Adv. Appl. Prob. **42,** 1013–1027 (2010)
Printed in Northern Ireland

ON ZERO-TRUNCATING AND MIXING POISSON DISTRIBUTIONS

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Abstract

The distributions that result from zero-truncating mixed Poisson (ZTMP) distributions and those obtained from mixing zero-truncated Poisson (MZTP) distributions are characterised based on their probability generating functions. One consequence is that every ZTMP distribution is an MZTP distribution, but not vice versa. These characterisations also indicate that the size-biased version of a Poisson mixture and, under certain regularity conditions, the shifted version of a Poisson mixture are neither ZTMP distributions nor MZTP distributions.

Keywords: Count variable; Poisson mixture; zero-truncated distribution; probability generating function; size-biased version; shifted version

2010 Mathematics Subject Classification: Primary 60E05
Secondary 60E10

1. Introduction

Nonnegative integer (count) data are frequently overdispersed, which means that the variability is larger than that expected under the Poisson model. This overdispersion is often a consequence of a lack of homogeneity of the sampled population, and can be modelled through a two-stage process in which the distribution of each count would be Poisson but with an expected value that changes randomly from count to count. By modelling the distribution of that expectation, one is naturally led to the use of Poisson mixture models.

Poisson mixture models frequently used in practice include finite mixture models in which the mixing distribution has finite support, as well as infinite mixture models, such as the negative binomial model, the generalized inverse Gaussian–Poisson model, or the Tweedie–Poisson model. The literature dealing with these models and their applications goes back more than fifty years and is far too extensive to be covered adequately in this introduction.

Apart from being overdispersed, count data often have a proportion of zeros that is larger than that expected from the initial model, which leads to the need for *zero-modified* models. There are also many instances where the value zero cannot be observed as a consequence of the method of ascertainment, which leads to the need for *zero truncated* count models.

Received 5 October 2009; revision received 16 August 2010.
Research partially supported by the Spanish Ministry of Education and Science and FEDER grants TIN2009-14560-C03-03, MTM2006-09920 and MTM2006-01477, and Generalitat de Catalunya through grant SGR-1187.
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When overdispersion and zero truncation appear together, it is most natural to resort to zero truncated mixed Poisson (ZTMP) models. The characterisation of the ZTMP models is very relevant, because these models mimic the mechanistic way in which positive integered data, such as word or species frequency count data, are typically generated. In these instances, using mixed Poisson models arises naturally because different words or species have different frequencies of appearance, and the mixing distribution can be interpreted as the word or species frequency distribution. Conditioning on the zeros not being observed, and hence truncating zero, arises naturally from the fact that most often we do not know the total number of different words in the vocabulary or species in the ecosystem. Using ZTMP models in these instances is what allows us to estimate the mixing distribution and the probability at zero of the corresponding untruncated Poisson mixture model, which are features that help characterise the vocabulary or the population. Among many other references, in [2], [5], [11], and [19] the ZTMP models are used to fit species frequency count data in ecology, and in [1], [8], [14], [17], and [18] they are used in word frequency count data in stylometry and in bibliometry data.

Mixed zero-truncated Poisson (MZTP) distributions are also distributions of random variables (RVs) on the strictly positive integers that are more dispersed than expected from the zero-truncated Poisson distributions.

The main goal of this paper is to characterise the ZTMP distributions and the MZTP distributions by means of their probability generating functions (PGFs), and to provide tools to help recognize whether or not a distribution belongs to any of these two families. Recognizing which models are ZTMP and which are not is especially relevant when one intends to use them on data generated through ZTMP mechanisms.

Böhning and Kuhnert [3] proved that when the mixing distribution is finite, the set of ZTMP distributions and the set of MZTP distributions are equivalent. Here it is found that when the mixing distribution is allowed to have infinite support, these two sets of distributions are no longer equivalent. From our characterisations, it also follows that the size-biased version of a mixed Poisson distribution and, under a certain regularity condition, the shifted version of a mixed Poisson distribution are neither ZTMP distributions nor MZTP distributions. Instead, an alternative transformation of mixed Poisson distributions into strictly positive count distributions that are guaranteed to be MZTP, and which, under regularity conditions, are also guaranteed to be ZTMP, is proposed.

This paper is organized as follows. In Section 2 we provide definitions and present a known characterisation of mixed Poisson distributions. In Section 3 we characterise the PGFs of ZTMP models, and in Section 4 we characterise the PGFs of MZTP models. In Section 5 we compare these two sets of models based on the characterisations obtained, and in Section 6 we present some consequences of these characterisations. For an application of these characterisations to help describe a natural extension of the zero truncated Tweedie–Poisson mixture model, see [20].

2. Background and definitions

2.1. Definitions

The Poisson distribution of parameter λ is the distribution of the number of occurrences of a random event in a given interval of time or a given area, under the assumptions that the events take place independently, and that the probability of one event occurring over a short time interval is λ times the length of the interval. These hypotheses are very often too restrictive to hold in practice.

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A common way to adapt the Poisson model in order to fit real data is to assume that its mean parameter, λ, is a random variable instead of an unknown constant, which leads to what is known as a *mixed Poisson distribution*. Let us assume that, for a given constant t, $\lambda = \Gamma t$, where Γ is a nonnegative RV with mean equal to 1 and distribution function U. Following Grandell's notation [7, p. 13], we say that a count RV, M, is *mixed Poisson distributed* with structure distribution U, $\mathrm{MP}(t, U)$, if and only if

$$\mathrm{P}(M = k) = \int_0^{+\infty} \frac{(lt)^k}{k!} \mathrm{e}^{-lt} \,\mathrm{d}U(l) \quad \text{for } k = 0, 1, 2, \ldots.$$

Note that if $M \sim \mathrm{MP}(t, U)$ then $M|\Gamma$ follows a Poisson distribution of parameter Γt.

The PGF of a count RV N with $\mathrm{P}(N = k) = p_k$, is defined as

$$h(s) = \mathrm{E}[s^N] = \sum_{k=0}^{+\infty} p_k s^k$$

for any s in the series convergence domain. The PGF of an RV depends only on its probability distribution, and two RVs have the same PGF if and only if they have the same probability distribution. Observe that any PGF is a power series verifying that $h(1) = 1$, $h(0) = p_0$, and $p_k = h^{(k)}(0)/k!$, where $h^{(k)}(s)$ denotes the kth derivative of $h(s)$, and that if the kth factorial moment of the corresponding RV exists then it is equal to $h^{(k)}(1)$.

The PGF of a Poisson distribution with parameter λ is $h(s) = \exp(\lambda(s - 1))$. If M follows an $\mathrm{MP}(t, U)$ and we denote by $\hat{u}(v)$ the Laplace transform of Γ,

$$\hat{u}(v) = \int_0^{+\infty} \mathrm{e}^{-lv} \,\mathrm{d}U(l), \tag{1}$$

the PGF of M is

$$h_M(s) = \hat{u}(t(1 - s)). \tag{2}$$

A function whose derivatives, $f^{(n)}(s)$, exist in an interval (a, b) is said to be *absolutely monotone* in (a, b) if and only if

$$f^{(n)}(s) \geq 0 \quad \text{for all } s \in (a, b) \text{ and } n = 0, 1, 2, \ldots, \tag{3}$$

and it is said to be *strictly absolutely monotone* if it satisfies (3) after replacing '$\geq$' by '$>$'. Furthermore, a function $f(s)$ is a PGF if and only if it is absolutely monotone in $(0, 1)$ and $f(1) = 1$ (see, e.g. [6, pp. 223, 439]).

2.2. Mixed Poisson characterisation

Puri and Goldie [15] provided a characterisation of a mixed Poisson distribution based on its PGF, which also appears in [7, Proposition 2.2, p. 24]. This result states that a nondegenerated count RV M with PGF $h_M(s)$ is mixed Poisson distributed if and only if $h_M(s)$ is absolutely monotone in $(-\infty, 1)$. A given function is called *analytical* in the neighbourhood of $x = a$ if an expansion of $f(x)$ as a convergent power series in $(x - a)$ is possible in this neighbourhood [4]. Given that a PGF is analytical, we could replace the absolute monotonicity condition by the strict absolute monotonicity condition in this characterisation, as long as we exclude from consideration the degenerate distribution at 0, as is done in the rest of the paper. Hence, this result can be reformulated as follows.

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Proposition 1. *A function $h_M(s)$ is the PGF of an RV M with a mixed Poisson distribution different from the degenerate distribution at* 0 *if and only if*

(a) $h_M(1) = 1$;

(b) *it is analytical in* $(-\infty, 1)$;

(c) *all the coefficients of the series expansion of $h_M(s)$ about any point s_0 in $(-\infty, 1)$ are strictly positive, or, equivalently, $h_M(s)$ is strictly absolutely monotone in* $(-\infty, 1)$.

Remark 1. Equations (1) and (2) imply that the PGF of a Poisson mixture, $h_M(s)$, is nonnegative. As a consequence, the limit of $h_M(s)$ at $-\infty$ is always nonnegative. In the case where the limit is strictly positive, the RV may be seen as a mixture of an RV degenerate at 0 and a mixed Poisson distribution with zero limit at $-\infty$. Hence, the PGF of a mixed Poisson distribution either has limit 0 at $-\infty$ or it is equal to a constant plus a PGF with limit 0 at $-\infty$.

The PGF of the negative binomial distribution is

$$h(s) = \left(\frac{1-\theta}{1-\theta s}\right)^{\alpha}, \quad 0 < \theta < 1,\ \alpha > 0, \tag{4}$$

the PGF of the logarithmic series distribution is

$$h(s) = \frac{\ln(1-\theta s)}{\ln(1-\theta)}, \quad 0 < \theta < 1,$$

and the PGF of the Hermite distribution is

$$h(s) = \mathrm{e}^{\alpha(s-1)+\beta(s^2-1)}, \quad \alpha, \beta > 0.$$

In Figure 1 we present these three functions for a particular set of values for θ, α, and β. Given that the first derivative of the PGF of the Hermite distribution takes the value 0 at a point $s_0 \in (-\infty, 1)$, and that the PGF of the logarithmic series distribution tends to $-\infty$ when s tends to $-\infty$, these two models cannot be mixed Poisson distributions.

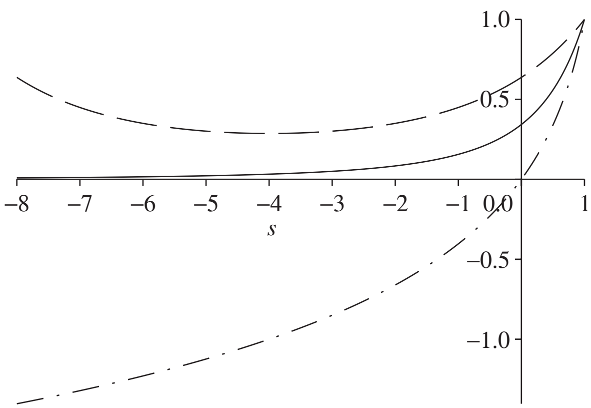

FIGURE 1: Probability generating functions of the negative binomial distribution (*solid line*) with $\theta = 0.3$ and $\alpha = 3$, the logarithmic series distribution (*dash–dot line*) with $\theta = 0.75$, and the Hermite distribution (*dashed line*) with $\alpha = 0.5$ and $\beta = \frac{1}{20}$.

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3. ZTMP distribution

The *zero-truncated version* of a count variable M, also known as its *positive version*, is the RV with probability at any given $k \geq 1$ equal to that which results from conditioning on the fact that the original RV does not take the zero value (see, e.g. [9, pp. 181–184]).

In particular, if M is mixed Poisson distributed with a PGF as in (2) then its zero-truncated version, N, which is a ZTMP, has a probability mass function equal to

$$\mathrm{P}(N = k) = \frac{1}{1 - \hat{u}(t)} \int_0^{+\infty} \frac{(lt)^k}{k!} \mathrm{e}^{-lt}\, \mathrm{d}U(l) \quad \text{for } k = 1, 2, 3, \ldots,$$

where $\hat{u}(t)$ is defined in (1) and corresponds to the probability at 0 of M. The PGF of N is equal to

$$h_N(s) = \frac{h_M(s) - h_M(0)}{1 - h_M(0)} = \frac{\hat{u}(t(1-s)) - \hat{u}(t)}{1 - \hat{u}(t)}. \tag{5}$$

The following theorem characterises the PGF of a ZTMP distribution.

Theorem 1. *A function $h_N(s)$ is the PGF of a ZTMP distribution with finite mean if and only if*

(a) $h_N(0) = 0$*,* $h_N(1) = 1$*, and* $h'_N(1) < +\infty$*;*

(b) *it is analytical in* $(-\infty, 1)$*;*

(c) *all the coefficients of the series expansion of $h_N(s)$ about any point s_0 in $(-\infty, 1)$ are strictly positive except the constant term that may be negative or zero, or, equivalently, the first derivative of $h_N(s)$ is strictly absolutely monotone in $(-\infty, 1)$;*

(d) $\lim_{t\to-\infty} h_N(s) = -L$*, with L being a finite strictly positive number.*

Proof. Assume first that $h_N(s)$ is the PGF of a zero-truncated version of a mixed Poisson distribution, as in (5). It is clear that $h_N(0) = 0$, $h_N(1) = 1$, and that $h'_N(1) = h'_M(1)/(1 - h_M(0))$, which is finite because it is equal to a constant times the mean of the Poisson mixture. That $h_N(s)$ is analytical in $(-\infty, 1)$ is a consequence of the fact that the PGF of any Poisson mixture is analytical in this interval. The constant term of the series expansion of $h_N(s)$ about $s_0 \in (-\infty, 1)$ is

$$h_N(s_0) = \frac{\hat{u}(t(1-s_0)) - \hat{u}(t)}{1 - \hat{u}(t)},$$

which is positive if $s_0 \in (-\infty, 0)$, zero if $s_0 = 0$, and negative otherwise, since $\hat{u}(v)$ is a decreasing function. Given that if $n \geq 1$ then $h_N^{(n)}(s_0) = h_M^{(n)}(s_0)$, the rest of the coefficients of the series expansion of $h_N(s)$ at any point in $(-\infty, 1)$ must be positive because of Proposition 1. Finally,

$$\lim_{s\to-\infty} h_N(s) = \frac{\lim_{s\to-\infty} h_M(s) - \hat{u}(t)}{1 - \hat{u}(t)} = \frac{-\hat{u}(t)}{1 - \hat{u}(t)} = -L,$$

where L is a finite and strictly positive real number.

To prove the reverse implication, let $h_N(s)$ be a function verifying the four conditions of the theorem and define

$$\tilde{h}_N(s) = \frac{h_N(s) - L}{1 - L}.$$

Clearly, $\tilde{h}_N(1) = 1$, and given that $h_N(s)$ is analytical in $(-\infty, 1)$, $\tilde{h}_N(s)$ is also analytical in $(-\infty, 1)$. Furthermore, $h_N(s)$ is an increasing function and, therefore, $\tilde{h}_N(s_0)$ is strictly

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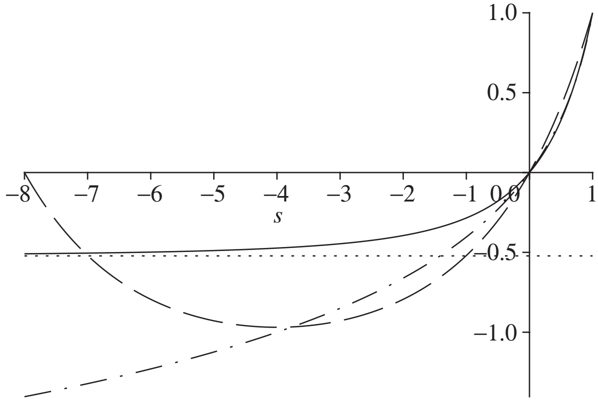

FIGURE 2: Probability generating functions of the zero-truncated versions of the negative binomial distribution (*solid line*) with $\theta = 0.3$ and $\alpha = 3$, the logarithmic series distribution (*dash–dot line*) with $\theta = 0.75$, and the Hermite distribution (*dashed line*) with $\alpha = 0.5$ and $\beta = \frac{1}{20}$. The dotted line corresponds to the limit at $-\infty$ of the PGF of the negative binomial distribution.

positive for any s_0 in $(-\infty, 1)$. Given that $h'_N(s)$ is strictly absolutely monotone, the derivatives of $\tilde{h}_N(s)$ at s_0,

$$\tilde{h}_N^{(n)}(s_0) = \frac{h_N^{(n)}(s_0)}{1 - L},$$

are also strictly positive for $n = 1, 2, \ldots$. As a consequence, $\tilde{h}_N(s)$ is a function that verifies the three conditions of Proposition 1 and, therefore, it is the PGF of a mixed Poisson distribution. For this reason, if $\hat{u}(t)$ is the Laplace transform of the corresponding mixing distribution, the positive version of that mixed Poisson distribution will have a PGF equal to

$$\frac{\tilde{h_N}(s) - \hat{u}(t)}{1 - \hat{u}(t)} = \frac{\tilde{h}_N(s) - \tilde{h}_N(0)}{1 - \tilde{h}_N(0)} = \frac{h_N(s) - h_N(0)}{1 - h_N(0)} = h_N(s).$$

Hence, $h_N(s)$ is the PGF of the positive version of a mixed Poisson distribution. This completes the proof.

From this result, it follows that the PGF of the truncation at 0 of any mixed Poisson distribution is increasing, takes the value 0 at 0, takes negative values on the negative real line, and has a limit at $-\infty$ that must be negative and finite.

In Figure 2 we present the PGF of the positive versions of the negative binomial, the logarithmic series, and the Hermite distributions presented in Figure 1. As can be seen, the last two distributions cannot be a ZTMP distribution, because the limit at $-\infty$ of the Hermite PGF is positive and the limit for the logarithmic series distribution is $-\infty$. In the logarithmic series case, the truncated and untruncated versions are the same because this distribution takes only strictly positive values.

4. MZTP distributions

If we assume that the parameter λ of a zero-truncated Poisson distribution is such that $\lambda = t\Gamma$, where Γ is a nonnegative RV with mean equal to 1 and distribution function equal

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to U, the resulting RV, N, follows a mixture of zero-truncated Poisson (MZTP) distributions with probability mass function equal to

$$\mathrm{P}(N = k) = \int_0^{+\infty} \frac{(lt)^k \mathrm{e}^{-lt}}{k!} \frac{1}{1 - \mathrm{e}^{-lt}} \,\mathrm{d}U(l) \quad \text{for } k = 1, 2, \ldots$$

and with PGF equal to

$$h_N(s) = \int_0^{+\infty} \frac{\mathrm{e}^{-tl(1-s)} - \mathrm{e}^{-tl}}{1 - \mathrm{e}^{-tl}} \,\mathrm{d}U(l) = \int_0^{+\infty} \frac{\mathrm{e}^{tls} - 1}{\mathrm{e}^{tl} - 1} \,\mathrm{d}U(l). \tag{6}$$

The next theorem characterises the MZTP distributions by means of their PGFs. The characterisation is analogous to the one established in Theorem 1, with the exception of condition (d), which is no longer required since the PGF at $-\infty$ may have an infinite limit.

Theorem 2. *A function $h_N(s)$ is the PGF of an MZTP distribution with finite mean if and only if*

(a) $h_N(0) = 0$, $h_N(1) = 1$, *and* $h'_N(1) < +\infty$*;*

(b) *it is analytical in* $(-\infty, 1)$*;*

(c) *all the coefficients of the series expansion of $h_N(s)$ about any point s_0 in $(-\infty, 1)$ are strictly positive except the constant term that may be negative or zero, or, equivalently, the first derivative of $h_N(s)$ is strictly absolutely monotone in* $(-\infty, 1)$.

Proof. Assume first that $h_N(s)$ is the PGF of a mixture of a zero-truncated Poisson distribution as in (6). It is easy to check that $h_N(0) = 0$, $h_N(1) = 1$, and that $h'_N(1) < +\infty$, because it is assumed that N has a finite mean. Moreover,

$$h_N(s_0) = \int_0^{+\infty} \frac{\mathrm{e}^{tls_0} - 1}{\mathrm{e}^{tl} - 1} \,\mathrm{d}U(l) \begin{cases} > 0 & \text{if } s_0 > 0, \\ = 0 & \text{if } s_0 = 0, \\ < 0 & \text{if } s_0 < 0, \end{cases}$$

and, thus, its domain contains $(-\infty, 1)$ and it is analytical in that domain. The nth derivative of $h_N(s)$ is equal to

$$h_N^{(n)}(s_0) = \int_0^{+\infty} \frac{\mathrm{e}^{tls_0}}{\mathrm{e}^{tl} - 1} (tl)^n \,\mathrm{d}U(l), \qquad n = 1, 2, \ldots,$$

and, thus, $h_N^{(n)}(s)$ is always positive, because it is the integral of a positive function. As a consequence, all the coefficients of its series expansion are positive with the exception of the constant term that is negative for negative values of s_0. Finally, note that

$$\lim_{s \to -\infty} h_N(s) = \int_0^{+\infty} \frac{1}{1 - \mathrm{e}^{tl}} \,\mathrm{d}U(l),$$

which is finite if the expected value of $(1 - \mathrm{e}^{t\Gamma})^{-1}$ exists, and is infinite otherwise.

In order to prove that conditions (a)–(c) are sufficient, two cases must be distinguished, depending on whether or not the variance of N is finite.

Case 1: $h_N^{(2)}(1) < +\infty$. Let us define $\mu = h'_N(1)$. It is easy to see that the function $\tilde{h}_N(s) = h'_N(s)/\mu$ verifies the conditions of Proposition 1 and, hence, it is the PGF of a mixed

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Poisson distributed RV. As a consequence, $\tilde{h}_N(s) = \hat{u}(t(1-s))$ for a given Laplace transform $\hat{u}$. Given that $h'_N(s) = \mu\hat{u}(t(1-s))$,

$$\begin{aligned} h_N(s) &= \int_0^s \mu\hat{u}(t(1-x))\,\mathrm{d}x \\ &= \int_0^s \bigg(\int_0^{+\infty} \mu\mathrm{e}^{-tl(1-x)}\,\mathrm{d}U(l)\bigg)\,\mathrm{d}x \\ &= \int_0^{+\infty} \bigg(\int_0^s \mu\mathrm{e}^{-lt(1-x)}\,\mathrm{d}x\bigg) U(\mathrm{d}l) \\ &= \int_0^{+\infty} \mu\frac{\mathrm{e}^{-lt(1-s)} - \mathrm{e}^{-lt}}{tl}\,\mathrm{d}U(l) \\ &= \int_0^{+\infty} \frac{\mathrm{e}^{lts}-1}{\mathrm{e}^{lt}-1}\mu\frac{1-\mathrm{e}^{-lt}}{tl}\,\mathrm{d}U(l). \end{aligned} \tag{7}$$

Given that $h_N(1) = 1$ and (7),

$$\int_0^{+\infty} \frac{1-\mathrm{e}^{-lt}}{tl}\mu\,\mathrm{d}U(l) = 1,$$

and $h_N(s)$ is the PGF of a mixture of a zero-truncated Poisson distribution, as in (6) with

$$\frac{\mu}{tl}(1-\mathrm{e}^{-lt})U'(l)$$

as the density function of Γ.

Case 2: $h_N^{(2)}(1)$ is not finite. For a given value of $a \in (0, 1)$, let

$$h_a(s) = \frac{h_N(as)}{h_N(a)}.$$

In what follows it is checked that $h_a(s)$ verifies the three conditions of the theorem and that $h_a^{(2)}(1) < +\infty$. Clearly, $h_a(0) = h_N(0) = 0$, $h_a(1) = 1$, and

$$\mu_a = h'_a(1) = \frac{ah'_N(a)}{h_N(a)} < +\infty$$

since $h'_N(s)$ is a strictly increasing function. Given that $h_N(s)$ verifies condition (b), $h_a(s)$ is analytical in $(-\infty, 1/a) \supseteq (-\infty, 1)$. In particular, it is analytical in a neighbourhood of 1 and, consequently, $h_a^{(2)}(1) < +\infty$. For any $s_0 \in (-\infty, 1)$, the coefficients of the series expansion of $h_a(s_0)$ have the same sign as the coefficients of $h_N(s)$ and, therefore, condition (c) holds. As a consequence, for any $a \in (0, 1)$, $h_a(s)$ is the PGF of an RV Y_a with a distribution that is a mixture of the zero-truncated Poisson. Therefore, for any $a \in (0, 1)$, there exists a distribution function U_a such that

$$h_a(s) = \int_0^{+\infty} \frac{\mathrm{e}^{lts-1} - \mathrm{e}^{-lt}}{\mathrm{e}^{lt}-1}\,\mathrm{d}U_a(l).$$

Taking into account the definition of $h_a(s)$ and letting $r = as$, we have

$$h_N(r) = \int_0^{+\infty} h_N(a)\frac{\mathrm{e}^{ltr/a}-1}{\mathrm{e}^{lt}-1}\,\mathrm{d}U_a(l),$$

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which, by applying the change of variable $l = ap$, becomes

$$\begin{aligned} h_N(r) &= \int_0^{+\infty} h_N(a) \frac{\mathrm{e}^{ptr} - 1}{\mathrm{e}^{pta} - 1} a \,\mathrm{d}U_a(ap) \\ &= \int_0^{+\infty} \frac{\mathrm{e}^{ptr} - 1}{\mathrm{e}^{pt} - 1} \frac{\mathrm{e}^{pt} - 1}{\mathrm{e}^{pta} - 1} a h_N(a) \,\mathrm{d}U_a(ap) \\ &= \int_0^{+\infty} \frac{\mathrm{e}^{pt(r-1)} - \mathrm{e}^{-pt}}{1 - \mathrm{e}^{-pt}} \,\mathrm{d}U(p), \end{aligned}$$

where

$$\mathrm{d}U(p) = \frac{\mathrm{e}^{pt} - 1}{\mathrm{e}^{pta} - 1} a h_N(a) \,\mathrm{d}U_a(ap) \geq 0.$$

Note that $\mathrm{d}U(p)$ does not depend on a since $h_N(r)$ does not depend on a. Given (6), we need only to prove that U is a probability distribution function on the positive real line, but this is a consequence of the fact that

$$h_N(1) = 1 = \int_0^{+\infty} \mathrm{d}U(p).$$

Hence, $h_N(s)$ is the PGF of a mixture of the zero-truncated Poisson distribution. This completes the proof.

The following result, which is a direct consequence of Theorem 2, provides extra tools to help recognize PGFs of MZTP distributions and will be used subsequently.

Corollary 1. (a) *A strictly positive count RV N with a finite mean, μ, and PGF $h_N(s)$ follows an MZTP distribution if and only if $h'_N(s)/\mu$ is the PGF of a mixed Poisson distribution.*

(b) *A real function $h(s)$ such that $h(0) = 0$, $h(1) = 1$, and $h'(1) < +\infty$ is the PGF of an MZTP distribution with finite mean if and only if $h'(s)/h'(1)$ is the PGF of a mixed Poisson.*

5. The class of ZTMP distributions as a subset of the class of MZTP distributions

If we just consider finite mixing distributions, Böhning and Kuhnert [3] proved that the class of MZTP distributions and the class of ZTMP distributions are the same. A consequence of Theorems 1 and 2 is that this does not hold in general, when the mixing distributions are allowed to have infinite support, because we can find distributions that are MZTP but not ZTMP.

Given that a PGF cannot oscillate when s tends to ∞, the only difference between the PGFs of ZTMP distributions and the PGFs of MZTP distributions is that, in this latter case, the value $-\infty$ is accepted as the limit of these functions at $-\infty$.

Corollary 2. *Any ZTMP distribution is also an MZTP distribution.*

Corollary 3. *An MZTP distribution is a ZTMP distribution if and only if its PGF has a finite limit at $-\infty$.*

Figure 2 indicates that the zero-truncated negative binomial and logarithmic series distributions are MZTP distributions but that the zero-truncated Hermite distribution is not. In the following example we have a set of PGFs that correspond to the extended truncated negative binomial model of [5]. In part of the parameter space of that model it is both a ZTMP and an MZTP distribution, as described in [20], while in the rest of the parameter space, is an MZTP but not a ZTMP distribution.

Example 1. Let $\theta \in (0, 1)$, and consider the following set of PGF functions:

$$h(s) = \begin{cases} \dfrac{1-(1-\theta s)^{1-\alpha}}{1-(1-\theta)^{1-\alpha}} & \text{with } \alpha > 0 \text{ and } \alpha \neq 1, \\ \dfrac{\ln(1-\theta s)}{\ln(1-\theta)} & \text{with } \alpha = 1. \end{cases} \tag{8}$$

Given that

$$\frac{h'(s)}{h'(1)} = \left(\frac{1-\theta}{1-\theta s}\right)^{\alpha}$$

is the PGF of a negative binomial with parameters α and θ, by applying Corollary 1, we find that, for any $\theta \in (0, 1)$ and any $\alpha \in (0, \infty)$, $h(s)$ is the PGF of an MZTP distribution. Given that

$$\lim_{s\to-\infty} h(s) = \begin{cases} -\infty & \text{when } \alpha \in (0, 1], \\ \dfrac{1}{1-(1-\theta)^{1-\alpha}} & \text{when } \alpha \in (1, \infty), \end{cases}$$

when $\alpha \in (0, 1]$, $h(s)$ is not the PGF of a ZTMP distribution owing to Corollary 3.

6. Some other consequences of the characterisations

Here three alternatives to the zero truncation for transforming a mixed Poisson distribution into a strictly positive distribution are explored. From the characterisations obtained, it is proved that the size-biased version of a Poisson mixture considered in Subsection 6.2 and, under regularity conditions, the shifted version of a Poisson mixture considered in Subsection 6.1 are neither ZTMP distributions nor MZTP distributions. In particular, given that they are not ZTMP distributions, neither of these two transformations are convenient when we intend to use the resulting model on data generated through a ZTMP mechanism, because it does not allow us to estimate the mixing distribution or the probability at 0 of the corresponding untruncated mixed Poisson model, which is typically of interest.

Instead, an alternative transformation of a Poisson mixture into a strictly positive count distribution guaranteed to be MZTP and, under regularity conditions, guaranteed to be ZTMP is proposed in Subsection 6.3.

6.1. Shifted version of a mixed Poisson distribution

The shifted version is the simplest way to transform a count RV into a strictly positive count RV. The definition is as follows. Given any count RV M, we define the shifted version of M to be $N = M + 1$, and, hence,

$$\mathrm{P}(N = k) = \mathrm{P}(M = k - 1) \quad \text{for all } k \geq 1.$$

If $h_M(s)$ is the PGF of M, the PGF of its shifted version is $h_N(s) = s h_M(s)$.

Proposition 2. *Let N be the shifted version of a mixed Poisson distribution M with PGF $h_M(s)$, and denote by L the limit of $h_M(s)$ at $-\infty$. If*

$$\lim_{s\to-\infty} s(h_M(s) - L) = 0 \tag{9}$$

holds then the distribution of N is neither a ZTMP nor an MZTP.

Proof. The function $h_N(s) = sh_M(s)$ verifies conditions (a) and (b) of Theorems 1 and 2. Therefore, to prove this result, it is enough to check that if (9) holds, $h_N(s)$ does not satisfy condition (c) of Theorems 1 and 2. Given that M is mixed Poisson distributed, from (2) we have $L \in [0, 1)$.

If $L = 0$ we have

$$\lim_{s\to-\infty} h_N(s) = \lim_{s\to\infty} sh_M(s) = 0,$$

and the mean value theorem indicates that there exists an $s_0 < 0$ for which $h'_N(s_0) = 0$ and, thus, condition (c) is not verified.

If $L > 0$, M can be written as $M = L + (1 - L)M^*$, where M^* is mixed Poisson distributed with $\lim_{s\to-\infty} h_{M^*}(s) = 0$. Moreover, given that

$$h_M(s) = L + (1 - L)h_{M^*}(s),$$

M verifies (9) if and only if M^* verifies the same condition with $L = 0$, and, by condition (a) of Theorems 1 and 2, there exists an $s_0 < 0$ such that $h^{(2)}_{M^*}(s_0) < 0$. Given that, for $n \geq 2$, the sign of $h^{(n)}_M(s)$ is equal to the sign of $h^{(n)}_{M^*}(s)$, it follows that condition (c) is not verified. This completes the proof.

In what follows it is illustrated by means of a particular example that the shifted version of a mixed Poisson distribution that does not verify condition (9) might be

(A) MZTP distributed but not ZTMP distributed;

(B) both ZTMP distributed and MZTP distributed; and

(C) neither ZTMP distributed nor MZTP distributed.

Example 2. Let M be a negative binomial distribution, which has PGF as in (4), with a zero limit at $-\infty$. Then the following assertions hold.

1. If $\alpha > 1$, condition (9) holds and the shifted version is neither a ZTMP nor an MZTP.
2. If $\alpha < 1$, condition (9) is not verified. Nevertheless, the first derivative of the PGF of the shifted distribution is

$$h'_N(s) = \frac{\alpha}{1-\theta}\left(\frac{1-\theta}{1-\theta s}\right)^{\alpha+1} + (1-\alpha)\left(\frac{1-\theta}{1-\theta s}\right)^{\alpha}, \tag{10}$$

   which verifies condition (c) of Theorem 2 because it is a linear combination of the PGFs of two negative binomial distributions with positive coefficients. As a consequence, the shifted version is MZTP distributed but it is not ZTMP distributed because the limit of its PGF at $-\infty$ is equal to $-\infty$ and, therefore, it constitutes an example of case (A).
3. If $\alpha = 1$, the limit at $-\infty$ of $h_M(s)$ is 0 and the limit of $h_N(s)$ is $(1 - \theta)/\theta$. Hence, condition (9) is not verified but condition (c) of Theorem 2 is true because the first derivative is obtained from (10) with $\alpha = 1$. Consequently, the shifted version has a ZTMP and an MZTP distribution and it constitutes an example of case (B).

To obtain an example of case (C), we consider an M with a distribution that is a mixture of two negative binomial distributions: one with $\alpha > 1$ and the other one with $\alpha \leq 1$. A mixture such as this does not verify (9), and its shifted version is neither a ZTMP nor an MZTP.

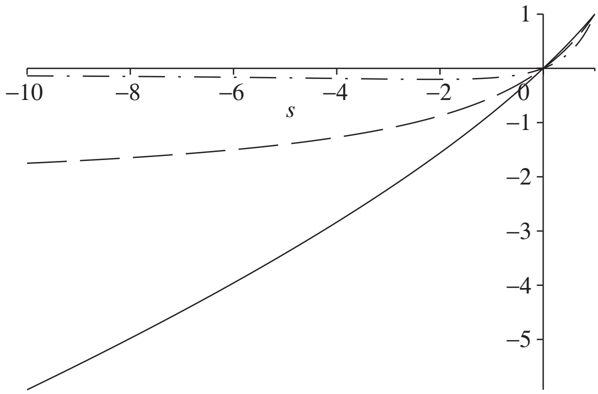

FIGURE 3: Probability generating functions for the shifted negative binomial with $\theta = 0.3$ and $\alpha = 0.3$ (*solid line*), the shifted negative binomial with $\theta = 0.3$ and $\alpha = 1$ (*dashed line*), and the shifted mixture of these two negative binomials with weights 0.95 and 0.05 (*dash–dot line*).

Figure 3 presents the PGFs of two shifted negative binomial distributions and the PGF of the distribution that results from shifting a mixture of these two negative binomials. None of the three un-shifted distributions verifies condition (9), and their shifted versions constitute an example of each one of the three possibilities listed above.

6.2. Size-biased version of a mixed Poisson distribution

The size-biased version of a discrete distribution is a special case of weighted distribution, defined in [16] as the sampling distribution when the probability to select an observation is proportional to the size of the observation; in particular, its probability at 0 is always 0. This corresponds, for example, to the sampling scheme used when we analyse uncommon contagious diseases or genetic characteristics; first we sample some families and then we select all the individuals in them. See, for example, [12] for an application on wildlife populations. Recently, in [10] the size-biased version of the negative binomial distribution was used as an alternative to the zero-truncated negative binomial. Here it was proved that size-biased versions of mixed Poisson distributions are not a good alternative when one intends to use them on data yielded through a ZTMP mechanism and needs to estimate the corresponding mixing distribution.

The size-biased version of a given count RV M with finite mean μ and PGF $h_M(s)$ is the weighted version of M with weight function $w(k) = k$, and, hence, probability mass function

$$\mathrm{P}(N = k) = \frac{k\,\mathrm{P}(M = k)}{\mu}$$

and PGF

$$h_N(s) = \frac{s h'_M(s)}{h'_M(1)}. \tag{11}$$

Note that N is a strictly positive RV and that if M is Poisson distributed, the shifted version coincides with the size-biased version.

Corollary 4. *The size-biased version of a mixed Poisson distribution with finite mean is neither ZTMP distributed nor MZTP distributed.*

Proof. Let M be mixed Poisson distributed, and let N be its size-biased version. By (11), N is the shifted version of an RV, M_0, with PGF $h_{M_0}(s) = h'_M(s)/h'_M(1)$. Given that M is mixed Poisson distributed, M_0 is also mixed Poisson distributed because its PGF verifies the conditions of Proposition 1. Assume that N is MZTP distributed and that

$$\lim_{s \to -\infty} h_N(s) \neq 0,$$

which is equivalent to

$$\lim_{s \to -\infty} s h'_M(s) \neq 0.$$

Given that $h_N(s)$ is a negative increasing function in $(-\infty, 0)$, there exists $s_0 < 0$ and $\alpha_0 < 0$ such that

$$s h'_M(s) < \alpha_0 \quad \Longleftrightarrow \quad h'_M(s) > \frac{\alpha_0}{s} \quad \text{for all } s,\ s < s_0;$$

observe that both sides of the inequality are positive in $(-\infty, s_0)$, and that integrating them between s and s_0 we obtain

$$h_M(s) < h_M(s_0) - \alpha_0 \ln(-s_0) + \alpha_0 \ln(-s).$$

Hence, $\lim_{s \to -\infty} h_M(s) = -\infty$, which contradicts the fact that M is mixed Poisson distributed. So, if N was MZTP distributed,

$$\begin{aligned}
\lim_{s \to -\infty} h_N(s) = 0 \quad &\Longleftrightarrow \quad \lim_{s \to -\infty} s h'_M(s) = 0 \\
&\Longrightarrow \quad \lim_{s \to -\infty} h'_M(s) = 0 \\
&\Longrightarrow \quad \lim_{s \to -\infty} h_{M_0}(s) = 0.
\end{aligned}$$

From the last expression we find that M_0 is mixed Poisson distributed with a PGF that verifies condition (9), and, hence, by Proposition 2, N cannot be MZTP distributed and, by Corollary 2, it cannot be ZTMP distributed either.

6.3. On the RV with a PGF obtained by normalising an integrated PGF

Given a count RV M with PGF $h_M(s)$, we can obtain a strictly positive RV by defining N to be the RV with PGF

$$h_N(s) = \frac{\int_0^s h_M(t)\,\mathrm{d}t}{\int_0^1 h_M(t)\,\mathrm{d}t}. \tag{12}$$

For simplicity, N is denoted here as the *integral transformation* of M. By definition, $h_N(0) = 0$ and, thus, the new RV takes the zero value with probability 0. Observe that the set of distributions considered in Example 1 corresponds to the integral transformation of the negative binomial model. The next result proves that, unlike the shifted and size-biased versions, the integral transformation of a mixed Poisson distribution is always an MZTP distribution.

Proposition 3. *If N is the integral transformation of a count RV M then*

(a) *N is MZTP distributed if and only if M is mixed Poisson distributed with a finite mean;*

(b) *N is ZTMP distributed if and only if M is mixed Poisson distributed and $\int_{-\infty}^0 h_M(t)\,\mathrm{d}t$ is finite.*

Proof. From (12), it follows that $h_N(s)$ verifies the conditions of Theorem 2 if and only if $h_M(s)$ verifies the conditions of Proposition 1, and, therefore, (a) holds. That (b) holds is a straight consequence of Corollary 3.

As pointed out in Example 1, the integral transformation of a negative binomial distribution with $\alpha \in (0, 1]$ constitutes an example of an integral transformation that is MZTP distributed but not ZTMP distributed.

7. Conclusions

In this paper the probability models obtained by zero-truncating mixed Poisson (ZTMP) and mixing zero-truncated Poisson (MZTP) distributions have been characterised by means of their PGFs. Recognizing which models are in the ZTMP class and which are not helps identify the models suitable for data generated through a ZTMP mechanism, where one is usually interested in the estimation of features associated with the mixing distribution.

One consequence of our characterisations is that all ZTMP distributions are MZTP distributions, but the opposite is not true. Furthermore, it turns out that if one needs to modify a mixed Poisson random variable in order to make it into a strictly positive count variable, its shifted version and its size-biased version may not be suitable options, because the resulting distributions are neither ZTMP distributions nor MZTP distributions. On the other hand, the integral transformation of a mixed Poisson RV is always MZTP distributed and, under a certain condition, it is also ZTMP distributed.

It might be of interest to extend these results for distributions that are mixtures of models for nonnegative integer data partially closed under addition (see [13]), because their PGFs have an exponential form and the mixing parameter appears only in the exponent as a multiple of a function of s, similarly to that for Poisson mixtures. It might also be worth trying to extend these results to the more general left-truncated case.

References

[1] Baayen, R. H. (2001). *Word Frequency Distributions*. Kluwer, Dordretch.

[2] Brian, M. V. (1953). Species frequencies in random samples from animal populations. *J. Animal Ecol.* **22,** 57–64.

[3] Böhning, D. and Kuhnert, R. (2006). Equivalence of truncated count mixture distributions and mixtures of truncated count distributions. *Biometrics* **62,** 1207–1215.

[4] Courant, R. and John, F. (1965). *Introduction to Calculus and Analysis*, Vol. I. Springer, New York.

[5] Engen, S. (1974). On species frequency models. *Biometrika* **61,** 263–270.

[6] Feller, W. (1971). *An Introduction to Probability Theory and Its Applications*, Vol II, 2nd edn. John Wiley, New York.

[7] Grandell, J. (1997). *Mixed Poisson Processes* (Monogr. Statist. Appl. Prob. **77**). Chapman and Hall, London.

[8] Holmes, D. I. (1992). A stylometric analysis of Mormon scripture and related texts. *J. R. Statist. Soc. A* **155,** 91–120.

[9] Johnson, N. L., Kemp, A. W. and Kotz, S. (2005). *Univariate Discrete Distributions*, 3rd edn. John Wiley, Hoboken, NJ.

[10] Mir, K. A. (2009). On size-biased negative binomial distribution and its use in zero-truncated cases. *Measurement Sci. Rev.* **9,** 33–35.

[11] Ord, J. K. and Whitmore, G. A. (1986). The Poisson-inverse Gaussian distribution as a model for species abundance. *Commun. Statist. Theory Meth.* **15,** 853–871.

[12] Patil, G. P. and Rao, C. R. (1978). Weighted distributions and size-biased sampling with applications to wildlife populations and human families. *Biometrics* **34,** 179–189.

[13] Puig, P. and Valero, J. (2006). Count data distributions: some characterizations with applications. *J. Amer. Statist. Assoc.* **101,** 332–340.

[14] Puig, X., Ginebra, J. and Pérez-Casany, M. (2009). Extended truncated inverse Gaussian–Poisson model. *Statist. Modelling* **9,** 151–171.

[15] Puri, P. S. and Goldie, C. M. (1979). Poisson mixtures and quasi-infinite divisibility of distributions. *J. Appl. Prob.* **16,** 138–153.
[16] Rao, C. R. (1965). On discrete distributions arising out of methods of ascertaiment. In *Classical and Contagious Discrete Distributions*, ed. G. P. Patil, Statistical Publishing Society, Calcutta, pp. 320–332.
[17] Sichel, H. S. (1975). On a distribution law for words frequencies. *J. Amer. Statist. Assoc.* **70,** 542–547.
[18] Sichel, H. S. (1992). Anatomy of the generalized inverse Gaussian–Poisson distribution with special applications to bibliometric studies. *Inf. Processing Manag.* **28,** 5–17.
[19] Sichel, H. S. (1997). Modelling species-abundance frequencies and species-individual functions with the generalized inverse Gaussian–Poisson distribution. *S. Afr. Statist. J.* **31,** 13–37.
[20] Valero, J., Pérez-Casany, M. and Ginebra, J. (2010). Extended truncated Tweedie–Poisson model. Submitted.