J. Austral. Math. Soc. **70** (2001), 76–87

m-WIELANDT SERIES IN INFINITE GROUPS

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(Received 12 February 2000; revised 28 June 2000)

Communicated by R. B. Howlett

Abstract

In a group G, $u_m(G)$ denotes the subgroup of the elements which normalize every subnormal subgroup of G with defect at most m. The m-Wielandt series of G is then defined in a natural way. G is said to have finite m-Wielandt length if it coincides with a term of its m-Wielandt series. We investigate the structure of infinite groups with finite m-Wielandt length.

2000 *Mathematics subject classification*: primary 20F22, 20F16.

1. Introduction

If G is a group, the Wielandt subgroup $w(G)$ of G is defined to be the intersection of the normalizers of all the subnormal subgroups of G. The Wielandt series of G is defined recursively by setting $w_0(G) = 1$ and $w_i(G)/w_{i-1}(G) = w(G/w_{i-1}(G))$, for $i \geq 1$. If for some integer n, $w_n(G) = G$, then G is said to have finite Wielandt length, and the minimal of such n is said the *Wielandt length* of G. Following Casolo [4], we denote by $\mathscr{W}$ the class of all groups with finite Wielandt length. Wielandt [19] proved that the socle of a finite group is contained in the Wielandt subgroup; thus any finite group belongs to $\mathscr{W}$. When we consider infinite groups the situation is more complex. Clearly, $w(G)$ contains the centre $Z(G)$, and so nilpotent groups have finite Wielandt length. Also the class Min-sn (that is, groups satisfying the minimal condition on the subnormal subgroups) and the class of soluble groups with the property that each subnormal subgroup has finitely many conjugates and there is a bound on their number, are contained in $\mathscr{W}$ (see [3, 13, 17]). Nevertheless the example of the infinite dihedral group shows that $w(G)$ can be trivial, also in polycyclic groups. Actually a polycyclic

https://doi.org/10.1017/S1446788700002299 Published online by Cambridge University Press

group belongs to $\mathscr{W}$ if and only if it is finite-by-nilpotent (see [5, 12]). Furthermore, McDougall [11] proved that a minimax soluble group with finite Wielandt length is (abelian divisible)-by-(nilpotent torsion-free)-by-finite and Casolo in an unpublished paper has generalized this result by proving that an $\mathfrak{S}_1$-group with finite Wielandt length is (nilpotent divisible)-by-(nilpotent torsion-free)-by-finite. Recall that $\mathfrak{S}_1$ denotes the class of soluble groups of finite abelian section rank, whose elements have only finitely many distinct prime orders (see [15, Part 2]).

In 1990, Bryce [2] introduced a new family of subgroups, which generalize the concept of Wielandt subgroup. For each integer $m \geq 1$, he denoted by $u_m(G)$ the intersection of the normalizers of all the subnormal subgroups of G with defect at most m, and he defined inductively the subgroups $u_{m,i}(G)$ by setting $u_{m,0}(G) = 1$ and for $i \geq 1$, $u_{m,i}(G)/u_{m,i-1}(G) = u_m(G/u_{m,i-1}(G))$.

Then for each $m \geq 1$, we have a characteristic series of G, which we shall call the *m-Wielandt series* of G. If for some integer n, $u_{m,n}(G) = G$, then G is said to have finite m-Wielandt length, and the least of such integers is the *m-Wielandt length* of G.

We denote by $\mathscr{W}_m^*$ the class of groups with finite m-Wielandt length for a fixed m, and by $\mathscr{W}^*$ the class of groups with finite m-Wielandt length for each $m \geq 1$. Thus $\mathscr{W}^* = \bigcap_{m=1}^{\infty} \mathscr{W}_m^*$. Since $w_i(G) \leq u_{m,i}(G)$ for each $m \geq 1$, $i \geq 0$ (see [2, 6]), we have that $\mathscr{W}$ is contained in $\mathscr{W}^*$. The infinite dihedral group shows that this inclusion is a strict one.

The aim of this paper is principally to investigate the structure of groups which belong to $\mathscr{W}^*$. In Section 2 we prove the following

THEOREM 1. *If G is a nilpotent-by-finite group, then $G \in \mathscr{W}^*$.*

In Section 3 we characterize $\mathfrak{S}_1$-groups which belong to $\mathscr{W}^*$ and to $\mathscr{W}$. Moreover, we prove that $\mathscr{W}_m^* \cap \mathfrak{S}_1 = \mathscr{W}^* \cap \mathfrak{S}_1$, for each $m \geq 2$. Denote by $\gamma_\infty(G)$ the *nilpotent residual* of a group G. We have

THEOREM 2. *An $\mathfrak{S}_1$-group belongs to $\mathscr{W}^*$ if and only if it is (nilpotent divisible)-by-(nilpotent torsion-free)-by-finite.*

THEOREM 3. *An $\mathfrak{S}_1$-group G belongs to $\mathscr{W}$ if and only if it is (nilpotent divisible)-by-nilpotent-by-finite, with the property that if D is the finite residual of G and F/D is the Fitting subgroup of G/D, then $(S/D)/\gamma_\infty(S/D)$ is nilpotent, for each subnormal subgroup $S \geq F$.*

For notation and general properties of subnormal subgroups and groups of finite rank we refer to [9, 15].

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2. Groups with finite m-Wielandt length

The proof of Theorem 1 is based on an idea of Casolo and uses the following lemma.

LEMMA 1. *Let $G = NS$ be a group, with N a normal nilpotent subgroup of G with nilpotency class at most c, and S a subnormal subgroup of G with defect at most m. Then $S \geq \gamma_{mc+1}(G)$.*

PROOF. The proof is similar to the proof of [3, Lemma 4.6], with obvious modifications. □

PROOF OF THEOREM 1. First, note that if H is a finite group and $\mathscr{S}$ denotes the set of all subnormal subgroups of H, then it is possible to order $\mathscr{S} = \{H = H_0, H_1, \ldots, H_n = 1\}$ in such a way that, for each $i = 1, \ldots, n$, the subgroup H_i permutes, by conjugation, the elements of the set $\{H_0, H_1, \ldots, H_{i-1}\}$.

Now let G be a nilpotent-by-finite group and fix an integer $m \geq 1$. We want to show that G has finite m-Wielandt length. Let N be the Fitting subgroup of G: then N is nilpotent and G/N is finite. Denote by $\mathscr{L}$ the set of all subnormal subgroups of G containing N. Clearly, $\mathscr{L}$ is finite and $NS \in \mathscr{L}$ for each subnormal subgroup S of G. By the previous remark, we can fix an ordering $\mathscr{L} = \{G = H_0, H_1, \ldots, H_n = N\}$ of $\mathscr{L}$ in such a way that H_i fixes, by conjugation, the set $\{H_0, H_1, \ldots, H_{i-1}\}$, for each $i = 1, \ldots, n$.

For $i = 0, \ldots, n$, set $U(0, i) = 1$, and for $j > 0$, define inductively the subgroup $U(j + 1, i)$ as the intersection of the normalizers in N of all subnormal subgroups S of G with defect at most m, such that $S \geq U(j, i)$ and $NS = H_r$, where $r \leq i$. We show by induction on i that $U(d^{i+1}, i) = N$, where $d = cm$ and c is the nilpotency class of N. Therefore, for $i = n$ we have $N = U(d^{n+1}, n) \leq u_{m,d^{n+1}}(G)$, and thus, since G/N is finite, we get that G has finite m-Wielandt length.

Let $i = 0$ and set $N_0 = 1$ and $N_j = N \cap \gamma_{d+1-j}(G)$, for $j = 1, \ldots, d$. It can be proved by induction on j that $N_j \leq U(j, 0)$, for each $j = 0, 1, \ldots, d$, using Lemma 1 to start the induction. Hence $N = N_d = U(d, 0)$.

Now let $i \geq 1$ and assume by induction that $N = U(d^i, i - 1)$. Set $D_0 = 1$ and $D_j = N \cap \gamma_{d+1-j}(H_i)$, for $j = 1, \ldots, d$. For $0 \leq k \leq d^{i+1}$ write $k = qd + r$, where q and r are the quotient and the remainder, respectively, of the division of k by d (thus obviously $0 \leq q \leq d^i$ and $0 \leq r \leq d-1$), and set $X_k = U(q, i-1)D_r \cap U(q+1, i-1)$. Note that for $k = d^{i+1}$, we have $X_{d^{i+1}} = U(d^i, i - 1) = N$. We claim that if $S \geq X_k$, $0 \leq k < d^{i+1}$, is a subnormal subgroup with defect at most m in G such that $NS = H_t$, with $t \leq i$, then $[X_{k+1}, S] \leq S$. To see this we have to distinguish two cases.

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(a) $r < d-1$. Then $k+1 = qd + (r+1)$ and $X_{k+1} = U(q, i-1)D_{r+1} \cap U(q+1, i-1)$, whence

$$[X_{k+1}, S] \leq [U(q, i-1)D_{r+1}, S] \cap [U(q+1, i-1), S] =: L.$$

Now, if $t < i$, then

$$L \leq [U(q+1, i-1), S] \leq U(q, i-1)S \leq X_k S \leq S.$$

If $t = i$, then H_i normalizes each $U(j, i-1)$, $j = 0, \dots, d^i$, since H_i fixes the set $\{H_0, \dots, H_{i-1}\}$. Moreover, $S \geq \gamma_{d+1}(H_i)$, by Lemma 1, and thus $[D_{r+1}, S] \leq D_r$ when $r > 0$, and $[D_1, S] \leq S$. Hence we have

$$L \leq U(q, i-1)[D_{r+1}, S] \cap U(q+1, i-1) \leq X_k S \leq S.$$

(b) $r = d-1$. Then $k+1 = (q+1)d$ and $X_{k+1} = U(q+1, i-1)$. Set $L = [X_{k+1}, S] = [U(q+1, i-1), S]$. Thus if $t < i$, then $L \leq U(q, i-1)S \leq X_k S \leq S$. If $t = i$, then, as above, $U(q+1, i-1)$ is normalized by H_i and $[N, S] \leq [H_i, H_i] \cap N = D_{d-1}$. Hence

$$L \leq U(q+1, i-1) \cap D_{d-1} \leq U(q, i-1)D_{d-1} \cap U(q+1, i-1) = X_k \leq S.$$

This completes the proof that $[X_{k+1}, S] \leq S$. Now if $k = 0$, then $X_0 = U(0, i-1)D_0 \cap U(1, i-1) = 1$, and so by a recursive argument we have that $X_k \leq U(k, i)$, for each $0 \leq k \leq d^{i+1}$. It follows that $X_{k+1} \leq N_G(S)$ for every S which goes into the definition of $U(k+1, i)$, so $X_{k+1} \leq U(k+1, i)$. Thus for $k = d^{i+1}$ we have $N \leq U(d^{i+1}, i)$ and the theorem follows. □

The following lemma contains a result similar to that of Theorem 1. In particular, it implies that *if a group G is the extension of a nilpotent group by a T-group with finitely many normal subgroups, then it belongs to $\mathscr{W}^*$*. Recall that a T-group is a group in which every subnormal subgroup is normal.

LEMMA 2. *Let G be a group, and let N be a normal subgroup of G.*

(i) *If $G/C_G(N)$ has only a finite number n of normal subgroups, then $N \leq u_{2,n}(G)$.*

(ii) *If $G/C_G(N)$ is a T-group with a finite number n of normal subgroups, then $N \leq u_{m,n(m-1)}(G)$, for each $m \geq 1$.*

PROOF. Fix m to be either equal to 2 if (i) holds or an integer greater than 0 if (ii) holds. Set $C = C_G(N)$ and let $\{S_1, \dots, S_n\}$ be the set of all normal subgroups of G containing C. For each $1 \leq r \leq n(m-1)$ denote by Λ_r the set of all r-tuples

$\lambda_r = (i_1, \dots, i_r)$ such that $1 \le i_j \le n$ and each symbol appears at most $m-1$ times. Set $\lambda_0 = 0$ and $\Lambda_0 = \{\lambda_0\}$. Then for each $\lambda_r = (i_1, \dots, i_r) \in \Lambda_r$, $1 \le r \le n(m-1)$, define $X_{\lambda_r} = [N, S_{i_1}, \dots, S_{i_r}]$ and for $r = 0$, $X_{\lambda_0} = N$. Finally, for $0 \le r \le n(m-1)$, set $N_r = \langle X_{\lambda_r} \mid \lambda_r \in \Lambda_r \rangle$.

We prove, by a recursive argument, that $N_{n(m-1)-l} \le u_{m,l}(G)$, for each $l = 0, \dots, n(m-1)$. If $l = 0$, then by definition $X_{\lambda_{n(m-1)}} \le [N, C] = 1$, for each $\lambda_{n(m-1)} \in \Lambda_{n(m-1)}$. Hence $N_{n(m-1)} = 1$.

Let $l > 0$ and assume that $N_{n(m-1)-l+1} \le u_{m,l-1}(G)$. Then in order to prove that $N_{n(m-1)-l} \le u_{m,l}(G)$ it is enough to show that $[X_{\lambda_{n(m-1)-l}}, R] \le RN_{n(m-1)-l+1}$, for each subnormal subgroup R with defect at most m, and $\lambda_{n(m-1)-l} \in \Lambda_{n(m-1)-l}$. So let R be a subnormal subgroup of G with defect at most m and let $R^{G,m-1}C = S_j$, where $R^{G,m-1} := R[G,_{m-1} R]$. If $\lambda_{n(m-1)-l}$ has at most $m-2$ entries equal to j, then $[X_{\lambda_{n(m-1)-l}}, R] \le [X_{\lambda_{n(m-1)-l}}, S_j] \le N_{n(m-1)-l+1}$. Otherwise, $\lambda_{n(m-1)-l}$ has exactly $m-1$ entries equal to j and then $X_{\lambda_{n(m-1)-l}} \le [N,_{m-1} S_j] = [N,_{m-1} R^{G,m-1}] \le N \cap R^{G,m-1} \le N_G(R)$.

Therefore, $N_{n(m-1)-l} \le u_{m,l}(G)$ as claimed. In particular, for $l = n(m-1)$ we have $N = N_0 \le u_{m,n(m-1)}(G)$. □

Recall that by a theorem of Wielandt [19], $w(G)$ contains every minimal normal subgroup of G which satisfies the minimal condition on the normal subgroups. Let M be a minimal normal subgroup of G and let R be a subnormal subgroup with defect at most 2. Then either $M \cap R^G = 1$, and thus M centralizes R, or $M \le R^G \le N_G(R)$. Therefore, *$u_2(G)$ contains every minimal normal subgroup of G.*

We conclude this section with an example of a group which does not belong to $\mathscr{W}^*$. Another example is given in Section 3.

Let $G = M \rtimes A$, where $M = M(\mathbb{Q}, GF(p))$ is the McLain group (see [15, Part 2, page 14]), and A is the group of automorphisms of M induced by the affine group of $\mathbb{Q}$. Then by [15, Theorem 6.21.iv], M is a minimal normal subgroup of G. Thus $M \le u_2(G)$. Furthermore A is metabelian and $[a, A'] = A'$, for each $a \in A \setminus A'$. Hence $A' \le u_2(A)$ and then $G \in \mathscr{W}_2^*$. Now if N is a normal subgroup of G not containing M, then $[N, M] \le N \cap M = 1$. Suppose by contradiction that $N \ne 1$ and choose an element $1 \ne ma \in N$, with $m \in M$, $1 \ne a \in A$. Then there exist $\lambda, \mu \in \mathbb{Q}$ such that $1 + e_{\lambda\mu} \in C_M(m)$ and hence $1 + e_{\lambda\mu} = (1 + e_{\lambda\mu})^a$. It follows that $a = 1$, a contradiction. Therefore, $N = 1$ and G is a monolithic group, with monolith M. Since $u_3(G)$ is a T-group (see [2, 6]), then $M \not\le u_3(G)$, and so $u_3(G) = 1$.

3. $\mathfrak{S}_1$-groups

Let $u_\chi(G)$ denote the normalizer of all subgroups of a group G lying between a characteristic subgroup and its derived subgroup. Then $u_\chi(G)$ is a characteristic

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subgroup of G and $u_\chi(G) \geq u_m(G)$, for each $m \geq 2$. We define by iteration an ascending series of G, $1 \leq u_\chi(G) = u_{\chi,1}(G) \leq u_{\chi,2}(G) \leq \ldots$, where for $i \geq 1$, $u_{\chi,i+1}(G)/u_{\chi,i}(G) = u_\chi(G/u_{\chi,i}(G))$. Let $\mathscr{W}_\chi$ be the class of all groups G such that $G = u_{\chi,n}(G)$, for some $n \geq 0$. Trivially $\mathscr{W}_2^* \subseteq \mathscr{W}_\chi$. Moreover, for each characteristic subgroup $H \leq G$, we have that $u_\chi(G) \cap H \leq u_\chi(H)$ and $u_\chi(G)H/H \leq u_\chi(G/H)$. Therefore, the class $\mathscr{W}_\chi$ is closed for characteristic subgroups and factor groups by characteristic subgroups. In this section we prove the following stronger version of Theorem 2.

THEOREM 2′. *For an $\mathfrak{S}_1$-group G the following conditions are equivalent*:

(i) $G \in \mathscr{W}_\chi$.
(ii) $G \in \mathscr{W}^*$.
(iii) *G is (nilpotent divisible)-by-(nilpotent torsion-free)-by-finite.*

We begin with some preliminary easy lemmas about nilpotent groups.

LEMMA 3. *Let G be a group and let H be a subgroup of G. Let $x, y \in G$, $n \in \mathbb{N}$, and suppose that $[x, y] \in H$. If $[H, y] \leq H^n$, then $[x, y]^q[x, y^q]^{-1} \in H^n$ for each integer $q \geq 0$.*

From the Hall-Petresco identity (see [8]), it follows that if G is a nilpotent group of nilpotency class c and q is an integer greater than c, then $G^q = \{g^q \mid g \in G\}$ (see [1, Corollary 2.31]). In particular, we have that $G^{q^n} = \{g^{q^n} \mid g \in G\}$ and $(G^{q^n})^{q^m} = G^{q^{n+m}}$, for each $q > c$, $n, m \geq 0$.

LEMMA 4. *Let G be a group, and let F be a nilpotent subgroup of G of nilpotency class c. Let $x, y \in G$ be such that $[x, y] \in F$ and $[F, y] \leq F^{q^n}$, where $q > c$ and $n \geq 1$. Then for each $k \geq 0$, $l \geq 0$ we have*

(i) $[x, y^{q^l}] \in F^{q^l}$;
(ii) $[F^{q^k}, y^{q^l}] \leq F^{q^{n+l+k}}$.

PROOF. First, we show that $[F^{q^k}, y] \leq F^{q^{n+k}}$, for each $k \geq 0$. Let a be an element of F. Then $a^y = ab^{q^n}$, for some $b \in F$. Thus by the Hall-Petresco identity, we have

$$(a^{q^k})^y = (a^y)^{q^k} = (ab^{q^n})^{q^k} = a^{q^k} b^{q^{n+k}} e_2^{q^k} \cdots e_c^{q^k}$$

with $e_i \in \langle a, b^{q^n}\rangle' \leq F^{q^n}$, $i = 2, \ldots, c$. Therefore, each e_i is a q^n-th power and $[F^{q^k}, y] \leq F^{q^{n+k}}$.

Then we prove (i) by induction on l. If $l = 0$, this is the hypothesis. Let $l \geq 1$ and assume that $[x, y^{q^{l-1}}] \in F^{q^{l-1}}$. Then by the previous considerations $[F^{q^{l-1}}, y^{q^{l-1}}] \leq F^{q^{n+l-1}}$, and hence Lemma 3 yields that $[x, y^{q^{l-1}}]^q[x, y^{q^l}]^{-1} \in F^{q^{n+l-1}}$. Therefore, $[x, y^{q^l}] \in F^{q^l}$ and we are done.

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Finally, since $[a, y] \in F^{q^{n+k}}$ when $a \in F^{q^k}$, and $[F^{q^{n+k}}, y] \leq F^{q^{2n+k}}$, we can apply (i) to $F^{q^{n+k}}$ and a in the place of F and x, and we get (ii). $\square$

We now state a first result about the action of some elements of $u_\chi(G)$ on the group G, which play an important role in the sequel.

PROPOSITION 5. *Let A and F be characteristic subgroups of G, such that F is nilpotent torsion-free with class c and $A' \leq F$. Assume that $A \leq C_G(F/F^{q^n})$, where $n \geq 2$, $q > c$. Then $[x, y] \in \bigcap_{k \geq 0} F^{q^k}$, for each $y \in A$ such that $\langle y \rangle \cap F = 1$ and for each $x \in u_\chi(G) \cap F$.*

PROOF. First note that $(A^{q^h})' \leq F^{q^{2h}}$, for each integer $h \geq 0$. In fact, by Lemma 4.i, we have $[a^{q^h}, b^{q^h}] \in F^{q^{2h}}$, for each $a, b \in A$.

Now let $y \in A$ be such that $\langle y \rangle \cap F = 1$. By definition, if $x \in u_\chi(G) \cap F$, then x normalizes the subgroup $\langle y^{q^h} \rangle (A^{q^h})'$ and thus

$$[x, y^{q^h}] \in F \cap \langle y^{q^h} \rangle (A^{q^h})' \leq F \cap \langle y^{q^h} \rangle F^{q^{2h}} = F^{q^{2h}}.$$

Suppose now that $[x, y] \in F^{q^{k-1}}$ for some $k \geq 1$. Then by Lemma 4, $[x, y^{q^{k-s}}] \in F^{q^{2k-s-1}}$, for each $0 \leq s \leq k$. Let us show, by a recursive argument, that $[x, y^{q^{k-s}}] \in F^{q^{2k-s}}$. If $s = 0$, by the previous considerations, there is nothing left to prove. So let $s \geq 1$ and assume that $[x, y^{q^{k-s+1}}] \in F^{q^{2k-s+1}}$. Then, since by Lemma 4.ii, $[F^{q^{2k-s-1}}, y^{q^{k-s}}] \leq F^{q^{2k-s-1+n+k-s}} = F^{q^{n+3k-2s-1}}$, Lemma 3 applied to $F^{q^{2k-s-1}}$ and $y^{q^{k-s}}$ yields that $[x, y^{q^{k-s}}]^q [x, y^{q^{k-s+1}}]^{-1} \in F^{q^{n+3k-2s-1}}$. Therefore, since $n+3k-2s-1 \geq 2k-s+1$, we have that $[x, y^{q^{k-s}}]^q \in F^{q^{2k-s+1}}$. F is a nilpotent torsion-free group and thus $a^r = b^r$ implies $a = b$, for each $a, b \in F$, $r \in \mathbb{Z}$. Hence $[x, y^{q^{k-s}}] \in F^{q^{2k-s}}$, as claimed.

To prove now the statement we proceed by induction on k. If $k = 0$, trivially $[x, y] \in F$. Let $k \geq 1$ and suppose that $[x, y] \in F^{q^{k-1}}$. Then the previous considerations with $s = k$ show that $[x, y] \in F^{q^k}$ and we are done. $\square$

COROLLARY 6. *Let G be an $\mathfrak{S}_1$-group, and let $F \leq A$ be characteristic subgroups of G. Assume that F is nilpotent torsion-free, $Z(F)$ is reduced and A/F is an abelian torsion-free group. Then $C_A(u_\chi(G) \cap Z(F))$ has finite index in A.*

PROOF. Let c be the nilpotency class of F. Since $F \in \mathfrak{S}_1$ and $Z(F)$ is reduced, by a theorem of Robinson ([14, Theorem E]), F is residually a finite π-group, for a suitable finite set of primes $\pi = \{p_1, \dots, p_t\}$. Therefore, if $q = (p_1 \cdots p_t)^c$, then $\bigcap_{k \geq 0} F^{q^k} = 1$.

Now F/F^{q^2} is finite since it is an $\mathfrak{S}_1$-group with finite exponent, and thus $C = C_A(F/F^{q^2})$ is a characteristic subgroup of G with finite index in A. Moreover $C/C \cap F$ is an abelian torsion-free group. Then by applying Proposition 5 to subgroups C and F of G, we get that $[x, y] \in \bigcap_{k \geq 0} F^{q^k} = 1$, for each $x \in u_\chi(G) \cap F$ and $y \in C \setminus F$. Therefore, $u_\chi(G) \cap Z(F)$ is centralized by C and $|A : C_A(u_\chi(G) \cap Z(F))|$ is finite. $\square$

Note that if in Proposition 5 and Corollary 6 we replace $u_{\mathfrak{X}}(G)$ by $u_2(G)$, the corresponding statements can be proved under the weakened hypothesis that A and F are normal subgroups of G.

We give now an example of a polycyclic group G in which $u_{\mathfrak{X}}(G) = 1$. Let $G = F \rtimes \langle y \rangle$, where $F = \langle a, b \rangle$ is a free abelian group of rank 2 and y is an element of infinite order such that $a^y = ab^4$ and $b^y = a^4 b^{17}$. Then F is the Fitting subgroup of G (since for each integer r, y^r induces on F a fixed-point-free automorphism), $G' \leq F$ and F/F^4 is a central section of G. Thus, by Proposition 5, $[x, y] = 1$ for each $x \in u_{\mathfrak{X}}(G) \cap F$, that is to say $u_{\mathfrak{X}}(G) \cap F \leq Z(G)$. Since $Z(G) = 1$, we have that $u_{\mathfrak{X}}(G) \cap F = 1$ and hence $u_{\mathfrak{X}}(G) = 1$.

The next lemma contains probably well-known properties of $\mathfrak{S}_1$-groups.

LEMMA 7. *Let G be an $\mathfrak{S}_1$-group.*

(i) *If G is residually finite and H is a normal subgroup of G of finite index, then $G/Z(H)$ is a residually finite $\mathfrak{S}_1$-group.*

(ii) *If G has no quasicyclic subgroups and H is a nilpotent divisible normal subgroup of G, then G/H is an $\mathfrak{S}_1$-group with no quasicyclic subgroups.*

PROOF. Let G and H be as in (i). Then, since H is residually finite, the centre of the Baer radical of H is reduced by [15, Theorem 9.37]. So $\operatorname{Aut} H$ has a normal torsion-free subgroup K of finite index by the remark after Corollary on page 139 in [15, Part 2]. Then $\operatorname{Inn} H \cap K$ is a torsion-free subgroup of finite index of $\operatorname{Inn} H$, and it is normal in $\operatorname{Aut} H$. Let N be the set of all elements of H which induce inner automorphisms belonging to K. Then N contains $Z(H)$ and $N/Z(H)$ is the image of $\operatorname{Inn} H \cap K$ with respect to the isomorphism $\operatorname{Inn} H \simeq_{\operatorname{Aut} H} H/Z(H)$. Hence N is a characteristic subgroup of H of finite index and $N/Z(H)$ is torsion-free. It follows that N is a normal subgroup of finite index in G, and $G/Z(H)$ is an $\mathfrak{S}_1$-group. Moreover, by Corollary to Theorem 9.37 in [15], the holomorph of H is residually finite, and so also $H/Z(H)$ is residually finite. Then $G/Z(H)$ is residually finite.

Let G and H be as in (ii). Then by Theorem 10.33 and Theorem 9.39.3 in [15], H is contained in a nilpotent torsion-free characteristic subgroup N of G, such that G/N is polycyclic. Hence we may assume without loss of generality that G is a nilpotent torsion-free group.

Let $T/Z(H)$ denote the torsion-subgroup of $G/Z(H)$, and set $C = C_T(Z(H))$. Then, since by [15, Theorem 9.23], $Z(H)$ is an abelian divisible group of finite rank, T/C is isomorphic with a periodic group of matrices over the field of rational numbers; so it is finite by a classical theorem of Schur (see [15, Part 1 page 85]). Since clearly $Z(H) \leq Z(C)$, $C/Z(C)$ is periodic and consequently locally finite. The corollary to Theorem 4.12 in [15] implies that C' is locally finite. Hence $C' = 1$ and C is abelian. Then C splits over $Z(H)$ and therefore $C = Z(H)$, since G is torsion-free. Thus

$G/Z(H)$ is an $\mathfrak{S}_1$-group with no quasicyclic subgroups. Now if H is abelian, we are done. If H is not abelian, then $H/Z(H)$ is a nilpotent divisible normal subgroup of $G/Z(H)$ and so by induction on the class of H we can complete the proof. $\square$

For the proof of Theorem 2′ we need the following lemma of Casolo. I am grateful to him for allowing me to include it here.

LEMMA 8 (C. Casolo). *Let A be a normal abelian divisible subgroup of a group G. If A has finite rank n, then $A \leq w_n(G)$.*

PROOF. Let us show first that if D is a normal abelian divisible subgroup of finite rank of a group G such that no non-trivial proper divisible subgroup of D is normal in G, then $D \leq w(G)$. Let H be a subnormal subgroup of G. To prove that $D \leq N_G(H)$ we proceed by induction on the defect d of H in G. If $d = 1$, then H is normal in G and there is nothing to prove. Let $d > 1$ and set $K = H^G$. If $[K, D] = 1$, then D centralizes H. Otherwise, $[K, D]$ is a non-trivial normal divisible subgroup of G contained in D, whence by our assumption $K \geq [K, D] = D$. Now D has no non-trivial proper characteristic divisible subgroups and so it is the direct product of finitely many copies either of $\mathbb{Q}$ or of the Prüfer group C_{p^∞}, for some prime p. Hence there exists a divisible subgroup D_1 of D, minimal with the property of being non-trivial and normal in K. The defect of H in K is $d - 1$, and so by inductive hypothesis D_1 as well as all its conjugates in G normalizes H. Hence D_1^G normalizes H and $D_1^G = D$, since it is a non-trivial divisible normal subgroup of G contained in D.

Now let us assume that A is the direct product of finitely many copies either of $\mathbb{Q}$ or of C_{p^∞} for some prime p, and proceed by induction on the rank n of A. If $n = 0$, then $A = 1$ and we are done. Let $n \geq 1$ and take $D \leq A$ to be a minimal non-trivial divisible normal subgroup of G. Then by the previous case $D \leq w(G)$. Moreover, A/D has rank at most $n - 1$, hence $A/D \leq w_{n-1}(G/D)$ by the inductive hypothesis. Therefore, $A \leq w_n(G)$.

To consider the general case write $A = R \times \mathrm{Dr}_{p\in\pi} T_p$, where π is the set of all prime divisors of the orders of the elements of A, T_p is the direct product of n_p copies of C_{p^∞}, R is the direct product of n_0 copies of $\mathbb{Q}$, and $n = n_0 + \max\{n_p \mid p \in \pi\}$. Now T_p is normal in G for each $p \in \pi$. Thus $T_p \leq w_{n_p}(G)$ by the above and $T = \mathrm{Dr}_{p\in\pi} T_p \leq w_k(G)$, where $k = \max\{n_p \mid p \in \pi\}$. Similarly, $A/T \leq w_{n_0}(G/T)$. Hence $A \leq w_{k+n_0}(G) = w_n(G)$ as wanted. $\square$

PROOF OF THEOREM 2′. Let G be an $\mathfrak{S}_1$-group and let D be its finite residual. By [15, Theorem 9.31], D is the maximal nilpotent divisible subgroup of G. Suppose first that G is (nilpotent divisible)-by-nilpotent-by-finite. Then G/D is nilpotent-by-finite. By [15, Theorem 9.23], each factor of the upper central series of D is abelian divisible of finite rank; thus by Lemma 8, $D \leq w_k(G)$, for some $k \geq 1$, and $G/w_k(G) \in \mathscr{W}^*$ by Theorem 1. Hence $G \in \mathscr{W}^* \subseteq \mathscr{W}_\chi$. This shows that (iii) implies (ii) and (i).

To prove that condition (i) implies (iii), assume that $G \in \mathscr{W}_\chi$. If R is the subgroup generated by the quasicyclic subgroups of G, then $R \leq D$ and by [15, Theorem 10.33], G/R is an $\mathfrak{S}_1$-group with no quasicyclic subgroups. Hence by Lemma 7.ii, G/D is a residually finite $\mathfrak{S}_1$-group. Then [15, Theorem 9.39.3] yields that G/D has a characteristic completely infinite subgroup of finite index, say T/D. Since $\mathscr{W}_\chi$ is closed for characteristic subgroups and factor groups by characteristic subgroups, we have that T/D is an $\mathfrak{S}_1$-group in $\mathscr{W}_\chi$. Therefore to prove (iii) it is enough to show that a residually finite completely infinite $\mathfrak{S}_1$-group $H \in \mathscr{W}_\chi$ is (nilpotent torsion-free)-by-finite.

To this end, let r denote the torsion-free rank of H (see [16, page 407]), and let us proceed by induction on r, the case $r = 0$ being trivial. Let $r > 0$, and let F be the Fitting subgroup of H. By [15, Theorem 10.33], F is nilpotent (torsion-free) and there exists a characteristic subgroup $A \geq F$ of finite index in H, such that A/F is abelian torsion-free. If $A = F$ we are done. If $A > F$, set $C = C_A(u_\chi(H) \cap Z(F))$ and note that $u_\chi(H) \cap Z(F) \neq 1$ since F is nilpotent and $u_\chi(H) \cap F$ is a non-trivial normal subgroup of F. Then, by Corollary 6, C has finite index in A, and by Lemma 7.i, $C/Z(C)$ is a residually finite $\mathfrak{S}_1$-group. As above, it follows that it has a characteristic completely infinite subgroup $C_1/Z(C)$ of finite index. Hence $C_1/Z(C)$ is a residually finite completely infinite $\mathfrak{S}_1$-group in $\mathscr{W}_\chi$, with torsion-free rank less than r, since $Z(C)$ is a non-trivial torsion-free group. Therefore, by induction, $C_1/Z(C)$ is (nilpotent torsion-free)-by-finite. Hence C_1, and then H, is (nilpotent torsion-free)-by-finite. □

PROOF OF THEOREM 3. Suppose that G is an $\mathfrak{S}_1$-group with finite Wielandt length. Then by Theorem 2′, G is (nilpotent divisible)-by-nilpotent-by-finite. Let D and F be as in the statement, and let $S \geq F$ be a subnormal subgroup of G. Then the defect of the subnormal subgroups of each nilpotent quotient of S/D is bounded by the Wielandt length of G. Since, by a theorem of Roseblade [18], a group in which each subgroup is subnormal with defect at most s is nilpotent with class bounded by a function of s, it follows that there is a bound on the class of the nilpotent quotients of S/D and hence $(S/D)/\gamma_\infty(S/D)$ is nilpotent.

Conversely, if G is as in the statement, then it has finite m-Wielandt length for each $m \geq 1$, by Theorem 2′. Therefore, in order to show that G has finite Wielandt length it is sufficient to prove that there is a bound on the defects of the subnormal subgroups of G. Let R be a subnormal subgroup of G. As in the proof of Theorem 2′, we have that $D \leq w_k(G)$, for some $k \in \mathbb{N}$, and thus the defect of R in RD is at most k. Moreover, RF is a subnormal subgroup of G and by Lemma 1, $RD/D \geq \gamma_\infty(RF/D)$. Now

https://doi.org/10.1017/S1446788700002299 Published online by Cambridge University Press

since by hypothesis $(RF/D)/\gamma_\infty(RF/D)$ is nilpotent and G/F is finite, there exists $h \geq 1$ such that $\gamma_\infty(RF/D) = \gamma_h(RF/D)$ for each subnormal subgroup R of G. Therefore, the defect of RD in RF is at most h. Hence, if $n = |G/F|$, we can deduce that the defect of R in G is at most $k + h + n$. $\square$

Note that $\mathfrak{S}_1$-groups with finite Wielandt length need not to be (nilpotent divisible)-by-finite-by-nilpotent, as follows from consideration of the T-group $G = A \rtimes \langle x \rangle$, where A is the group of rational numbers with denominators powers of 2 and x is an element of order 2 mapping each element of A into its inverse.

The following example shows that Theorem 2 and Theorem 3 cannot be extended to soluble groups of finite rank.

By a theorem of Dirichlet [7, Theorem 15] there exists an infinite family of distinct primes $\{p_n, q_n\}_{n\in\mathbb{N}}$ such that, for each $n \geq 0$, q_n divides $p_n - 1$. For each $n \geq 0$, let C_n be a cyclic group of order p_n and let A_n be a group of automorphisms of C_n of order q_n. Set $G_n = C_n \rtimes A_n$ and $G = \mathrm{Dr}_{n\in\mathbb{N}}\, G_n$. It is easy to see that G is a residually finite soluble T-group of rank 2, but it is not nilpotent-by-finite.

I wish to thank Prof. C. Casolo for making his unpublished results about this topic available to me, and Prof. F. Napolitani for many valuable suggestions and discussions.

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