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ON THE INTENSITY OF CROSSINGS BY A SHOT NOISE PROCESS

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Abstract

The crossing intensity of a level by a shot noise process with a monotone response is studied, and it is shown that the intensity can be naturally expressed in terms of a marginal probability.

Consider the shot noise process

$$X(t)=\sum_{\tau\leqq t} h(t-\tau), \qquad t\in\mathbb{R},$$

where the τ's are the points of a stationary Poisson process η on $\mathbb{R}$ with mean rate $\lambda>0$, and h, the impulse response, is a non-negative function on $[0,\infty)$ such that

(i) h is non-increasing,
(ii) h is finite except possibly at 0,
(iii) $\int_u^\infty h(x)\,dx<\infty$ for some large u.

By Theorem 1 of Daley (1971), the conditions (ii) and (iii) ensure that $X(t)<\infty$ almost surely for each t.

Observe that the sample function of X increases only at the points of η. Thus it is clear to define that X upcrosses the level u at t, where $u\geqq 0$, if $X(t-)\leqq u$ and $X(t)>u$. For $u\geqq 0$, write N_u for the point process (cf. Kallenberg (1976)) that consists of the points at which upcrossings of level u by X occur. Thus for each Borel set B, $N_u(B)$ denotes the number of upcrossings of u by X in B. N_u is a stationary point process, which may be viewed as a thinned process of η. The purpose of this communication is to derive the following result.

Theorem 1. For each $u\geqq 0$, $\mathscr{E}N_u(B)=\lambda m(B)P[u-h(0)<X(0)\leqq u]$ for each Borel set B, where m is Lebesgue measure.

To prove Theorem 1, first enumerate the points of η in $(-\infty,0)$ by letting ρ_i be the ith largest point of η to the left of 0 for $i=1,2,3,\cdots$. The ρ_i are almost surely well defined, and $-\rho_1, \rho_1-\rho_2, \rho_2-\rho_3,\cdots$ are independent and identically distributed random variables. The following result is useful.

Lemma 2. For each $i=1,2,\cdots$, $P[X(\rho_i-)=\sum_{j\geqq i+1}h(\rho_i-\rho_j)]=1$ where $X(\rho_i-)$ denotes the left-hand limit of X at ρ_i. From this it follows immediately that $X(\rho_i-)$ is independent of ρ_i, and $X(\rho_i-)$ has the same distribution as $X(0)$.

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Proof. Let $i \geqq 1$ be fixed. Since h is monotone, it is almost everywhere continuous. Using the continuity of $\rho_i - \rho_j$, $j \geqq i+1$, we obtain

$$\lim_{\varepsilon \to 0} h(\rho_i - \rho_j - \varepsilon) = h(\rho_i - \rho_j) \quad \text{a.s. for } j \geqq i+1.$$

Also by the monotonicity of h, $h(\rho_i - \rho_j - \varepsilon) \leqq h(\rho_{i+1} - \rho_j)$ for $0 < \varepsilon < \rho_i - \rho_{i+1}$, $j \geqq i+2$, where $\sum_{j \geqq i+2} h(\rho_{i+1} - \rho_j)$ is almost surely finite since it has the same distribution as $X(0)$. Thus it follows from dominated convergence that almost surely

$$\lim_{\varepsilon \downarrow 0} X(\rho_i - \varepsilon) = \lim_{\varepsilon \downarrow 0} \sum_{j \geqq i+1} h(\rho_i - \rho_j - \varepsilon) = \sum_{j \leqq i+1} h(\rho_i - \rho_j).$$

Proof of Theorem 1. By stationarity, it apparently suffices to show that $N_u(B)$ equals $\lambda m(B) P[u - h(0) < X(0) \leqq u]$ for each Borel set B in $(-\infty, 0)$, where $m(B)$ denotes the Lebesgue measure of B. Since

$$X(\rho_i) = h(0) + \sum_{j \geqq i+1} h(\rho_i - \rho_j),$$

Lemma 2 implies that almost surely

$$N_u(B) = \sum_{i \geqq 1} 1(u - h(0) < X(\rho_i -) \leqq u,\ \rho_i \in B),$$

where $1(\cdot)$ is the indicator function. Applying the facts that $X(\rho_i)$ is independent of ρ_i and $X(\rho_i -)$ is equal in distribution to $X(0)$, we get

$$\begin{aligned} \mathscr{E} N_u(B) &= \sum_{i \geqq 1} \mathscr{E} 1(u - h(0) < X(\rho_i -) \leqq u) \mathscr{E} 1(\rho_i \in B) \\ &= P[u - h(0) < X(0) \leqq u] \lambda m(B). \end{aligned}$$

We mention the following for completeness.

(a) By stationarity, the downcrossing intensity of a level by X is also given by Theorem 1.

(b) We assumed, for simplicity of illustration, that the impulse response h is deterministic. Lifting this restriction, it is readily seen that Theorem 1 continues to hold if the impulse responses brought about by the points of η are independent of η, and are independent and identically distributed.

(c) For methods of obtaining the marginal distribution of X see Gilbert and Pollak (1960).

(d) The crossing intensities of some other shot noise processes were studied by Rice (1944), and Bar–David and Nemirovsky (1972). A result in the latter paper can be reduced to one which is similar to Theorem 1. However, our assumptions on h are considerably simpler.

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