ORTHOGONAL MATRICES IN FOUR-SPACE

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Every proper orthogonal matrix A can be written

$$A = e^Q$$

where Q is a skew matrix [6], and conversely every such matrix A is orthogonal. It is also known that every proper orthogonal transformation in real Euclidean four-space may be characterized in term of quaternions [1, 3] by the equation

$$x' = axb, \qquad \mathrm{N}a = \mathrm{N}b = 1.$$

Here the quaternion

$$x = x_0 + x_1 i + x_2 j + x_3 k$$

determines with the origin a vector having the coordinates (x_0, x_1, x_2, x_3). The relationship between these two representations was clearly shown by Murnaghan [5].

The present paper employs the first and second regular representations of quaternions by matrices in place of Murnaghan's "special matrices," with the result that known properties of the regular representations can be applied directly to this problem. Incidentally an easy method not using infinite series is found for finding the skew matrix Q when the orthogonal matrix A is given.

1. The first and second regular representations of the real quaternion

$$\alpha = a_0 + a_1 i + a_2 j + a_3 k$$

are, respectively,

$$R(\alpha) = a_0 I + a_1 R_1 + a_2 R_2 + a_3 R_3, \qquad S(\alpha) = a_0 I + a_1 S_1 + a_2 S_2 + a_3 S_3$$

where

$$R_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

(1)

$$S_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad S_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

Let S^T denote the transpose of S. The six matrices R_1, R_2, R_3, S_1^T, S_2^T, S_3^T are all skew and are linearly independent. The most general 4 by 4 skew matrix

Received February 25, 1948.

$$(2) \qquad Q = \begin{bmatrix} 0 & q_{01} & q_{02} & q_{03} \\ -q_{01} & 0 & q_{12} & -q_{31} \\ -q_{02} & -q_{12} & 0 & q_{23} \\ -q_{03} & q_{31} & -q_{23} & 0 \end{bmatrix}$$

is therefore a linear combination of them. In fact

$$\begin{aligned} Q = & -\tfrac{1}{2}(q_{01} + q_{23})R_1 - \tfrac{1}{2}(q_{02} + q_{31})R_2 - \tfrac{1}{2}(q_{03} + q_{12})R_3 \\ & -\tfrac{1}{2}(q_{01} - q_{23})S_1^T - \tfrac{1}{2}(q_{02} - q_{31})S_2^T - \tfrac{1}{2}(q_{03} - q_{12})S_3^T. \end{aligned}$$

Note the analogy of the q's to Plücker line coordinates [2].

If we let $-\tfrac{1}{2}(q_{01} + q_{23}) = r_1$, $-\tfrac{1}{2}(q_{01} - q_{23}) = s_1$ etc., we may write

$$\rho = r_1 i + r_2 j + r_3 k, \qquad \sigma = s_1 i + s_2 j + s_3 k.$$

That is, every skew matrix can be written

$$Q = R(\rho) + S^T(\sigma),$$

where ρ and σ are pure quaternions. Therefore ρ satisfies the quadratic equation

$$(3) \qquad x^2 + N\rho = 0, \qquad N\rho = r_1^2 + r_2^2 + r_3^2,$$

and similarly for σ.

The matrix e^Q is defined as a power series which converges for every matrix Q. In every associative algebra, every matrix of the first regular representation is commutative with the transpose of every matrix of the second regular representation [4]. It follows upon multiplying power series that

$$e^Q = e^{R(\rho)} e^{S^T(\sigma)} = e^{S^T(\sigma)} e^{R(\rho)}.$$

Write R for $R(\rho)$. Then

$$e^R = I + \frac{1}{2}R^2 + \frac{1}{4!}R^4 + \cdots + R\left[I + \frac{1}{3!}R^2 + \frac{1}{5!}R^4 + \cdots\right].$$

From (3), $R^2 = -v^2 I$ where $v^2 = N\rho$, $v \geqq 0$. Hence

$$e^R = \cos v \cdot I + \frac{R}{v} \sin v.$$

If we define the quaternion

$$(4) \qquad \alpha = \cos v + \frac{\rho}{v} \sin v,$$

then clearly

$$R(\alpha) = e^{R(\rho)}, \qquad N\alpha = 1.$$

By means of (4) every pure quaternion ρ determines a unit quaternion α and vice versa. Similarly

$$e^{S^T(\sigma)} = S^T(\beta), \qquad N\beta = 1.$$

We have proved

THEOREM 1. *Every real proper 4 by 4 orthogonal matrix can be written*

$$A = R(\alpha) \cdot S^T(\beta) = S^T(\beta) \cdot R(\alpha)$$

where α and β are unit quaternions. Every such product is orthogonal and proper.

Let us assume a second such representation,

$$A = R(\gamma)\cdot S^T(\delta), \qquad \mathrm{N}\gamma = \mathrm{N}\delta = 1.$$

Then

$$R^{-1}(\gamma)\cdot R(\alpha) = S^T(\delta)\cdot S^{-T}(\beta), \quad R(\gamma^{-1}\alpha) = S^T(\beta^{-1}\delta).$$

The skew components of these matrices vanish, since the skew matrices in (1) are linearly independent. Thus

$$R(\gamma^{-1}\alpha) = S^T(\beta^{-1}\delta) = kI, \qquad k \text{ real},$$

so that $\alpha = k\gamma$, $\delta = k\beta$. Since $\mathrm{N}\alpha = \mathrm{N}\gamma = 1$, $\mathrm{N}k = k^2 = 1$, $k = \pm 1$. We have

THEOREM 2. *The pair of quaternions* α, β *of Theorem* 1 *is unique except that it may be replaced by* $-\alpha$, $-\beta$.

2. The unit quaternion

$$\alpha = a_0 + a_1 i + a_2 j + a_3 k, \qquad \mathrm{N}\alpha = 1, \tag{5}$$

satisfies the quadratic equation

$$x^2 - 2a_0 x + 1 = 0, \tag{6}$$

whose roots are the characteristic roots of $R(\alpha)$. Since the discriminant is $-4(a_1^2 + a_2^2 + a_3^2)$, these characteristic roots are real only if $\alpha = \pm 1$. That is, unless the orthogonal matrix $R(\alpha)$ is $\pm$ I, the orthogonal transformation which it defines leaves no vector through the origin invariant. But if v is any vector through the origin, the plane of vectors $k_1 v + k_2 R(\alpha)\cdot v$ is invariant. For by (6)

$$R(\alpha)[k_1 v + k_2 R(\alpha)\cdot v] = -k_2 v + (k_1 + 2a_0 k_2) R(\alpha)\cdot v.$$

Thus $R(\alpha)$ is the matrix of a left Clifford translation.

Coxeter [1] has shown that in quaternion coordinates the left Clifford translation is given by

$$x' = \alpha x, \qquad N\alpha = 1,$$

where α is given by (5), and

$$x = x_0 + x_1 i + x_2 j + x_3 k.$$

Upon multiplying out and equating the coefficients of 1, i, j and k, we have

$$\begin{aligned} x'_0 &= a_0 x_0 - a_1 x_1 - a_2 x_2 - a_3 x_3, \\ x'_1 &= a_1 x_0 + a_0 x_1 - a_3 x_2 + a_2 x_3, \\ x'_2 &= a_2 x_0 + a_3 x_1 + a_0 x_2 - a_1 x_3, \\ x'_3 &= a_3 x_0 - a_2 x_1 + a_1 x_2 + a_0 x_3. \end{aligned}$$

If we denote by v the column vector with components x_0, x_1, x_2, x_3, this may be written

$$v' = R(\alpha)\cdot v, \qquad N\alpha = 1.$$

In the same notation the right Clifford translations may be written

$$v' = S^T(\beta)\cdot v, \qquad N\beta = 1.$$

3. It has been shown that if A is proper orthogonal,

$$A = R(\alpha)\cdot S^T(\beta),$$

where α is given by (5) and β is given similarly. We shall show how α and β can be determined from A. From (1)

$$A = a_0 b_0 I + \sum_{i,j=1}^{3} a_i b_j R_i S_j^T + a_0 \sum_{i=1}^{3} b_i S_i^T + b_0 \sum_{j=1}^{3} a_j R_j . \tag{7}$$

Since R_i and S_j^T are both skew and commutative, their product is symmetric. Thus the first ten terms above are symmetric and the last six are skew. Hence the unique skew part of A is

$$\tfrac{1}{2}(A - A^T) = a_0[b_1 S_1^T + b_2 S_2^T + b_3 S_3^T] + b_0[a_1 R_1 + a_2 R_2 + a_3 R_3].$$

Since the R_i and S_j^T are linearly independent, we can determine uniquely the numerical values of

$$a_0 b_1,\ a_0 b_2,\ a_0 b_3,\ b_0 a_1,\ b_0 a_2,\ b_0 a_3.$$

With the aid of the relations

$$a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1, \qquad b_0^2 + b_1^2 + b_2^2 + b_3^2 = 1,$$

we obtain quadratic equations for a_0^2 and b_0^2 and hence the values of the eight a_i and b_j. It is known from Theorem 2 that just two sets of values can satisfy (7).

When α and β are known, ρ and σ can be found from (4), and then Q from (2).

References

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