\left(h(\mu)' S_N\right) V_N\right] = N^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N-\ell}\sum_{k=0}^{\ell}\tilde{k}_{jN}\mathbb{E}\left(\check{Y}_i\check{Y}_j\check{Y}_{j+k}\right).$$

Furthermore, let Ξ_N denote the covariance matrix of the $(d+1)\times 1$-dimensional vector $W_{1N} \equiv \left(\left(h(\mu)' S_N\right)\sigma^{-1}_N, S'_N\right)'$ and let a_γ's be constants defined by the identity

$$\sum^{*}_{|\gamma|=2} W^{\gamma}_{1N} = (2\sigma_N)^{-1}\sum_{|\alpha|=2} D^{\alpha} H(\mu) S^{\alpha}_N - \sigma^{-3}_N\left(N^{-1/2}\sum_{i=1}^{N}\check{Y}_i\right) S'_N\left[D^2 H(\mu)\Sigma_\infty h(\mu)\right], \quad \textbf{(16)}$$

where $\Sigma_\infty \equiv \Sigma = \lim_{N\to\infty} Var(S_N)$. Note that in the left-hand side of the identity in (16), the index $\gamma \in \mathbb{Z}^{d+1}_+$ while on the right-hand side, the index $\alpha \in \mathbb{Z}^d_+$. Similarly, for the WBDD, we let

$$\mu^*_{3,N} \equiv N^{1/2}\mathbb{E}^*\left(W^*_N\right)^3 = \frac{Q}{N}\left(\ell\mathbb{E}\left(u^3\right)\right)\cdot\mu^{*(\text{MBB})}_{3,N}, \quad \textbf{(17)}$$

$$\pi^*_N \equiv \mathbb{E}^*\left(W^*_N V^*_N\right) = \mu^*_{3,N}, \quad \textbf{(18)}$$

where $\mu^{*(\text{MBB})}_{3,N} \equiv \ell^{1/2} Q^{-1}\sum_{j=1}^{Q} B^3_j$ is the MBB skewness term estimate of $\mu_{3,N}$ (i.e., $\mu^{*(\text{MBB})}_{3,N}$ is the analog of $\mu^*_{3,N}$ based on the MBB resampling approach, see for example, Götze and Künsch (1996) (cf. page 1920)). With these notations, we follow Götze and

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Künsch (1996) see also Lahiri (2003) cf. page 167 and define the formal first-order Edgeworth expansion Ψ_N of $T_{N,stud}$ in terms of its Fourier transformation as follows:

$$\tilde{\Psi}_N(t) = 1 + N^{-1/2}\sigma_N^{-3}\left[\left(\frac{\mu_{3,N}}{6} - \frac{\pi_N}{2}\right)(\iota t)^3 - \frac{\pi_N}{2}\iota t\right]\exp\left[-t^2/2\right]$$
$$+ N^{-1/2}(\iota t)\sum_{\gamma}^{*} a_\gamma (-1)^{|\gamma|} D^\gamma \exp\left[-\varpi' \Xi_N \varpi/2\right]|_{\varpi=(t,0,\ldots,0)}. \quad \textbf{(19)}$$

Similarly, for the WBDD, we define the formal first-order Edgeworth expansion Ψ_N^* of $T^*_{N,stud}$ in terms of its Fourier transformation

$$\tilde{\Psi}^*_N(t) = 1 + N^{-1/2}\sigma_N^{*-3}\mu^*_{3,N}\left[-\frac{1}{3}(\iota t)^3 - \frac{1}{2}\iota t\right]\exp\left[-t^2/2\right]$$
$$+ N^{-1/2}(\iota t)\sum_{\gamma}^{*} a^*_\gamma (-1)^{|\gamma|} D^\gamma \exp\left[-\varpi' \Xi^*_N \varpi/2\right]|_{\varpi=(t,0,\ldots,0)}, \quad \textbf{(20)}$$

where $\Xi^*_N = Var^*\left(W^*_{1N}\right)$, with $W^*_{1N} \equiv \left(\left(h\left(\bar{X}_N\right)' S^*_N\right)\sigma_N^{*-1}, S^*_N\right)'$ and a^*_γ's are defined in analogy to the a_γ's of (19), with μ replaced by $\bar{X}_N$.

B. Auxiliary results and Proof of Theorem 3.1

The following result is well known in the literature (see e.g., Theorem 4.1 of Götze and Künsch (1996) and Theorem 6.8 of Lahiri (2003). This result is only given here for completeness.

THEOREM 5.1. *Assume that conditions (C.1)-(C.7) (in Appendix A) hold. Furthermore, suppose that ℓ satisfies*

$$(\log(N))^C \ll \ell \leq N^{1/3},$$

with C large enough and $\tilde{k} \equiv 1$. It follows that as $N \to \infty$ then the Edgeworth approximation for $T_{N,stud}$ defined in (19) holds, that is for $s \geq 8$,

$$\sup_{x\in\mathbb{R}}\left|P\left\{T_{N,stud} \leq x\right\} - \Psi_N(x)\right| = O\left(\ell N^{-1+2/s}\right). \quad \textbf{(21)}$$

Proof of Theorem 5.1. Result follows under our assumed conditions by using Theorem 4.1 of Götze and Künsch (1996).

Proof of Theorem 3.1. We first show that

$$\sup_{x\in\mathbb{R}}\left|\Psi_N(x) - \Psi^*_N(x)\right| = O_p\left(\ell N^{-1}\right) + O_p\left(\ell^{-1}N^{-1/2}\right), \quad \textbf{(22)}$$

and

$$\sup_{x\in\mathbb{R}}\left|P^*\left\{T^*_{N,stud} \leq x\right\} - \Psi^*_N(x)\right| = O_p\left(\ell N^{-1+\epsilon}\right), \quad \textbf{(23)}$$

for any $\epsilon > 0$. Then, the desired result follows by the triangular inequality, given (21), (22), and (23).

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Note that (giving results in Götze and Künsch (1996) (cf. page 1920)), we have

$$\pi_N = \mu_{3,N} + O\left(N^{-1}\right) = \mu_{3,\infty} + O\left(N^{-1}\right)$$
$$\mathbb{E}\left(\mu_{3,N}^{*(\mathrm{MBB})}\right) = \mu_{3,\infty} + O\left(\ell^{-1}\right), \text{ and}$$
$$\mu_{3,N}^{*(\mathrm{MBB})} - \mathbb{E}\left(\mu_{3,N}^{*(\mathrm{MBB})}\right) = O_p\left(\ell N^{-1/2}\right),$$

where $\mu_{3,\infty} \equiv \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\mathbb{E}\left(\check{Y}_0\check{Y}_i\check{Y}_j\right)$. Hence, using (17), (18) and Assumption WBDD, we have

$$\mathbb{E}\left(\pi_N^*\right) = \mathbb{E}\left(\mu_{3,N}^*\right) = \underbrace{\frac{Q}{N}}_{\to 1}\underbrace{\left(\ell\mathbb{E}\left(u^3\right)\right)}_{\to 1}\mathbb{E}\left(\mu_{3,N}^{*(\mathrm{MBB})}\right) = \mu_{3,\infty} + O\left(\ell^{-1}\right), \text{ and}$$

$$\pi_N^* - \mathbb{E}\left(\pi_N^*\right) = \mu_{3,N}^* - \mathbb{E}\left(\mu_{3,N}^*\right) = \underbrace{\frac{Q}{N}}_{\to 1}\underbrace{\left(\ell\mathbb{E}\left(u^3\right)\right)}_{\to 1}\cdot\left(\mu_{3,N}^{*(\mathrm{MBB})} - \mathbb{E}\left(\mu_{3,N}^{*(\mathrm{MBB})}\right)\right)$$
$$= O_p\left(\ell n^{-1/2}\right).$$

Therefore, the order of the difference between the two Edgeworth expansions as given by (22) follows directly giving (19) and (20).

To show (23), we follow the same arguments as in the proof of Theorems 4.1 and 4.2 of Götze and Künsch (1996). That is it suffices to show that the conditions ((C.1)-(C.7)) on $R_j \equiv \left(X_j, \check{Y}_j\right)$ required for the Edgeworth expansion of Theorem 5.1 are also satisfied for $R_j^* = \tilde{R}_j \cdot \sqrt{\ell}u_j$, with $\tilde{R}_j \equiv \left(B_j, A_j'\right)'$ for $j = 1,\ldots,Q$ conditionally on the sample $\mathcal{X}_N = \{X_t\}_{t=1}^N$, uniformly for all $\mathcal{X}_N$ in a set whose probability tends to 1 as $N \to \infty$.

Condition (C.1) holds directly given the definition of $\tilde{R}_j$ (which is a function of $\mathcal{X}_N$) and under Assumption WBDD $\mathbb{E}\left(u_j\right) = 0$. In order to check Condition (C.2), note that

$$\mathbb{E}^*\left\|R_j^*\right\|^s = \left\|\tilde{R}_j\right\|^s \cdot \mathbb{E}\left(\left|\sqrt{\ell}u_j\right|^s\right).$$

Therefore, Condition (C.2) holds provided that for any $x > 0$,

$$P\left\{\left|\left\|\tilde{R}_j\right\|^s - \mathbb{E}\left(\left\|\tilde{R}_j\right\|^s\right) > x\right|\right\} = o_p(1), \tag{24}$$

and

$$\mathbb{E}\left(\left|\sqrt{\ell}u_j\right|^s\right) \le U_{N,s} \to C_s < \infty, \tag{25}$$

as $N \to \infty$. From the proof of Theorem 4.2 of Götze and Künsch (1996) (cf. page 1931), (24) is satisfied. (25) holds directly given that in the statement of part (b) of Theorem 3.1 we assume this condition.

Next, recall that by construction, R_j^* are independent random vectors conditionally on the sample $\mathcal{X}_N$. Therefore, the Conditions (C3), (C4), and (C6) are trivially satisfied (by independence) with a probability tending to one using a sigma field $\mathcal{D}_j = \sigma\left(u_j\right)$, (the σ-field generated by u_j) for $j = 1,\ldots,Q$ conditionally on the sample $\mathcal{X}_N$. By the same reason, we can replace Condition C.5 by the Cramér's condition (14). This concludes the proof of part (b) of Theorem 3.1.

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SUPPLEMENTARY MATERIAL

To view the supplementary material for this article, please visit: http://dx.doi.org/10.1017/S0266466621000487.

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