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Inner E_0-Semigroups on Infinite Factors

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Abstract. This paper is concerned with the structure of inner E_0-semigroups. We show that any inner E_0-semigroup acting on an infinite factor M is completely determined by a continuous tensor product system of Hilbert spaces in M and that the product system associated with an inner E_0-semigroup is a complete cocycle conjugacy invariant.

Introduction

The study of continuous semigroups of unital $*$-endomorphisms acting on von Neumann algebras, or E_0-semigroups, was initiated by R. T. Powers in the late eighties [P1, P2], and since then it has seen rapid and impressive progress with contributions from W. Arveson, G. Price, B. Tsirelson and others. Having connections with quantum field theory [A2, A3] and quantum probability theory [B] and raising challenging and difficult problems, the research in this topic has been mainly focused on the classification of E_0-semigroups acting on a factor of type I_∞.

In his seminal work [A1], W. Arveson proved that any E_0-semigroup acting on a type I_∞ factor is completely determined by a continuous tensor product system of Hilbert spaces, briefly called a product system. Moreover, product systems are complete invariants for cocycle conjugacy: two E_0-semigroups acting on a type I_∞ factor are cocycle conjugate if and only if their associated product systems are isomorphic. Hence, the problem of classifying E_0-semigroups up to cocycle conjugacy was reduced to the problem of classifying product systems.

Regarding E_0-semigroups acting on factors that are not of type I_∞, W. Arveson has suggested that an effective theory should have similar properties [A3]:

(a) it should associate a continuous tensor product of Hilbert spaces with every E_0-semigroup and, conversely, every continuous tensor product in this category of objects should be associated with an E_0-semigroup;
(b) the continuous tensor product of Hilbert spaces associated with an E_0-semigroup should be a complete cocycle conjugacy invariant.

The goal of this paper is to initiate an investigation of the class of inner E_0-semigroups, based on Arveson's product systems approach, in the framework of arbitrary infinite factors. More specifically, we shall show that the inner E_0-semigroups satisfy Arveson's properties discussed above.

The class of inner E_0-semigroups was introduced in [F2] and plays an important role in the analysis of the structure of arbitrary E_0-semigroups. It was shown in

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[F2] that any E_0-semigroup acting on a von Neumann algebra can be uniquely decomposed as the central direct sum of an inner E_0-semigroup and a properly outer E_0-semigroup. Moreover, this decomposition is stable up to conjugacy and cocycle conjugacy. Hence the problem of classifying arbitrary E_0-semigroups reduces to the classification of these two canonical classes.

This paper is organized as follows. In the first section, we introduce and discuss some properties of the class of inner E_0-semigroups. In the second section, we define the concept of product system in an infinite factor, and show that the inner E_0-semigroups are completely determined by such product systems. In the last section, we show that the product system associated with an E_0-semigroup is a complete cocycle conjugacy invariant.

We close this introduction with a few remarks on notation and on the concepts used in this paper. Let $M \subset \mathfrak{B}(\mathcal{H})$ be a von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$. When it is convenient to do so, we shall consider M endowed with its natural standard Borel structure, *i.e.,* the Borel structure whose σ-field is that generated by the weak operator topology. Let $\mathrm{End}(M)$ denote the semigroup of all unital normal $*$-endomorphisms acting on M. An E_0-semigroup of M is a family $\widetilde{\rho} = \{\rho_t \mid t \geq 0\} \subset \mathrm{End}(M)$ which obeys the semigroup properties $\rho_{t+s} = \rho_t \circ \rho_s$, $\rho_0 = \mathrm{Id}$ and which is continuous in the sense that for every $x \in M$, the mapping $t \mapsto \rho_t(x)$ is continuous in the weak topology on M. This is equivalent to the continuity in the σ-weak, strong, and σ-strong topologies [B], or to the fact that the function $t \mapsto \varphi(\rho_t(x))$ is measurable for all normal states φ on M and for all $x \in M$ (*cf.* [A1, Proposition 2.5]).

1 Preliminaries on Inner E_0-Semigroups

We start this section by recalling some basic definitions and properties of inner endomorphisms and inner E_0-semigroups. For a more detailed treatment of this matter, we refer the reader to [R, F1, F2].

Let M be an infinite factor. By a Hilbert space in M (see [R]) we understand a norm-closed linear subspace H of M that satisfies the following relations:

(1) $u^*u \in \mathbb{C} \cdot 1$, for every $u \in H$;
(2) $xH \neq \{0\}$, for every $x \in M$, $x \neq 0$.

The scalar product on the Hilbert space H is then given by

$$\langle u\,, v\rangle_M \cdot 1_M = v^*u, \quad u, v \in H.$$

Thus, an orthonormal basis for H is the family $\{U_i\}_{i=1}^k$ of isometries of M satisfying the Cuntz relations [C]:

$$u_i^* u_j = \delta_{i,\,j} \cdot 1_M \quad \text{and} \quad \sum_{i=1}^k u_i u_i^* = 1_M \tag{1.1}$$

where, if $k = \infty$, then the sum is understood to converge with respect to the strong topology of M.

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Let $k \in \mathbb{N} \cup \{\infty\}$ be fixed. An endomorphism $\rho \in \operatorname{End}(M)$ is said to be k-inner if the space of intertwiners

$$\operatorname{Hom}_M(\mathrm{Id}, \rho) = \{u \in M \mid \rho(x)u = ux \text{ for all } x \in M\}$$

is a k-dimensional Hilbert space in M. Equivalently, ρ is a k-inner endomorphism if and only if it has the form

$$\rho(x) = \sum_{i=1}^{k} u_i x u_i^*, \quad x \in M,$$

where $\{u_i\}_{i=\overline{1,k}}$ is a set of isometries of M satisfying relations (1.1). Such a set will be called an implementing set for the k-inner endomorphism ρ.

Now let $\widetilde{\rho} = \{\rho_t \mid t \geq 0\}$ be an E_0-semigroup acting on M. It was shown in [F2] that if for any $t > 0$ there exists $k(t) \in \mathbb{N} \cup \{\infty\}$ such that ρ_t is a $k(t)$-inner endomorphism, then $k(t) = \infty$, for all $t > 0$. This property has motivated the following definition.

Definition 1.1 An E_0-semigroup $\widetilde{\rho} = \{\rho_t \mid t \geq 0\}$ acting on an infinite factor M is said to be inner if for any $t > 0$, ρ_t is an ∞-inner endomorphism of M.

We note that if M is a type I_∞ factor, then every E_0-semigroup acting on M is an inner E_0-semigroup [A1, Proposition 2.1]. Moreover, every inner E_0-semigroup can be extended to an E_0-semigroup acting on a factor of type I_∞.

Proposition 1.2 *Let $M \subset \mathfrak{B}(\mathcal{H})$ be an infinite factor, and let $\widetilde{\rho} = \{\rho_t \mid t \in \mathbb{R}_+\}$ be an inner E_0-semigroup of M. Then $\widetilde{\rho}$ can be extended to an E_0-semigroup of $\mathfrak{B}(\mathcal{H})$.*

Proof For each $t \in \mathbb{R}_+$, let $\{v_i(t)\}_i$ be an implementing set of the ∞-inner endomorphism ρ_t. Since M is a factor,

$$\operatorname{Hom}_M(\mathrm{Id}\,, \rho_t) = \overline{\operatorname{span}}\{v_i(t) \mid i = 1, 2, \dots\}$$

is a Hilbert space in M. Moreover, we can easily check that $\operatorname{Hom}_M(\mathrm{Id}\,, \rho_t)$ is also a Hilbert space in $\mathfrak{B}(\mathcal{H})$. It then follows that each endomorphism ρ_t can be extended to an endomorphism of $\mathfrak{B}(\mathcal{H})$, denoted also ρ_t, by the formula

$$\rho_t(X) = \sum_{i=1}^{\infty} v_i(t) X v_i(t)^*, \quad X \in \mathfrak{B}(\mathcal{H}),$$

and that for any $t > 0$, we have

$$\operatorname{Hom}_{\mathfrak{B}(\mathfrak{H})}(\mathrm{Id}\,, \rho_t) = \operatorname{Hom}_M(\mathrm{Id}\,, \rho_t). \tag{1.2}$$

We claim that $\{\rho_t \mid t \geq 0\}$ is an E_0-semigroup of $\mathfrak{B}(\mathcal{H})$. Indeed, if $x \in M$ and $y \in M'$, we have

$$\begin{aligned}\rho_t \circ \rho_s(xy) &= \sum_{i,j=1}^{\infty} v_i(t)v_j(s)xyv_j(s)^* v_i(t)^* \\ &= \Big(\sum_{i,j=1}^{\infty} v_i(t)v_j(s)xv_j(s)^* v_i(t)^*\Big)\, y \\ &= \rho_t \circ \rho_s(x)y = \rho_{t+s}(x)y \\ &= \Big(\sum_{i=1}^{\infty} v_i(t+s)xv_i(t+s)^*\Big)\, y \\ &= \rho_{t+s}(xy).\end{aligned}$$

Since M is a factor, $M \cup M'$ is weakly dense in $\mathfrak{B}(\mathcal{H})$ and, as our endomorphisms are normal, it follows that

$$\rho_{s+t}(X) = \rho_s \circ \rho_t(X), \quad X \in \mathfrak{B}(\mathcal{H}), s, t \geq 0.$$

Moreover, for $x \in M$, $y \in M'$ and $\xi, \eta \in \mathcal{H}$, the function

$$t \mapsto \langle \rho_t(xy)\xi\,, \eta\rangle$$

is continuous. Using again the weak density of $M \cup M'$ in $\mathfrak{B}(\mathcal{H})$, it follows that the function $t \mapsto \langle \rho_t(X)\xi\,, \eta\rangle$ is measurable for all $X \in \mathfrak{B}(\mathcal{H})$. Hence $\{\rho_t \mid t \in \mathbb{R}_+\}$ is an E_0-semigroup on $\mathfrak{B}(\mathcal{H})$. ∎

We close this section with a few remarks on the existence of inner E_0-semigroups. If M is an arbitrary infinite factor, then we can construct inner E_0-semigroups acting on M as follows: since M is infinite, M is $*$-isomorphic to $M \otimes \mathfrak{B}(\mathcal{K})$, where $\mathcal{K}$ is a separable infinite dimensional Hilbert space. Let $\widetilde{\sigma} = \{\sigma_t \mid t \geq 0\}$ be an an E_0-semigroup on $\mathfrak{B}(\mathcal{K})$, and let $\{u_t\}_{t \geq 0}$ be a semigroup of unitaries in M. Then we can easily check that the one-parameter family

$$\widetilde{\rho} = \{\theta^{-1} \circ \mathrm{Ad}(u_t) \otimes \sigma_t \circ \theta \mid t \geq 0\}$$

is an inner E_0-semigroup of M, where $\theta \colon M \to M \otimes \mathfrak{B}(\mathcal{K})$ is an arbitrary $*$-isomorphism.

2 Inner E_0-Semigroups and Product Systems

We define the concept of continuous tensor product of Hilbert spaces in an infinite factor, briefly called product system, in analogy with Arveson's concept of concrete product system [A1]:

Definition 2.1 A product system in an infinite factor M is a Borel subset $\mathcal{E} = \{H_t \mid t > 0\}$ of M consisting of Hilbert spaces H_t in M that satisfy the following conditions:

(a) $H_{t+s} = \overline{\text{span}} H_t \cdot H_s$, for all $s, t > 0$;

(b) there exists an infinite dimensional Hilbert space H_0 such that $\mathcal{E}$ and $(0, \infty) \times H_0$ are isomorphic as measurable families of Hilbert spaces, *i.e.,* there exists a Borel isomorphism $\theta \colon \mathcal{E} \to (0, \infty) \times H_0$ such that for any $t > 0$, the restriction

$$\theta|_{H_t} \colon H_t \to \{t\} \times H_0$$

is a unitary isomorphism of Hilbert spaces.

There exists a natural concept of isomorphic product systems: two product systems $\mathcal{E} = \{H_t \mid t > 0\}$ and $\mathcal{F} = \{K_t \mid t > 0\}$ in M are said to be isomorphic if there exists a Borel isomorphism $\Theta \colon \mathcal{E} \to \mathcal{F}$ such that

(i) for any $t > 0$, $\Theta|_{H_t} \colon H_t \to K_t$ is a unitary operator;

(ii) $\Theta(xy) = \Theta(x)\Theta(y)$, for all $x \in H_t$, $y \in H_s$, and all $s, t > 0$.

Observation 2.2 If M is a type I_∞ factor, then our concept of product system in M and Arveson's concept of concrete product system are the same. In particular, a product system in a von Neumann algebra M that acts on a Hilbert space $\mathcal{H}$ is a concrete product system (in $\mathfrak{B}(\mathcal{H})$).

In [A1, Proposition 2.2], W. Arveson proved that if $\widetilde{\rho} = \{\rho_t \mid t \geq 0\}$ is an E_0-semigroup acting on $\mathfrak{B}(\mathcal{H})$, then

$$\mathcal{E}_{\widetilde{\rho}} = \{\text{Hom}_{\mathfrak{B}(\mathcal{H})}(\text{Id}\,, \rho_t) \mid t > 0\}$$

is a concrete product system. Using the above observation, Proposition 1.2 and relation (1.2), we can easily obtain the following general result:

Proposition 2.3 *Let M be an infinite factor and $\widetilde{\rho} = \{\rho_t \mid t \geq 0\}$ be an inner E_0-semigroup on M. Then the set*

$$\mathcal{E}_{\widetilde{\rho}} = \{\text{Hom}_M(\text{Id}\,, \rho_t) \mid t > 0\} \tag{2.1}$$

is a product system in M.

Conversely, any product system gives rise to an inner E_0-semigroup:

Theorem 2.4 *Let M be an infinite factor and $\mathcal{E} = \{H_t \mid t > 0\}$ be a product system in M. Then there exists a unique inner E_0-semigroup $\widetilde{\rho} = \{\rho_t \mid t \geq 0\}$ on M such that $H_t = \text{Hom}_M(Id\,, \rho_t)$, for all $t > 0$.*

Proof First of all, we assert that there exists a sequence of measurable mappings $\{u_i\}_i$ such that for any $t > 0$, $\{u_i(t)\}_i$ is an orthonormal basis of the Hilbert space H_t. Indeed, by Definition 2.1, there exist an infinite dimensional Hilbert space H_0 and a Borel isomorphism $\theta \colon \mathcal{E} \to (0, \infty) \times H_0$ such that

$$U(t) = \theta|_{H_t} \colon H_t \to \{t\} \times H_0$$

is a unitary operator. The sequence of measurable mappings $\{u_i\}_i$ is then given by

$$u_i(t) = U(t)^*(e_i),$$

where $\{e_i\}_i$ is a fixed orthonormal basis of the Hilbert space H_0.

Since H_t is a Hilbert space in M with orthonormal basis $\{u_i(t)\}_i$, it follows that the formula

$$\rho_t(x) = \sum_{i=1}^{\infty} u_i(t)xu_i(t)^*, \quad x \in M,$$

defines an ∞-inner endomorphism of M for all $t > 0$. Moreover, since M is a factor, we have

$$H_t = \mathrm{Hom}_M(\mathrm{Id}\,, \rho_t), \quad \text{for all } t > 0.$$

We define $\rho_0 = \mathrm{Id}$ and claim that $\{\rho_t \mid t \geq 0\}$ is an E_0-semigroup of M. Indeed, if $x \in M$ and $s, t \geq 0$, then we have

$$\begin{aligned}
\rho_s \circ \rho_t(x)u_i(s)u_j(t) &= \sum_{l=1}^{\infty} u_l(s)\rho_t(x)u_l(s)^* u_i(s)u_j(t) \\
&= \Big(\sum_{l=1}^{\infty} u_l(s)u_l(s)^*\Big) u_i(s)\rho_t(x)u_j(t) \\
&= u_i(s)\rho_t(x)u_j(t) \\
&= u_i(s)u_j(t)x \\
&= \rho_{t+s}(x)u_i(s)u_j(t),
\end{aligned}$$

where the last equality follows from the fact that

$$u_i(s)u_j(t) \in H_s \cdot H_t \subset H_{s+t} = \mathrm{Hom}_M(\mathrm{Id}\,, \rho_{s+t}).$$

Therefore we have

$$\begin{aligned}
\rho_s \circ \rho_t(x) &= \rho_s \circ \rho_t(x) \cdot 1_M \\
&= \rho_s \circ \rho_t(x)\Big(\sum_{i,j=1}^{\infty} u_i(s)u_j(t)u_j(t)^* u_i(s)^*\Big) \\
&= \sum_{i,j=1}^{\infty} \big(\rho_s \circ \rho_t(x)u_i(s)u_j(t)\big)\, u_j(t)^* u_i(s)^* \\
&= \sum_{i,j=1}^{\infty} \rho_{s+t}(x)u_i(s)u_j(t)u_j(t)^* u_i(s)^* \\
&= \rho_{s+t}(x).
\end{aligned}$$

Thus $\{\rho_t\}_{t\geq 0}$ is a semigroup of endomorphisms.

Finally, since the mapping $t \mapsto u_i(t)$ is measurable for every i, it follows that the function

$$(0,\infty) \ni t \mapsto \varphi(\rho_t(x)) \in \mathbb{R}$$

is also measurable for every normal state φ of M and every $x \in M$. Hence $\widetilde{\rho} = \{\rho_t \mid t \geq 0\}$ is an E_0-semigroup of M.

The uniqueness follows immediately from the fact that an ∞-inner endomorphism acting on a factor is completely determined by its space of intertwiners [L, Proposition 2.1]. ■

3 Conjugacy and Cocycle Conjugacy

Let $\widetilde{\rho} = \{\rho_t \mid t \geq 0\}$ and $\widetilde{\sigma} = \{\sigma_t \mid t \geq 0\}$ be E_0-semigroups acting on M. We shall say $\widetilde{\rho}$ and $\widetilde{\sigma}$ are conjugate if there exists $\theta \in \mathrm{Aut}(M)$ such that

$$\theta \circ \rho_t = \sigma_t \circ \theta, \quad t \geq 0,$$

and that $\widetilde{\rho}$ and $\widetilde{\sigma}$ are cocycle conjugate if there exists a strongly continuous family of unitaries $\{u_t \mid t \geq 0\}$ of M such that

(i) $u_{s+t} = u_s \sigma_s(u_t), \quad s, t \geq 0;$

(ii) $\widetilde{\rho}$ is conjugate to the E_0-semigroup $\{\mathrm{Ad}(u_t) \circ \sigma_t \mid t \geq 0\}$.

A strongly continuous family of unitaries satisfying relation (i) will be called a $\widetilde{\sigma}$-cocycle.

Our goal, in this section, is to show that the product system associated to an inner E_0-semigroup as in Proposition 2.3 is a complete cocycle conjugacy invariant. Before proving this result, let us mention that if $\widetilde{\rho} = \{\rho_t \mid t \geq 0\}$ and $\widetilde{\sigma} = \{\sigma_t \mid t \geq 0\}$ are two conjugate inner E_0-semigroups acting on an infinite factor M, then their associated product systems $\mathcal{E}_{\widetilde{\rho}}$ and $\mathcal{E}_{\widetilde{\sigma}}$ given by Proposition 2.3 are isomorphic. Indeed, if $\theta \in \mathrm{Aut}(M)$ is such that $\theta \circ \rho_t = \sigma_t \circ \theta$, for all $t \geq 0$, then it is easily seen that the mapping $\Theta\colon \mathcal{E}_{\widetilde{\rho}} \to \mathcal{E}_{\widetilde{\sigma}}$, defined for all $u(t) \in \mathrm{Hom}_M(\mathrm{Id}, \rho_t)$ and all $t > 0$ by

$$\Theta(u(t)) = \theta(u(t))$$

is an isomorphism of product systems.

We prove the main result of this section:

Theorem 3.1 *Let $\widetilde{\rho} = \{\rho_t \mid t \geq 0\}$ and $\widetilde{\sigma} = \{\sigma_t \mid t \geq 0\}$ be two inner E_0-semigroups acting on an infinite factor M. Then the following statements are equivalent:*

(1) *$\widetilde{\rho}$ and $\widetilde{\sigma}$ are cocycle conjugate;*

(2) *the associated product systems $\mathcal{E}_{\widetilde{\rho}}$ and $\mathcal{E}_{\widetilde{\sigma}}$ are isomorphic.*

Proof Suppose that $\widetilde{\rho}$ and $\widetilde{\sigma}$ are cocycle conjugate E_0-semigroups. By using the above remark, we may assume that there exists a $\widetilde{\sigma}$-cocycle $\{u_t \mid t \geq 0\}$ such that

$$\rho_t(x) = u_t \sigma_t(x) u_t^*, \quad x \in M, t \geq 0$$

As in [A1], we define a mapping $\Theta\colon \mathcal{E}_{\widetilde{\sigma}} \to \mathcal{E}_{\widetilde{\rho}}$ by

$$\Theta(a) = u_t a, \quad a \in \operatorname{Hom}_M(\operatorname{Id}, \rho_t), \quad t \geq 0,$$

and we claim that Θ is an isomorphism of product system.

First of all, we notice that Θ is well defined and a Borel isomorphism. Indeed, we can easily check that

$$u_t \operatorname{Hom}_M(\operatorname{Id}, \sigma_t) = \operatorname{Hom}_M(\operatorname{Id}, \rho_t), \quad t \geq 0,$$

so Θ is well defined and bijective. Moreover, since $\{u_t \mid t \in \mathbb{R}_+\}$ is a strongly continuous family of unitaries, it follows that Θ and Θ^{-1} are measurable mappings, so Θ is a Borel isomorphism.

Secondly, if $a \in \operatorname{Hom}_M(\operatorname{Id}, \sigma_s)$, $b \in \operatorname{Hom}_M(\operatorname{Id}, \sigma_t)$, where $s, t \geq 0$, then we have:

$$\Theta(ab) = u_{s+t}ab = u_s\sigma_s(u_t)ab = u_s a u_t b = \Theta(a)\Theta(b).$$

Therefore Θ is multiplicative.

Finally, by definition $\Theta|_{\operatorname{Hom}_M(\operatorname{Id},\sigma_t)}$ is a unitary operator. Hence Θ is an isomorphism of product systems.

Conversely, assume that the associated product systems $\mathcal{E}_{\widetilde{\rho}}$ and $\mathcal{E}_{\widetilde{\sigma}}$ are isomorphic, and let $\Theta\colon \mathcal{E}_{\widetilde{\sigma}} \to \mathcal{E}_{\widetilde{\rho}}$ be such an isomorphism. As in the proof of Theorem 2.4, for each $t > 0$, let $\{v_i(t)\}_i$ be an implementing set of the ∞-inner endomorphism σ_t such that the mapping $t \mapsto v_i(t)$ is measurable for every $i \in \mathbb{N}$.

For any fixed t, we define

$$u_t = \sum_i \Theta(v_i(t))v_i(t)^* \in M.$$

Then for any $j \in \mathbb{N}$, we have

$$u_t v_j(t) = \sum_i \Theta(v_i(t))v_i(t)^* v_j(t) = \Theta(v_j(t)),$$

and since $\operatorname{Hom}_M(\operatorname{Id}, \sigma_t) = \overline{\operatorname{span}}\{v_i(t) \mid i \in \mathbb{N}\}$, we obtain that

$$\Theta(a) = u_t a, \text{ for all } a \in \operatorname{Hom}_M(\operatorname{Id}, \sigma_t), \quad t \geq 0.$$

We claim that $\{u_t \mid t \geq 0\}$ is a $\widetilde{\sigma}$-cocycle.

First of all, we note that for any $t \geq 0$, u_t is a unitary of M. Indeed,

$$\begin{aligned} u_t^* u_t &= \Bigl(\sum_i \Theta(v_i(t))v_i(t)^*\Bigr)^* \Bigl(\sum_j \Theta(v_j(t))v_j(t)^*\Bigr) \\ &= \sum_{i,j} v_i(t)\Theta(v_i(t))^*\Theta(v_j(t))v_j(t)^* = \sum_{i,j} \langle \Theta(v_j(t)), \Theta(v_i(t))\rangle_M \cdot v_i(t)v_j(t)^* \\ &= \sum_{i,j} \langle v_j(t), v_i(t)\rangle_M \cdot v_i(t)v_j(t)^* = \sum_{i,j} v_i(t)v_j(t)^* v_i(t)v_j(t)^* \\ &= \sum_i v_i(t)v_i(t)^* = 1_M, \end{aligned}$$

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and similarly $u_t u_t^* = 1_M$.

Secondly, we assert that $\{u_t\}_{t\in\mathbb{R}_+}$ satisfies the $\widetilde{\sigma}$-cocycle relation. For proving this, we let $i, j \in \mathbb{N}$. Then for all $s, t \in \mathbb{R}_+$, we have

$$\begin{aligned} u_s\sigma_s(u_t)v_i(s)v_j(t) &= u_s v_i(s) u_t v_j(t) = \Theta(v_i(s))\Theta(v_j(t)) \\ &= \Theta(v_i(s)v_j(t)) = u_{s+t}v_i(s)v_j(t), \end{aligned}$$

where the last equality follows from the fact that $v_i(s)v_j(t) \in \operatorname{Hom}_M(\operatorname{Id}, \sigma_{s+t})$.

We then have

$$\begin{aligned} u_s\sigma_s(u_t)v_i(s) &= u_s\sigma_s(u_t)v_i(s)\cdot 1_M = \sum_{j\in\mathbb{N}} u_s\sigma_s(u_t)v_i(s)v_j(t)v_j(t)^* \\ &= \sum_{j\in\mathbb{N}} u_{s+t}v_i(s)v_j(t)v_j(t)^* = u_{s+t}v_i(s), \end{aligned}$$

and again,

$$\begin{aligned} u_s\sigma_s(u_t) &= u_s\sigma_s(u_t)\cdot 1_M = \sum_i u_s\sigma_s(u_t)v_i(s)v_i(s)^* \\ &= \sum_i u_{s+t}v_i(s)v_i(s)^* = u_{s+t}. \end{aligned}$$

Finally, since the the mappings $t \mapsto v_i(t)$ are measurable, we obtain that the mapping $t \mapsto \varphi(u_t)$ is also measurable for every normal state φ of M. Thus, by applying [A1, Proposition 2.5], we obtain that $\{u_t \mid t \in \mathbb{R}_+^*\}$ is strongly continuous, so $\{u_t \mid t \in \mathbb{R}_+^*\}$ is a $\widetilde{\sigma}$-cocycle.

To complete the proof, we shall show that for any $t \geq 0$ and $x \in M$, we have $\rho_t(x) = u_t\sigma_t(x)u_t^*$. Indeed,

$$\begin{aligned} \rho_t(x)u_t &= \sum_i \rho_t(x)\Theta(v_i(t))v_i(t)^* = \sum_i \Theta(v_i(t))xv_i(t)^* \\ &= \sum_i \Theta(v_i(t))v_i(t)^*\sigma_t(x) = u_t\sigma_t(x). \end{aligned}$$

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