SOME SOLUBILITY CRITERIA IN FACTORISED GROUPS

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Dedicated to John Cossey on the occasion of his 70th birthday

Abstract

In this paper, solubility of groups factorised as a product of two subgroups which are connected by certain permutability properties is studied.

1. Introduction and statement of results

All groups considered in this paper are finite.

Groups which are factorised as a product of two subgroups have been the subject of many investigations in group theory. Let the group $G = AB$ be the product of two subgroups A and B. It is quite clear that not many properties carry over from A and B to G, and so a natural problem is: suppose that A and B belong to a certain class of groups, what can be said about G? This question has been extensively studied, with many results available spread over many papers (see [1, 3]). The fact that the product of two supersoluble groups is not supersoluble in general, even when both factors are normal, has opened the door to the study of groups which are factorised as a product of two subgroups connected by some permutability properties. The cases when A and B are totally permutable, that is, every subgroup of A permutes with every subgroup of B, or A and B are mutually permutable, that is, every subgroup of A permutes with B and every subgroup of B permutes with A, have produced many surprising and nice results (see [3]).

Sometimes it is interesting to impose permutability of one factor not with the family of all subgroups of the other, but with a smaller relevant family of subgroups, for instance, the family of all maximal subgroups. Then *mutually m-permutable products*, that is, groups $G = AB$ factorised as a product of two subgroups A and B such that

A permutes with every maximal subgroup of B and B permutes with every maximal subgroup of A, appear. A first example of these products was studied by Maier [9], who investigated subgroups H of a group G which permute with all maximal subgroups of G. In this case, G is the mutually m-permutable product of H and G. The second author, Guo and Pedraza-Aguilera have studied in [4] the behaviour of mutually m-permutable products with respect to saturated formations containing the class of all supersoluble groups.

THEOREM 1.1 [4, Theorems 1 and 2]. *Let the soluble group $G = G_1 G_2 \cdots G_r$ be the product of the pairwise mutually m-permutable subgroups $G_1, G_2, \ldots, G_r$. Suppose also that G' is nilpotent.*

(1) *Suppose that $\mathfrak{F}$ is a saturated formation containing the class $\mathfrak{U}$ of all supersoluble groups. If G_i is an $\mathfrak{F}$-group for each $i \in \{1, 2, \ldots, r\}$, then G belongs to $\mathfrak{F}$.*

(2) *Suppose that $\mathfrak{F}$ is a saturated formation and that G is an $\mathfrak{F}$-group. Then $G_i \in \mathfrak{F}$ for all $i \in \{1, 2, \ldots, r\}$.*

The second author, Cossey and Pedraza-Aguilera have proved the following result.

THEOREM 1.2 [2, Theorem 2]. *Let the group $G = AB$ be the mutually m-permutable product of its subgroups A and B. If A and B are supersoluble, then G is soluble.*

Ezquerro and Soler-Escrivà have extended Theorem 1.2 to mutually m-permutable products of soluble groups.

THEOREM 1.3 [7, Theorem 1.2]. *Let the group $G = AB$ be the product of the mutually m-permutable subgroups A and B. If A and B are soluble, then G is soluble.*

The main aim of this paper is to analyse the solubility of factorised groups whose factors are connected by more general permutability conditions. Our first result can be seen as a generalisation of Theorem 1.3 for factors of coprime orders.

THEOREM 1.4. *Assume that a group $G = AB$ is the product of the soluble subgroups A and B and let p be the smallest prime dividing the order of G. If $\gcd(|A|, |B|) = 1$, p divides the order of A, and B permutes with every maximal subgroup of A, then G is soluble.*

The following result can be regarded as a generalisation of Theorem 1.2 in which permutability is required only for the maximal subgroups whose index is not divisible by the smallest prime dividing the order of the group.

THEOREM 1.5. *Let the group $G = AB$ be the product of the supersoluble subgroups A and B and let p be the smallest prime dividing the order of G. If A permutes with every maximal subgroup B_1 of B such that p does not divide $|B : B_1|$ and B permutes with every maximal subgroup A_1 of A such that p does not divide $|A : A_1|$, then G is soluble.*

Finally, if in Theorem 1.5 one of the factors, B say, is nilpotent, the conclusion is achieved without requiring A to permute with the maximal subgroups of B whose index is not divisible by p.

THEOREM 1.6. *Let the group $G = AB$ be the product of the subgroups A and B and let p be the smallest prime dividing the order of G. Assume that A is supersoluble and B is nilpotent. If B permutes with every maximal subgroup A_1 of A such that p does not divide the index $|A : A_1|$, then G is soluble.*

The notation is standard and can be found in books such as [6] or [8].

2. Preliminary results

Let p be a prime number and let G be a group. Deskins [5], see also [8, Ch. III, Aufgabe 12], defined the subgroup $\Phi_p(G)$ as the intersection of all maximal subgroups of G with index not divisible by p. He proved the following theorem.

THEOREM 2.1. *Let p be a prime number and let G be a group. Then $\Phi_p(G)$ has a normal Sylow p-subgroup P such that $\Phi_p(G)/P$ is nilpotent.*

The following results of Ezquerro and Soler-Escrivà are needed in our proofs. The first one follows from the arguments contained in [7, proof of Lemma 2.2].

LEMMA 2.2. *Let the group $G = AB$ be the product of its subgroups A and B and let $D = A \cap B$. Assume that B permutes with each maximal subgroup A_1 of A such that $D \le A_1$. Then*

$$\left(\bigcap \{A_1 \mid A_1 \text{ maximal in } A, D \le A_1\}\right)B = \bigcap \{BA_1 \mid A_1 \text{ maximal in } A, D \le A_1\}$$

is a subgroup of G.

The second one is useful in proofs by minimal counterexample.

LEMMA 2.3 [7, Lemma 2.3]. *Let G be a primitive group with a unique minimal normal subgroup and suppose that $G = AB$, where A and B are nontrivial subgroups of G such that B permutes with all maximal subgroups of A. Let A_1 be a maximal subgroup of A such that BA_1 is a core-free maximal subgroup of G. Then:*

(1)   *H is a Sylow p-subgroup of G of order p, where p is the greatest prime dividing $|G|$; and*

(2)   *B is a core-free maximal subgroup of G such that $|G : B| = p$.*

The following result was proved by Maier.

LEMMA 2.4 [9, Lemma 2.10]. *Let G be a group and let X be a subgroup of G. If X permutes with all maximal subgroups of G whose index is coprime with the order of the Fitting subgroup $F(X)$, then $F(X) \le F(G)$.*

Our proof of Theorem 1.5 is based on the following lemma.

LEMMA 2.5. *Let the group $G = AB$ be the product of its nontrivial subgroups A and B and suppose that $A \cap B = 1$. Assume that G has a unique minimal normal subgroup, N say. If A_1 is a maximal subgroup of A such that B permutes with A_1^a for all $a \in A$ and $\operatorname{Core}_G(A_1 B) = 1$, then A has prime order or $N \le A$.*

PROOF. We prove first that B centralises A_1. By hypothesis $A_1^a B$ is a subgroup of G for each $a \in A$. Consider $g \in G$, then $g = ab$, where $a \in A$ and $b \in B$. Hence $(A_1^a B)^b = A_1^{ab} B^b = A_1^g B \leq G$. This implies that B permutes with A_1^g for every $g \in G$. By [1, Lemma 2.5.1] we have that $[B, A_1]$ is subnormal in G. Hence N normalises $[B, A_1]$ by [1, Lemma 7.5.5]. On the other hand, since $A \cap B = 1$, $A_1 B$ is a maximal subgroup of G and $\mathrm{Core}_G(A_1 B) = 1$. Therefore $G = (A_1 B)N$ and $[B, A_1]$ is a normal subgroup of G contained in $A_1 B$. Thus $[B, A_1] = 1$ and B centralises A_1. Consider now the normal closure $\langle A_1^G \rangle$ of A_1 in G, which is contained in A. Then $A_1^G = A_1$ or $\langle A_1^G \rangle = A$ and so A has prime order or $N \leq A_1^G = A$.   □

3. Proofs of the main theorems

PROOF OF THEOREM 1.4. Suppose that the result is not true. Let G be a counterexample of minimal order. We may assume that $p = 2$. Otherwise, by the well-known Feit–Thompson theorem, G would be soluble, which would contradict the choice of G. Let N be a minimal normal subgroup of G. If 2 divides the order of AN/N, the minimal choice of G implies that G/N is soluble. Moreover, if 2 does not divide the order of AN/N, then 2 does not divide the order of G/N and, as above, G/N is soluble. Thus any proper quotient group of G is soluble. Since the class of all soluble groups is a saturated formation, it follows that $\Phi(G) = 1$ and G has a unique minimal normal subgroup $N = \mathrm{Soc}(G)$, N is nonabelian, and $\mathrm{C}_G(N) = 1$.

On the other hand, we may suppose that $|A| \neq 2$ because otherwise G would be 2-nilpotent and hence soluble, in contradiction to the choice of G. Let A_1 be a maximal subgroup of A. Then, by hypothesis, $A_1 B$ is a maximal subgroup of G. If $\mathrm{Core}_G(A_1 B) = 1$, the application of Lemma 2.3 yields that A is a Sylow p-subgroup of G of order p for the largest prime p dividing $|G|$. This contradicts the hypothesis that 2 divides $|A|$. Therefore $\mathrm{Core}_G(A_1 B) \neq 1$ for every maximal subgroup A_1 of A. Consequently, N is contained in $A_1 B$ for every maximal subgroup A_1 of A. By Lemma 2.2, $N \leq B\Phi(A)$ and so $A \cap N \leq A \cap B\Phi(A) = \Phi(A)$. Let π denote the set of primes dividing $|A \cap N|$. Since $\gcd(|A \cap N|, |G : A|) = 1$, by [8, Ch. IV, Satz 4.6] there exists a normal subgroup R of N such that N/R is a π-group, $R \cap A = 1$, and $|N \cap A|$ divides $|N : R|$. Moreover, $|N : A \cap N|$ divides $|G : A|$, which is a π'-number. Hence $N = R(A \cap N)$. On the other hand, since $\gcd(|A|, |B|) = 1$, it follows that $N \cap A$ is a Hall π-subgroup of N and $N \cap B$ is a Hall π'-subgroup of N. Thus $N = (N \cap A)(N \cap B)$. By order considerations, bearing in mind that $R \cap A = 1$, we obtain that $|R| = |N \cap B|$. Now R is a normal π'-subgroup of N and $N \cap B$ is a Hall π'-subgroup of N, so $R = N \cap B$. But then R and $N/R \cong N \cap A$ are soluble. This leads to the solubility of N, the final contradiction which completes the proof of the theorem.   □

PROOF OF THEOREM 1.5. Assume that the result is not true and let G be a minimal counterexample. Let $1 \neq N$ be a normal subgroup of G. If p does not divide $|G/N|$, then, by hypothesis, AN/N permutes with every maximal subgroup of BN/N and BN/N permutes with every maximal subgroup of AN/N. Applying Theorem 1.2, we have that G/N is soluble. On the other hand, if p divides the order of G/N, the

hypotheses of the theorem hold in G/N. Therefore the minimal choice of G implies that G/N is soluble. Consequently, $\mathrm{Soc}(G) = N$ is the unique minimal normal subgroup of G, N is nonabelian, and $\mathrm{C}_G(N) = 1$.

By the fact that B permutes with every maximal subgroup A_1 of A such that p does not divide $|A : A_1|$, it follows that $A \cap B$ permutes with every maximal subgroup A_1 of A such that p does not divide $|A : A_1|$. Assume that A_1 is a maximal subgroup of A such that p divides $|A : A_1|$. Since A is supersoluble, it is clear that $|A : A_1| = p$. Moreover, the fact that p is the smallest prime dividing the order of G yields that A_1 is a normal subgroup of A. Consequently, $A \cap B$ permutes with A_1 and so $A \cap B$ permutes with all maximal subgroups of A. With the same arguments, $A \cap B$ permutes with every maximal subgroup of B of index p. By Lemma 2.4, $\mathrm{F}(A \cap B) \leq \mathrm{F}(A)$ and $\mathrm{F}(A \cap B) \leq \mathrm{F}(B)$. Thus $\mathrm{F}(A \cap B)$ is a subnormal subgroup of A and B, respectively. Now an application of [3, Theorem 1.1.7] yields that $\mathrm{F}(A \cap B)$ is a subnormal subgroup of G. Moreover, $\mathrm{F}(A \cap B)$ is nilpotent. Consequently, $\mathrm{F}(A \cap B) \leq \mathrm{F}(G) = 1$. Hence the solubility of $A \cap B$ implies that $A \cap B = 1$.

We will prove now that p divides the orders of A and B. It is clear that we may assume that p divides $|A|$. Assume that p does not divide $|B|$. Thus, by hypothesis, A permutes with every maximal subgroup of B. We may suppose that A is not nilpotent, otherwise G is soluble by [2, Theorem 1]. Let B_1 be a maximal subgroup of B. Then AB_1 is a maximal subgroup of G.

Assume that $\mathrm{Core}_G(AB_1) = 1$. By Lemma 2.5, $N \leq B$, which contradicts the solubility of B, or B has prime order. Suppose that $|B| = q$, where $q \neq p$ is a prime number. Then $|G : A| = q$ and $\mathrm{Core}_G(A) = 1$. This means that G is isomorphic to a subgroup of the symmetric group of degree q. Consequently, q is the largest prime dividing the order of G and B is a Sylow q-subgroup of G. Let $\mathcal{M}(A)$ be the set of all maximal subgroups of A with index in A not divisible by p. Since A is not nilpotent, it is clear that $\mathcal{M}(A)$ is not empty. Consider $A_1 \in \mathcal{M}(A)$. By hypothesis, BA_1 is a subgroup of G. Moreover, the fact that $A \cap B = 1$ implies that BA_1 is a maximal subgroup of G. By Lemma 2.5 we obtain that $\mathrm{Core}_G(BA_1) \neq 1$ for all $A_1 \in \mathcal{M}(A)$. In particular, $N \leq BA_1$ for all $A_1 \in \mathcal{M}(A)$. By Lemma 2.2, we have that $N \leq B\,[\bigcap\{A_1 \mid A_1 \in \mathcal{M}(A)\}] = B\Phi_p(A)$. Since $\Phi_p(A) \leq A$, it follows that it is supersoluble. In particular, $\Phi_p(A)$ is p-nilpotent. On the other hand, by Theorem 2.1, $\Phi_p(A)$ has a normal Sylow p-subgroup, P say, such that $\Phi_p(A)/P$ is nilpotent. Hence $\Phi_p(A)$ is the direct product of two nilpotent subgroups and thus it is nilpotent. Since $|B| = q$, $B\Phi_p(A)$ is soluble by [8, Ch. VI, Hauptsatz 4.3] and so N is soluble, a contradiction.

Thus $\mathrm{Core}_G(AB_1) \neq 1$ for all maximal subgroups B_1 of B. (Recall that A permutes with every maximal subgroup of B because p does not divide $|B|$.) Furthermore, if $\mathrm{Core}_G(BA_1) = 1$ for some $A_1 \in \mathcal{M}(A)$, it follows from Lemma 2.5 that A has prime order, which contradicts the fact that A cannot be nilpotent, or $N \leq A$, which contradicts the fact that N is not soluble. So we may assume that $\mathrm{Core}_G(BA_1) \neq 1$ for all $A_1 \in \mathcal{M}(A)$. Arguing as above and applying Lemma 2.2 again, we obtain that $N \leq A\Phi(B) \cap B\Phi_p(A) = \Phi(B)\Phi_p(A)$. As above, $\Phi_p(A)$ is nilpotent. Then, by [8, Ch. VI, Hauptsatz 4.3], $\Phi(B)\Phi_p(A)$ is soluble and so N is soluble, a contradiction. Consequently, p divides the order of A and B.

We now prove that A and B are nonnilpotent subgroups of G. Suppose that A is nilpotent. If B is nilpotent, then G would be soluble by [8, Ch. VI, Hauptsatz 4.3], a contradiction. Thus we may assume that B is not nilpotent and, in particular, that the number of distinct prime numbers dividing the order of B is at least two. Moreover, p divides the order of B as we have proved before. Consider $B_1 \in \mathcal{M}(B)$. By hypothesis and the fact that $A \cap B = 1$, we have that AB_1 is a maximal subgroup of G. Arguing as above, we obtain that if $\mathrm{Core}_G(AB_1) = 1$, then B has prime order, against the fact that B is not nilpotent, or $N \leq B$, which contradicts the fact that N is not soluble. Consequently, we may assume that $N \leq \mathrm{Core}_G(AB_1)$ for every $B_1 \in \mathcal{M}(B)$. With analogous arguments to those used before, we have $N \leq A\Phi_p(B)$ and $\Phi_p(B)$ is nilpotent. Therefore N is soluble, a contradiction. Hence A and, similarly, B are nonnilpotent subgroups of G.

Finally, note that if N is contained in all maximal subgroups of G of the form AB_1 and BA_1, where $B_1 \in \mathcal{M}(B)$ and $A_1 \in \mathcal{M}(A)$ (the fact that A and B are nonnilpotent yields that $\mathcal{M}(B)$ and $\mathcal{M}(A)$ are nonempty sets), then we would obtain $N \leq A\Phi_p(B) \cap B\Phi_p(A) = \Phi_p(A)\Phi_p(B)$. But $\Phi_p(A)\Phi_p(B)$ is a product of two nilpotent groups by the above arguments. Hence N is soluble by [8, Ch. VI, Hauptsatz 4.3], a contradiction. Consequently, we may assume that there exists $A_1 \in \mathcal{M}(A)$ such that BA_1 is a maximal subgroup of G with $\mathrm{Core}_G(BA_1) = 1$. Now Lemma 2.5 implies that A has prime order or $N \leq A$, the final contradiction.   □

PROOF OF THEOREM 1.6. Assume that the result is false and let G be a counterexample of minimal order. We argue that every proper quotient G/N, where $1 \neq N$ is a normal subgroup of G, is soluble. If p does not divide $|G : N|$, then G/N is soluble by the Feit–Thompson theorem. If p divides $|G : N|$, it is clear that the hypotheses of the theorem hold in G/N. By minimality of G, we have that G/N is soluble. Consequently, $\mathrm{Soc}(G) = N$ is the unique minimal normal subgroup of G, N is nonabelian, and $C_G(N) = 1$. Arguing as in Theorem 1.5, we obtain that $A \cap B$ permutes with every maximal subgroup of A. Moreover, since B is nilpotent, every maximal subgroup of B is a normal subgroup of B. Hence $A \cap B$ permutes with every maximal subgroup of B. Now a similar argument to that in used Theorem 1.5 yields $A \cap B = 1$.

Note that A is not nilpotent, otherwise G would be soluble by [8, Ch. VI, Hauptsatz 4.3], which contradicts the choice of G. Furthermore, if p does not divide the order of A, then B permutes with every maximal subgroup of A and so G is soluble by [2, Theorem 1], against the choice of G. Thus we may assume that A is not nilpotent and p divides the order of A. Consider now the set $\mathcal{M}(A)$ of all maximal subgroups of A whose index in A is not divisible by p. Since A is not nilpotent, $\mathcal{M}(A)$ is not empty. If N is contained in BA_1 for all $A_1 \in \mathcal{M}(A)$, then, arguing as in Theorem 1.5, we obtain that $N \leq B\Phi_p(A)$ and $\Phi_p(A)$ is nilpotent. Thus by [8, Ch. VI, Hauptsatz 4.3] N is soluble, against the choice of G. Therefore there exists a maximal subgroup A_1 of A such that BA_1 is a maximal subgroup of G with $\mathrm{Core}_G(BA_1) = 1$. By Lemma 2.5, A has prime order or $N \leq A$, the final contradiction.   □

4. Final remarks

The result of Theorem 1.4 does not hold when p is not the smallest prime dividing the order of G. For instance, the alternating group $G = A_5$ of degree five can be expressed as the product of a cyclic group $A \cong C_5$ of order five and the alternating group $B = A_4$ of degree four. In this case, $\gcd(|A|, |B|) = 1$, 5 divides $|A|$, and B permutes with the unique maximal subgroup of A, but G is not soluble.

In Theorem 1.6, the nilpotency of B is required for the proof. The group $G = \mathrm{PSL}_2(7)$ can be factorised as the product of a dihedral group A of order eight and a nonabelian group B of order 21. Trivially, B permutes with every maximal subgroup A_1 of A such that 2 does not divide $|A : A_1|$, but G is not soluble.

We do not know whether the hypothesis on p in Theorem 1.5 is necessary.

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