On certain expansions involving Whittaker's M-Functions

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1. The object of the present note is to obtain expansions of the square of Whittaker's M-Functions in series of M-Functions and also other expansions involving M-Functions.

2. Whittaker's M-Function[1] is defined as

$$M_{k,m}(x)=x^{\frac{1}{2}+m}e^{-\frac{1}{2}x}\left\{1+\frac{\frac{1}{2}+m-k}{1!\,(2m+1)}x+\frac{(\frac{1}{2}m+m-k)(\frac{3}{2}+m-k)}{2!\,(2m+1)(2m+2)}x^2+\ldots\right\} \quad (1)$$

where $2m$ is not a negative integer. The following recurrence-relations are explicitly known or else easily deducible from the well-known recurrence-relations for the confluent hypergeometric function.

$$(\tfrac{1}{2}-m-k)\,M_{k,m}(x)=(x-2k+2)\,M_{k-1,m}(x)+(k-\tfrac{3}{2}-m)\,M_{k-2,m}(x) \quad (2)$$

$$x\frac{d}{dx}M_{k,m}(x)=(k-\tfrac{1}{2}x)\,M_{k,m}(x)+(\tfrac{1}{2}+m-k)\,M_{k-1,m}(x) \quad (3)$$

$$x\frac{d}{dx}M_{k,m}(x)=\left\{\frac{k}{2m-1}x-(m-\tfrac{1}{2})\right\}M_{k,m}(x)+2mx\,M_{k,m-1}(x) \quad (4)$$

and[2]

$$x\frac{d}{dx}M_{k,m}(x)=\left(m+\tfrac{1}{2}-\frac{k}{2m+1}x\right)M_{k,m}(x)+\frac{(\frac{1}{2}+m+k)(\frac{1}{2}+m-k)}{(2m+1)^2(2m+2)}x\,M_{k,m+1}(x) \quad (5)$$

3. The functions $M_{k,\pm m}(x)$ satisfy the differential equation

$$\frac{d^2y}{dx^2}+\left\{-\tfrac{1}{4}+\frac{k}{x}+\frac{\frac{1}{4}-m^2}{x^2}\right\}y=0. \quad (6)$$

and $M^2_{k,\pm m}(x)$ satisfy the equation

$$\frac{d^3y}{dx^3}=\left(\frac{2k}{x^2}-\frac{4m^2-1}{x^3}\right)y+\left(1-\frac{4k}{x}+\frac{4m^2-1}{x^2}\right)\frac{dy}{dx} \quad (7)$$

To obtain a solution of (7), let us assume $y=\Sigma\,A_r\,x^{\frac{1}{2}}\,M_{r,-2m}(2x)$ and substitute this value in (7).

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[1] Whittaker and Watson, *Modern Analysis* (Cambridge, 1920), 337.

[2] *Tohoku Math. Journal*, 29 (1928), 321.

Then we get after considerable simplification

$$\Sigma A_r (x - r) M_{r, -2m} (2x) = \Sigma A_r (4r - 8k) x M'_{r, -2m} (2x). \qquad (8)$$

Making use of the recurrence-relations (2) and (3), we get after some simplification

$$\Sigma A_r (\tfrac{1}{2} - 2m - r)(4k + 1 - 2r) M_{r-1, -2m} (2x)$$
$$= \Sigma A_r (\tfrac{1}{2} - 2m + r)(-4k + 1 + 2r) M_{r+1, -2m} (2x). \qquad (9)$$

Hence

$$A_{r+2} = A_r \frac{(\frac{1}{2} - 2m + r)(-4k + 1 + 2r)}{(\frac{3}{2} + 2m + r)(-4k + 3 + 2r)}, \qquad (10)$$

and the initial value of r is $2k + \frac{1}{2}$ or $\frac{1}{2} - 2m$. Hence equating the coefficients of various powers of x, we easily find that

$$x^{-\frac{1}{2}} M^2_{k, -m} (x) = \frac{\Gamma(\frac{1}{2} + k + m)\,\Gamma(\frac{1}{2} + 2m)}{\Gamma(k + m + 1)\,\Gamma(2m)} 2^{2m - \frac{1}{2}} \times$$
$$\Big\{ M_{2k+\frac{1}{2}, -2m} (2x) + \frac{\frac{1}{2}(\frac{1}{2} - m + k)}{1!\,(k + m + 1)} M_{2k+\frac{5}{2}, -2m} (2x)$$
$$+ \frac{\frac{1}{2} \cdot \frac{3}{2} (\frac{1}{2} - m + k)(\frac{3}{2} - m + k)}{2!\,(k + m + 1)(k + m + 2)} M_{2k+\frac{9}{2}, -2m} (2x) + \ldots \Big\} \qquad (11)$$

Now applying the test that Σu_n is absolutely convergent if

$$\overline{\lim_{n \to \infty}}\, n \left\{ \left| \frac{u_{n+1}}{u_n} \right| - 1 \right\} = -1 - c$$

when c is positive, we can prove that the infinite series is absolutely convergent, provided that $m > 0$.

The other expansion which is valid for $m > 0$ is

$$x^{-\frac{1}{2}} M^2_{k, -m} (x) = \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2} + m - k)}{\Gamma(1 - k - m)\,\Gamma(2m)} 2^{2m - \frac{1}{2}} \Big[M_{\frac{1}{2} - 2m, -2m} (2x)$$
$$- \frac{(4m - 1)(4m + 4k - 2)}{2 \,.\, (4m + 4k - 4)} M_{\frac{5}{2} - 2m, -2m} (2x)$$
$$+ \frac{(4m - 1)(4m - 3)(4m + 4k - 2)(4m + 4k - 6)}{2 \,.\, 4\,(4m + 4k - 4)(4m + 4k - 8)} M_{\frac{9}{2} - 2m, -2m} (2x)$$
$$- \ldots \Big]. \qquad (12)$$

Thus we see that $x^{-\frac{1}{2}} M^2_{k, -m} (x)$ can be expressed in either of the two forms (11) and (12).

4. We have seen that $M_{k, m} (x)$ satisfies the differential equation (6). To find a solution, let us assume

$$y = \Sigma A_r x^{r + \frac{1}{2}} M_{\frac{1}{2}k, r} (x). \qquad (13)$$

Substituting in (6) and making use of the relations (4) and (5), we find after some simplification,

$$\Sigma A_r \{m^2 - (2r + \tfrac{1}{2})^2\} x^{r-\frac{1}{2}} M_{\frac{1}{2}k,\, r}(x)$$
$$= \Sigma A_r \frac{(\frac{1}{2} + \frac{1}{2}k + r)(\frac{1}{2} - \frac{1}{2}k + r)}{(2r+1)(2r+2)} x^{r-\frac{1}{2}} M_{\frac{1}{2}k,\, r+1}(x), \quad (14)$$

whence

$$A_{r+1} = A_r \frac{(\frac{1}{2} + \frac{1}{2}k + r)(\frac{1}{2} - \frac{1}{2}k + r)}{(2r+1)(2r+2)(m + 2r + \frac{5}{2})(m - 2r - \frac{5}{2})}, \quad (15)$$

and the initial value of r is $\frac{1}{2}m - \frac{1}{4}$. Hence we obtain the absolutely convergent infinite series,

$$M_{k,m}(x) = \Big\{ x^{\frac{1}{2}m+\frac{1}{4}} M_{\frac{1}{2}k,\, \frac{1}{2}m-\frac{1}{4}}(x)$$
$$- \frac{(\frac{1}{2}+m+k)(\frac{1}{2}+m-k)}{(2m+1)(2m+3)\,.\,2\,(2m+2)} x^{\frac{1}{2}m+\frac{5}{4}} M_{\frac{1}{2}k,\, \frac{1}{2}m+\frac{3}{4}}(x)$$
$$+ \frac{(\frac{1}{2}+m+k)(\frac{5}{2}+m+k)(\frac{1}{2}+m-k)(\frac{5}{2}+m-k)}{(2m+1)(2m+3)(2m+5)(2m+7)\,.\,2\,.\,4\,.\,(2m+2)(2m+4)} \times$$
$$x^{\frac{1}{2}m+\frac{9}{4}} M_{\frac{1}{2}k,\, \frac{1}{2}m+\frac{7}{4}}(x) - \ldots \Big\}. \quad (16)$$

This suggests the following expansion

$${}_1F_1(2a; 2b; x) = \Big\{ {}_1F_1(a; b; x) - \frac{(b-a)\,a}{b\,(b+1)\,.\,(b+\frac{1}{2})\,1!\,2^2} x^2\, {}_1F_1(a+1; b+2; x)$$
$$+ \frac{(b-a)(b-a+1)\,a\,(a+1)}{b\,(b+1)(b+2)(b+3)\,.\,(b+\frac{1}{2})(b+\frac{3}{2})\,2!\,2^4} x^4\, {}_1F_1(a+2; b+4; x) - \ldots \Big\}. \quad (17)$$

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