SPECTRA OF CONJUGATED IDEALS IN GROUP ALGEBRAS OF ABELIAN GROUPS OF FINITE RANK AND CONTROL THEOREMS

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(Received 15 March, 1995; revised 22 August, 1995)

1. Introduction. Throughout k will denote a field. If a group Γ acts on a set A we say an element is Γ-*orbital* if its orbit is finite and write $\Delta_\Gamma(A)$ for the subset of such elements. Let I be an ideal of a group algebra kA; we denote by I^+ the normal subgroup $(I+1)\cap A$ of A. A subgroup B of an abelian torsion-free group A is said to be *dense* in A if A/B is a torsion-group. Let I be an ideal of a commutative ring K; then the spectrum $\mathrm{Sp}(I)$ of I is the set of all prime ideals P of K such that $I \leq P$. If R is a ring, M is an R-module and $x \in M$ we denote by $\mathcal{A}_R(x)$ the annihilator of x in R. We recall that a group Γ is said to have finite *torsion-free* rank if it has a finite series in which each factor is either infinite cyclic or locally finite; its torsion-free rank $r_0(\Gamma)$ is then defined to be the number of infinite cyclic factors in such a series.

Let A be an abelian torsion-free group of finite rank acted upon by a group Γ and let I be an ideal of kA. The subgroup $S_\Gamma(A)$ of Γ of elements γ such that $I\cap kB = I^\gamma \cap kB$ for some finitely generated dense subgroup B of A is said to be the *standardiser* of I. We will say that an ideal I of kA is *locally prime* if $I \cap kB$ is a prime ideal of kB for some dense finitely generated subgroup B of A. It easily follows from Wilson's version [**18**, Section 3.11] of an important theorem of Brookes [**1**, Theorem A] that if $\Delta_\Gamma(A)=1$, I is a locally prime ideal of kA and $S_\Gamma(I)=\Gamma$, then $I^+\neq 1$. But, of course, I^+ may contain no non-trivial Γ-invariant subgroup.

Let G be a group with a torsion-free abelian normal subgroup A of finite rank. In [**12**, Theorem E] Nabney proved that if M is any kG-module which is not kA-torsion-free then there is an element $a\in M\setminus\{0\}$ such that $akG=(akS)\otimes_{kS}kG$, where $S=S_G(P)$ for some $P\in \mathrm{Sp}(\mathcal{A}_{kA}(a))$. But, generally, if G has finite torsion-free rank then $r_0(S/C_s(akS))$ may be the same as $r_0(G)$ for any $a\in M\setminus\{0\}$. However, it would be very useful to find such a subgroup H of G that $akG=(akH)\otimes_{kH}kG$ and $r_0(H/C_H(akH))<r_0(G)$ for some $a\in M\setminus\{0\}$, because it would be possible to use induction on $r_0(G)$ for the study of M then. The search for such a subgroup H is the main aim of this paper. In the case of a polycyclic group G this approach was applied by Roseblade in [**14**].

Let A be an abelian torsion-free group of finite rank acted upon by a group Γ and let I be an ideal of kA. We say that a subgroup Λ of $S_\Gamma(I)$ *separates* I if $\mathrm{Sp}(I)\cap \mathrm{Sp}(I^\gamma)=\varnothing$ for any $\gamma\in S_\Gamma(I)$ which is not contained in Λ. It is not difficult to note, that the intersection $\mathrm{Sep}_\Gamma(I)$ of all subgroups separating I also separates I; $\mathrm{Sep}_\Gamma(I)$ will be called the *separator* of I. Evidently, $\mathrm{Sep}_\Gamma(I)\leq S_\Gamma(I)$. We prove that if k is a field of characteristic zero, Γ is a soluble group of finite torsion-free rank and M is a kA-module such that $\mathcal{A}_{kA}(x)$ is a non-zero locally prime ideal of kA and $r_0(\Gamma)=r_0(\mathrm{Sep}_\Gamma(\mathcal{A}_{kA}(x))$ for some element $x\in M\setminus\{0\}$ then there is an element $y\in M\setminus\{0\}$ such that $\mathcal{A}^+_{kA}(y)$ has a non-trivial $\mathrm{Sep}_\Gamma(\mathcal{A}_{kA}(y))$-invariant subgroup (Theorem 3.8). This theorem allows us to obtain our main result—a control theorem for modules over group algebras of soluble gorups of finite torsion-free rank (which will be our Theorem 4.2).

Glasgow Math. J. **38** (1996) 309–320.

THEOREM. *Let G be a soluble group of finite torsion-free rank and let A be a torsion-free abelian normal subgroup of G such that $\Delta_G(A)=1$. Let k be a field of characteristic zero and let M be a kG-module. If M is not kA-torsion-free then there is an element $a \in M\setminus\{0\}$ such that $akG=(akH)\otimes_{kH}kG$ and $r_0(H/C_H(akH))<r_0(G)$, where $H=\mathrm{Sep}_G(\mathscr{A}_{kA}(a))$.*

We should note that some other approaches to control theorems for modules over group rings of infinite groups were developed by Brookes and Brown (see [**2**] and [**3**]).

We recall that a group G has finite *Prüfer rank* if there is an integer r such that each finitely generated subgroup of G can be generated by r elements; its Prüfer rank $r(G)$ is then the least integer r with this property. As an application of Theorem 4.2, we consider faithful irreducible representations of a finitely generated metabelian group G of finite Prüfer rank over a field k of characteristic zero. We prove that if G is not nilpotent-by-finite then each such representation is induced from an irreducible representation of a subgroup $H\leq G$ such that $r_0(H)<r_0(G)$ (Theorem 5.5). If G is an abelian-by-cyclic group it implies that any faithful irreducible representation of G over k is induced from an irreducible representation of an abelian subgroup of G (Corollary 5.6). Irreducible representations of some abelian-by-cyclic groups were considered by Musson in [**10**]. Irreducible representations of finitely generated nilpotent groups were considered by Harper [**7**] and by Segal [**15**], and irreducible representations of polycyclic groups were considered by Harper [**8**] and by Musson [**11**].

By [**6**], any finitely generated metabelian group of finite Prüfer rank is a minimax abelian-by-polycyclic group. A minimax group is a group with a finite series each of whose factors satisfies either the minimal condition or the maximal condition for subgroups. Irreducible representations of minimax abelian-by-polycyclic groups under certain additional conditions where considered by Nabney [**12**].

2. Some properties of Černikov modules. This section is auxiliary; its main result (Proposition 2.6) will be used in the proof of Theorem 3.5.

Let R be a ring. An R-module A is said to be cyclic if it is generated by one element. By the *socle* $\mathrm{Soc}(A)$ of an R-module A we mean the submodule of A which is generated by the minimal submodules of A; if A has no minimal submodule then $\mathrm{Soc}(A)=0$.

LEMMA 2.1. *Let A be a $\mathbb{F}_p[g]$-module. Then the module A is cyclic if and only if $\mathrm{Soc}(A)$ is cyclic.*

Proof. This assertion holds because $\mathbb{F}_p[g]$ is a principal ideals domain. □

Let R be a ring. An R-module A is said to be *Černikov* if its additive group is Černikov, that is, a direct sum of finitely many cyclic and quasi-cyclic groups (see [**9**]). If the additive group of A is a p-group then $\Omega_n(A)$ is the submodule of A which consists of all elements $x\in A$ such that $xp^n=0$, where $n\in\mathbb{N}$.

Let R be a ring. An infinite R-module A is said to be *minimal infinite* (or *m.i.-module*) if any proper submodule of A is finite. It is not difficult to show that if A is a Černikov m.i.-module then A is a divisible p-group.

LEMMA 2.2. *Let A be a Černikov $\mathbb{Z}[g]$-module. Suppose that the additive group of A is a p-group and the socle of A is cyclic. Then for any m.i.-submodule B of A the socle of the quotient module A/B is cyclic.*

Proof. Obviously, it is sufficient to show that the socle of $\Omega_1(A/B)$ is cyclic. Since B is a divisible group, it is not too difficult to show that $\Omega_1(A/B) = (\Omega_1(A) + B)/B \simeq \Omega_1(A)/(\Omega_1(A) \cap B)$. As $\Omega_1(A)$ has cyclic socle, it easily follows from Lemma 2.1 that the quotient module $\Omega_1(A/B) \simeq \Omega_1(A)/(\Omega_1(A) \cap B)$ has cyclic socle. □

LEMMA 2.3. *Let A be a Černikov $\mathbb{Z}[g]$-module and let $\mathscr{A}$ be the group of $\mathbb{Z}[g]$-automorphisms of A. Suppose that A is a divisible p-group. Then*:

(i) *if* $\mathrm{Soc}(A)$ *is cyclic then $\mathscr{A}$ is abelian*;

(ii) *for any finite submodule X of $A \oplus A$ the socle of the quotient module $(A \oplus A)/X$ is not cyclic.*

Proof. (*i*) Since $\mathrm{Soc}(A)$ is cyclic and, evidently, $\mathrm{Soc}(A) = \mathrm{Soc}(\Omega_1(A))$, by Lemma 2.1, $\Omega_1(A)$ is cyclic. It easily follows that $\Omega_n(A)$ is cyclic for each $n \in \mathbb{N}$. Then $\Omega_n(A) \simeq K_n = \mathbb{Z}[g]/I_n$ for each $n \in \mathbb{N}$, where I_n is an ideal of $\mathbb{Z}[g]$. Let $\mathscr{A}_n$ be the group of $\mathbb{Z}[g]$-automorphisms of $\Omega_n(A)$; it is well known that $\mathscr{A}_n \simeq U(K_n)$, where $U(K_n)$ is the group of units of K_n, and hence $\mathscr{A}_n$ is abelian. As $\mathscr{A}/C_{\mathscr{A}}(\Omega_n(A)) \leq \mathscr{A}_n$ and $\bigcap_{n \in \mathbb{N}} C_{\mathscr{A}}(\Omega_n(A)) = 1$, it follows that $\mathscr{A}$ is abelian.

(ii) Suppose that for some finite submodule X of $B = A \oplus A$ the socle of B/X is cyclic. Let $\mathscr{A}$ be the group of $\mathbb{Z}[g]$-automorphisms of B and $\mathscr{B}$ be the group of $\mathbb{Z}[g]$-automorphisms of B/X. Then, by (i), $\mathscr{B}$ is abelian. Let $\mathscr{N}$ be the normalizer of X in $\mathscr{A}$ then, as $X \leq \Omega_n(B)$ for some $n \in \mathbb{N}$, it is not difficult to show that $|\mathscr{A}:\mathscr{N}| < \infty$. As each $\nu \in \mathscr{N}$ induces a $\mathbb{Z}[g]$-automorphism of B/X, there is a homomorphism φ: $\mathscr{N} \to \mathscr{B}$ such that $\ker \varphi = C_{\mathscr{N}}(B/X)$. Let $\alpha \in \ker \varphi$; then $B(1-\alpha) \leq X$ and hence, as X is finite, there is $n \in \mathbb{N}$ such that $Bp^n(1-\alpha) = 0$. Therefore, as the additive group of A is divisible, $B(1-\alpha) = 0$ and hence $\alpha = 1$. So, $\mathscr{N} \leq \mathscr{B}$ because $\ker \varphi = 1$. Thus $\mathscr{N}$ is an abelian group and hence $\mathscr{A}$ is an almost abelian group.

On the other hand, evidently, $\mathscr{A}$ contains the linear group $GL_2(\mathbb{Z})$ and it is well known that $GL_2(\mathbb{Z})$ is not almost abelian. This is a contradiction. □

LEMMA 2.4. *Let A be a Černikov $\mathbb{Z}[g]$-module and let $\mathscr{M}$ be the set of all m.i.-submodules of A. If the socle of A is cyclic then $\mathscr{M}$ is finite.*

Proof. As any submodule of A is the direct sum of its Sylow components, any m.i.-submodule of A is contained in some Sylow component of A. Thus we may assume that the additive group of A is a p-group.

The proof is by induction on Prüfer rank of the additive group of A. Let B be an m.i.-submodule of A. Then, by Lemma 2.2, $\mathrm{Soc}(A/B)$ is a cyclic $\mathbb{Z}[g]$-module and hence, by the induction hypothesis, the set of all m.i.-submodules of A/B is finite. Thus it is sufficient to consider the case when A/B is an m.i.-module.

Suppose that $\mathscr{M}$ is infinite and let $A_i \in \mathscr{M}$, $A_i \neq B$, where $i = 1, 2$. Put $X_i = A_i \cap B$; then $|X_i| < \infty$ and

$$A_i/X_i \simeq (A_i + B)/B = A/B \tag{1}$$

where $i = 1, 2$. Put $X = X_1 + X_2$, $\hat{A} = A/X$ and $\hat{A}_i = (A_i + X)/X$; then, as $A_i \cap X = X_i$, by

https://doi.org/10.1017/S0017089500031736 Published online by Cambridge University Press

(1), $\hat{A}_1 \simeq \hat{A}_2$. Evidently, $\hat{A} = \hat{A}_1 + \hat{A}_2$; then it is not difficult to show that there is a finite submodule $Y \leq \hat{A}_1 \oplus \hat{A}_2$ such that

$$\hat{A} \simeq \hat{A}_1 \oplus \hat{A}_2/Y \tag{2}$$

On the other hand, there is $n \in \mathbb{N}$ such that $X \leq \Omega_n(A)$ and hence, as $A \simeq A/\Omega_n(A)$, there is a finite submodule $Z \leq \hat{A}$ such that $\hat{A}/Z \simeq A$. Then, by (2), $(\hat{A}_1 \oplus \hat{A}_2)/D \simeq A$ for some finite submodule $D \leq \hat{A}_1 \oplus \hat{A}_2$ but this contradicts Lemma 2.3 (ii). □

LEMMA 2.5. *Let A be a Černikov $\mathbb{Z}[g]$-module and let k be a field of characteristic zero. Let M be a kA-module, $x \in M$ and $P \in \mathrm{Sp}(\mathscr{A}_{kA}(x))$. Then;*

(i) *for any finite subgroup $B \leq A$ there is an element $y \in M$ such that $\mathscr{A}_{kA}(y) \cap kB = P \cap kB = D$ is a maximal ideal of kB and $P \geq \mathscr{A}_{kA}(y)$;*

(ii) *if A' is an m.i.-submodule of A and P^+ does not contain A' then there is an element $y \in M$ such that for any $L \in \mathrm{Sp}(\mathscr{A}_{kA}(y))$ L^+ does not contain A'.*

Proof. (i) Put $D = P \cap kB$ then $D \in \mathrm{Sp}(\mathscr{A}_{kB}(x))$. By Maschke's theorem, $T = xkB$ is a semisimple kB-module. Then there is a simple submodule $S \leq T$ which is annihilated by D. Thus y may be chosen as a non-zero element of S. Evidently, $\mathrm{Sp}(\mathscr{A}_{kA}(y))$ consists of all $L \in \mathrm{Sp}(\mathscr{A}_{kA}(x))$ such that $L \cap kB = D$ and hence $P \geq \mathscr{A}_{kA}(y)$.

(ii) Evidently, there is a finite subgroup $B \leq A'$ which is not contained in P^+. By (i), there is an element $y \in M$ such that $\mathscr{A}_{kA}(y) \cap kB = P \cap kB = D$. As D is a maximal ideal of kB, $P \cap kB = D$ for any $L \in \mathrm{Sp}(\mathscr{A}_{kA}(y))$. Therefore, for any $L \in \mathrm{Sp}(\mathscr{A}_{kA}(y))$, L^+ does not contain B and hence L^+ does not contain A'. □

PROPOSITION 2.6. *Let $A = \oplus_{i=1}^n A_i$ be a Černikov $\mathbb{Z}[g]$-module such that $\mathrm{Soc}(A_i)$ is cyclic for each i. Let k be a field of characteristic zero, and let M be a kA-module. Then there is an element $a \in M \setminus \{0\}$ such that for any $x \in akA$ and, for each $1 \leq i \leq n$, $kC_i \cap \mathscr{A}_{kA}(x) = P_i$ is a maximal ideal of kC_i, where $C_i/H_i = \mathrm{Soc}(A_i/H_i)$ and H_i is the maximal g-invariant subgroup of $\mathscr{A}^+_{kA}(x) \cap A_i$.*

Proof. The proof is by induction on n.

Consider first the case where $n = 1$. The proof is by induction on Prüfer rank of the additive group of A. Suppose that there is an element $x \in M \setminus \{0\}$ such that $\mathscr{A}^+_{kA}(x)$ has an m.i.-submodule A'. Then xkA may be considered as a $k(A/A')$-module and, by Lemma 2.2, we may use the induction hypothesis. Thus we may assume that $\mathscr{A}^+_{kA}(x)$ contains no m.i.-submodule for any $x \in M \setminus \{0\}$.

Let $\mathscr{M} = \{A_1, \ldots, A_m\}$ be the set of all m.i.-submodules of A; by Lemma 2.4, $\mathscr{M}$ is finite. We will show by induction on m that there is an element $y \in M$ such that for any $P \in \mathrm{Sp}(\mathscr{A}_{kA}(y))$ P^+ contains no m.i.-submodule. Suppose that there is $x \in M$ such that for any $P \in \mathrm{Sp}(\mathscr{A}_{kA}(x))$ P^+ does not contain submodules $A_1, \ldots, A_{m-1}$. It easily follows from Maschke's theorem that the quotient ring $kA/\mathscr{A}_{kA}(x)$ has no nilpotent element and hence it is semiprime. So, $\mathscr{A}_{kA}(x)$ is the intersection of all $P \in \mathrm{Sp}(\mathscr{A}_{kA}(x))$. Then, as $\mathscr{A}^+_{kA}(x)$ does not contain A_m, there is $P \in \mathrm{Sp}(\mathscr{A}_{kA}(x))$ such that P^+ does not contain A_m. Therefore, by Lemma 2.5(ii), there is an element $y \in xkA$ such that for any $P \in \mathrm{Sp}(\mathscr{A}_{kA}(y))$ P^+ does not contain A_m. As $y \in xkA$, $\mathscr{A}_{kA}(x) \leq \mathscr{A}_{kA}(y)$. Therefore, $\mathrm{Sp}(\mathscr{A}_{kA}(y)) \subseteq \mathrm{Sp}(\mathscr{A}_{kA}(x))$ and hence P^+ contains no m.i.-submodule for any $P \in \mathrm{Sp}(\mathscr{A}_{kA}(y))$. Evidently, $y \neq 0$.

Let $P \in \mathrm{Sp}(\mathscr{A}_{kA}(y))$ and let H be the maximal g-invariant subgroup of P^+. As P^+

contains no m.i.-submodule, H is finite. Put $C/H = \mathrm{Soc}(A/H)$ then $|C| < \infty$ and hence, by Lemma 2.5(i), there is an element $a \in M\setminus\{0\}$ such that $\mathcal{A}_{kA}(a) \cap kC = P \cap kC$ and, as $P \geq \mathcal{A}_{kA}(a)$, H is the maximal g-invariant subgroup of $\mathcal{A}_{kA}(a)$. Let $x \in akA$; then $\mathcal{A}_{kA}(x) \cap kC \geq \mathcal{A}_{kA}(a) \cap kC = D$ and, as D is a maximal ideal of kC, $\mathcal{A}_{kA}(x) \cap kC = D$. Let X be the maximal g-invariant subgroup of $\mathcal{A}_{kA}(x)$; then, as $\mathcal{A}_{kA}(x) \geq \mathcal{A}_{kA}(a)$, $X \geq H$. Suppose that $X \neq H$; then, as C/H is the socle of A/H, $L = C \cap X > H$. Evidently, $L \leq D^+ \leq P^+$ but this is a contradiction, because H is the maximal g-invariant subgroup of P^+.

Consider now the general case. By the induction hypothesis, there is an element $b \in M\setminus\{0\}$ such that for any element $x \in bKa$ $kC_i \cap \mathcal{A}_{kA}(x) = P_i$ is a maximal ideal in kC_i, where $C_i/H_i = \mathrm{Soc}(A_i/H_i)$, H_i is the maximal g-invariant subgroup of $\mathcal{A}^+_{kA}(x) \cap A_i$ and $2 \leq i \leq n$. By the same arguments, there is an element $a \in bkA\setminus\{0\}$, such that $kC_1 \cap \mathcal{A}_{kA}(x) = P_1$ is a maximal ideal in kC_1 for any element $x \in akA$, where $C_1/H_1 = \mathrm{Soc}(A_1/H_1)$ and H_1 is the maximal g-invariant subgroup of $\mathcal{A}^+_{kA}(x) \cap A_1$. Let $x \in akA$; then, as $a \in bkA$, $x \in bkA$. Thus $kC_i \cap \mathcal{A}_{kA}(x) = P_i$ is a maximal ideal in kC_i, where $C_i/H_I = \mathrm{Soc}(A_i/H_i)$, H_i is the maximal g-invariant subgroup of $\mathcal{A}^+_{kA}(x) \cap A_i$ and $1 \leq i \leq n$. □

3. On spectra of conjugated ideals of group algebras of abelian groups of finite rank.

LEMMA 3.1. *Let A be an abelian group acted upon by a group Γ, and B be a Γ-invariant subgroup of A. Let k be a field and let I be an ideal of kA. Then*:

(i) *if* $\mathrm{Sp}(I_1^\gamma) \cap \mathrm{Sp}(I_1) = \varnothing$ *then* $\mathrm{Sp}(I^\gamma) \cap \mathrm{Sp}(I) = \varnothing$, *where* $\gamma \in \Gamma$ *and* $I_1 = I \cap kB$;

(ii) *suppose that* $B \leq I^+$ *and let* $\Delta \leq I$ *be the ideal of* kA *generated by* $1 - B$. *Put* $\hat{I} = I/\Delta$. *Then* $\mathrm{Sp}(\hat{I}^\gamma) \cap \mathrm{Sp}(\hat{I}) = \varnothing$ *if* $\mathrm{Sp}(I^\gamma) \cap \mathrm{Sp}(I) = \varnothing$, *where* $\gamma \in \Gamma$.

Proof. (i) Suppose that there is $P \in \mathrm{Sp}(I^\gamma) \cap \mathrm{Sp}(I)$. Then, as $I_1^\gamma = I^\gamma \cap kB$, $P_1 \in \mathrm{Sp}(I_1^\gamma) \cap \mathrm{Sp}(I_1)$, where $P_1 = P \cap kB$. This is a contradiction.

(ii) Suppose that there is $\hat{P} \in \mathrm{Sp}(\hat{I}^\gamma) \cap \mathrm{Sp}(\hat{I})$; then $\hat{P} = P/\Delta = (P_1/\Delta)^\gamma$, where P, $P_1 \in \mathrm{Sp}(I)$. As B is a Γ-invariant subgroup of A, $(P_1/\Delta)^\gamma = P_1^\gamma/\Delta$ and hence $P = P_1^\gamma$. This is a contradiction. □

LEMMA 3.2. *Let A be an abelian torsion-free group of finite rank acted upon by a soluble group Γ such that $C_\Gamma(A) = 1$. Then*:

(i) *Γ has a torsion-free normal subgroup of finite index*;

(ii) *if $A \otimes_{\mathbb{Z}} \mathbb{Q}$ is a simple $\mathbb{Q}\Gamma$-module then Γ has a free abelian normal subgroup of finite index.*

Proof. These assertions are well known properties of linear soluble groups (see **[16]**). □

LEMMA 3.3. *Let F be a finitely generated abelian group and let R be a prime ideal of $\mathbb{Z}F$ such that $K = \mathbb{Z}F/R$ is a torsion-free group of finite rank. Let J be a dense subgroup of K. Then there are a $\mathbb{Z}F$-endomorphism g of K and a $\mathbb{Z}[g]$-submodule H of J such that the quotient module K/H is Černikov and* $\mathrm{Soc}(K/H)$ *is cyclic.*

Proof. Let $\hat{K}$ be the field of fractions of K. Since $\hat{K}$ is finite-dimensional over $\mathbb{Q}$, it is well known that there is an algebraic integer $\xi \in \hat{K}$ such that $\hat{K} = \mathbb{Q}(\xi)$. As ξ is an algebraic integer, there is $n \in \mathbb{N}$, such that any element $b \in \mathbb{Z}[\xi]$ may be written in the form $b = a_0 + a_1\xi + \ldots + a_n\xi^n$, where $a_i \in \mathbb{Z}$. Since K is a dense subgroup of $\hat{K}$ and J is a dense subgroup of K, J is a dense subgroup of $\hat{K}$ and hence for any ξ^i there is $m_i \in \mathbb{N}$, such that $\xi^i m_i \in J$. Then $\mathbb{Z}[\xi]m \le J$, where $m = \Pi_{i=0}^n m_i$. Put $g = \xi m$; then $m\mathbb{Z}[g] \le \mathbb{Z}[\xi]m \le J$. As $g \in K$ and K is a ring, g can be considered as a $\mathbb{Z}F$-endomorphism of K and K can be considered as a $\mathbb{Z}[g]$-module. Put $H = m\mathbb{Z}[g]$. Evidently, $\hat{K} = \mathbb{Q}(g)$ and hence H is a dense subgroup of K and, as, by [**6**, Lemma 5.1], the group K is minimax, the quotient module K/H is Černikov. We now show that $\mathrm{Soc}(K/H)$ is cyclic. Since $K/H \le \hat{K}/H$, it is sufficient to show that $\mathrm{Soc}(\hat{K}_p/H)$ is cyclic for any Sylow p-component $\hat{K}_p/H$ of the quotient module $\hat{K}/H$. Evidently, $H/Hp \simeq \Omega_1(\hat{K}_p/H)$. As H is a cyclic $\mathbb{Z}[g]$-module, H/Hp is a cyclic $\mathbb{Z}_p[g]$-module and, as $H/Hp \simeq \Omega_1(\hat{K}_p/H)$, by Lemma 2.1, $\mathrm{Soc}(\Omega_1(\hat{K}_p/H))$ is cyclic. □

LEMMA 3.4. *Let A be an abelian group acted upon by a group Γ let k be a field and let I be an ideal of kA. Let L be a subgroup of A such that I^+ does not contain L and suppose that $P = kL \cap I$ is a maximal ideal of kL. If $\gamma \in \Gamma$ and $L \le (I^+)^\gamma$ then* $\mathrm{Sp}(I^\gamma) \cap \mathrm{Sp}(I) = \varnothing$.

Proof. Evidently, $L \le (I^\gamma)^+$ and hence $kL \cap I^\gamma = \Delta = \langle h - 1 \mid h \in L\rangle$. Then, as Δ is a maximal ideal of kI, $\mathrm{Sp}(kL \cap I^\gamma) = \{\Delta\}$. Suppose that $\mathrm{Sp}(I^\gamma) \cap \mathrm{Sp}(I) \ne \varnothing$; then $\mathrm{Sp}(I^\gamma \cap kL) \cap \mathrm{Sp}(I \cap kL) \ne \varnothing$ and hence, as $\mathrm{Sp}(I \cap kL) = \{P\}$, $\Delta = P$. Then, as $P \le I$, $\Delta \le I$ and hence $L \le I^+$. This is a contradiction. □

THEOREM 3.5. *Let A be an abelian torsion-free group of finite rank acted upon by a soluble group Γ of finite torsion-free rank and let k be a field of characteristic zero. Let M be a kA-module which contains a non-zero element x such that $\mathcal{A}^+_{kA}(x)$ is a dense subgroup of A. Then there is an element $y \in M\setminus\{0\}$ such that $\mathcal{A}^+_{kA}(y)$ has a non-trivial subgroup W such that* $\mathrm{Sp}(\mathcal{A}_{kA}(y)) \cap \mathrm{Sp}(\mathcal{A}^\gamma_{kA}(y)) = \varnothing$ if *$\gamma \in \Gamma$ and γ is not contained in $N_\Gamma(W)$, where $N_\Gamma(W)$ is the normalizer of W in Γ.*

Proof. By Lemma 3.1(i), in the proof A may be changed to any of its proper Γ-invariant subgroups. So, we can assume that A is a $\mathbb{Z}\Gamma$-module generated by one element $z \in A$ and $A \otimes_{\mathbb{Z}} \mathbb{Q}$ is a simple $\mathbb{Q}\Gamma$-module. We can also assume that $C_\Gamma(A) = 1$. Then, by Lemma 3.2(ii), Γ has a finitely generated abelian normal subgroup F of finite index and, as A is a cyclic $\mathbb{Z}\Gamma$-module, $A \simeq \mathbb{Z}\Gamma/I$, where I is a right ideal of $\mathbb{Z}\Gamma$. By Schur's Lemma, $A \otimes_{\mathbb{Z}} \mathbb{Q}$ has a simple $\mathbb{Q}F$-submodule and hence the element z may be chosen such that $I \cap \mathbb{Z}F = R$ is a prime ideal of $\mathbb{Z}F$. Put $B = \mathbb{Z}\Gamma/R\mathbb{Z}\Gamma \simeq K \otimes_{\mathbb{Z}F} \mathbb{Z}\Gamma = \bigoplus_{i=1}^n Kt_i$, where $\{t_1, \ldots, t_n\}$ is a right transversal to F in Γ and $K \simeq \mathbb{Z}F/R$. Then $A \simeq B/X$ where $X = I/R\mathbb{Z}\Gamma$. Putting $X \le C_B(M)$ we may consider M as a kB-module. It is easy to check that $\mathcal{A}^+_{kB}(x)/X = \mathcal{A}^+_{kA}(x)$. Then $\mathcal{A}^+_{kB}(x)$ is a dense subgroup of B and hence $\mathcal{A}^+_{kB}(x) \cap Kt_i = J_i$ is a dense subgroup in Kt_i for each i. Put $J = \bigcap_{i=1}^n J_i t_i^{-1}$; then J is a dense subgroup in K. By Lemma 3.3, there is an endomorphism g of the $\mathbb{Z}F$-module K such that J has a $\mathbb{Z}[g]$-submodule H such that K/H is a Černikov $\mathbb{Z}[g]$-module with cyclic socle. As $H \le J \le J_i t_i^{-1}$ and $J_i \le \mathcal{A}^+_{kB}(x)$, $Ht_i \le \mathcal{A}^+_{kB}(x)$ for each i and hence $V = \bigoplus_{i=1}^n Ht_i \le \mathcal{A}^+_{kB}(x)$. Thus $V \le C_B(xkB)$ and hence xkB can be considered as a $k(B/V)$-module. Putting

$b^g = \sum_{i=1}^{n} b_i^g t_i$ for any $b = \sum_{i=1}^{n} b_i t_i \in B$, we can consider g as a $\mathbb{Z}\Gamma$-endomorphism of B. Thus, B is a $\mathbb{Z}[g]$-module and V is a submodule of B. Then $B/V = \bigoplus_{i=1}^{n} (K/H)t_i$, where $(K/H)t_i$ is a Černikov $\mathbb{Z}[g]$-module with cyclic socle for each i. So, by Proposition 2.6, there is an element $y \in xkB \setminus \{0\}$ such that $kC_i \cap \mathscr{A}_{kB}(y) = P_i$ is a maximal ideal of kC_i, where $C_i/D_i = \mathrm{Soc}(Kt_i/D_i)$ and D_i is the maximal g-invariant subgroup of $\mathscr{A}^+_{kB}(y) \cap Kt_i$. Put $D = \bigoplus_{i=1}^{n} D_i$.

Let γ be an element of Γ which is not contained in $N_\Gamma(D)$. It is not difficult to show that, for any i, $D_i^\gamma \le Kt_j$ for some j and hence, as $D^\gamma = \bigoplus_{i=1}^{n} D_i^\gamma$ and $D^\gamma \ne D$, $D_i^\gamma \le Kt_j$ and $D_i^\gamma \ne D_j$ for some i and j. As g is a $\mathbb{Z}\Gamma$-endomorphism of B, D_i^γ is a $\mathbb{Z}[g]$-submodule of Kt_j and hence, as $D_i^\gamma \ne D_j$, $L = C_j \cap D_i^\gamma \ne D_j$. Since $L \le C_j$, $kL \cap \mathscr{A}_{kB}(y) = P$ is a maximal ideal of kL.

Suppose that D_i^γ is not contained in D_j. Then, as D_j is the maximal g-invariant subgroup of $Kt_j \cap \mathscr{A}^+_{kB}(y)$, $\mathscr{A}^+_{kB}(y)$ does not contain L. Since $L \le D_i^\gamma$, $L \le (\mathscr{A}^+_{kB}(y))^\gamma$ and, by Lemma 3.4, $\mathrm{Sp}(\mathscr{A}_{kB}(y)) \cap \mathrm{Sp}(\mathscr{A}^\gamma_{kB}(y)) = \varnothing$. If $D_i^\gamma < D_j$ then $D_j^{\gamma^{-1}}$ is not contained in D_i and the same arguments show that $\mathrm{Sp}(\mathscr{A}_{kB}(y)) \cap \mathrm{Sp}(\mathscr{A}^{\gamma^{-1}}_{kB}(y)) = \varnothing$. Therefore, as $\mathrm{Sp}(\mathscr{A}_{kB}(y))^\gamma = \mathrm{Sp}(\mathscr{A}^\gamma_{kB}(y))$, $\mathrm{Sp}(\mathscr{A}_{kB}(y)) \cap \mathrm{Sp}(\mathscr{A}^\gamma_{kB}(y)) = \varnothing$.

As $X \le C_B(xkB)$ and $y \in xkB$, $X \le C_B(ykB)$ and hence $X \le \mathscr{A}^+_{kB}(y)$. Since X is a Γ-invariant subgroup of B, $N_\Gamma(D) \le N_\Gamma(XD) = N_\Gamma(W)$, where $W = XD/X$, and hence $\mathrm{Sp}(\mathscr{A}_{kB}(y)) \cap \mathrm{Sp}(\mathscr{A}^\gamma_{kB}(y)) = \varnothing$ if γ is not contained in $N_\Gamma(W)$. Let Δ be the ideal of kB generated by $1 - X$. Then it is not difficult to show that $\mathscr{A}_{kB}(y)/\Delta = \mathscr{A}_{kA}(y)$ and the theorem follows from Lemma 3.1(ii). □

LEMMA 3.6. *Let A be an abelian torsion-free group of finite rank acted upon by a soluble group Γ of finite torsion-free rank and let K be a subgroup of Γ such that $r_0(K) = r_0(\Gamma)$. If $\Delta_\Gamma(A) = 1$ then $\Delta_K(A) = 1$.*

Proof. Evidently, we may assume that $C_\Gamma(A) = 1$. Then it easily follows from Lemma 3.2(i) that Γ has an abelian normal torsion-free subgroup H. The proof is by induction on $r_0(\Gamma)$. Suppose that $\Delta_K(A) \ne 1$. As $r_0(K) = r_0(\Gamma)$, H/V is a torsion group, where $V = H \cap K$. Let $1 \ne d \in \Delta_K(A)$ and let $D = \langle d^h \mid h \in H \rangle$; it is not difficult to show that $D \le \Delta_V(A)$. Let B be a dense finitely generated subgroup of D. As $D \le \Delta_V(A)$, $|V : C_V(B)| < \infty$ and hence $H/C_H(D)$ is a torsion-group. Then, since D is an abelian torsion-free group of finite rank, by Lemma 3.2(i), $|H/C_H(D)| < \infty$. Hence there is $n \in \mathbb{N}$ such that $H^n \le C_H(D)$. Since H^n is a normal subgroup of Γ, $C_A(H^n) = C$ is a Γ-invariant subgroup of A. Then $r_0(\Gamma/C_\Gamma(C)) < r_0(\Gamma)$ and hence, by the induction hypothesis, $\Delta_K(C) = 1$ but this is a contradiction because $D \le C$. □

LEMMA 3.7. *Let A be an abelian torsion-free group of finite rank acted upon by a group Γ such that $\Delta_\Gamma(A) = 1$. Let k be a field and let M be a kA-module. Suppose that there is an element $x \in M$ such that $\mathscr{A}_{kA}(x)$ is a non-zero locally prime ideal of kA and $S_\Gamma(\mathscr{A}_{kA}(x)) = \Gamma$. Then there is a non-trivial Γ-invariant subgroup B of A such that* $B \cap \mathscr{A}^+_{kA}(x)$ is a dense subgroup of B.

Proof. By [**18**, Section 3.11], $\mathcal{A}^+_{kA}(x) \neq 1$ and B may be chosen as the isolator of $\mathcal{A}^+_{kA}(x)$ in A. □

THEOREM 3.8. *Let A be an abelian torsion-free group of finite rank acted upon by a soluble group Γ of finite torsion-free rank such that $\Delta_\Gamma(A) = 1$. Let k be a field of characteristic zero and let M be a kA-module. Suppose that there is an element $x \in M \setminus \{0\}$ such that $\mathcal{A}_{kA}(x)$ is a non-zero locally prime ideal of kA and $r_0(\mathrm{Sep}_\Gamma(\mathcal{A}_{kA}(x))) = r_0(\Gamma)$. Then there is an element $y \in M \setminus \{0\}$ such that $\mathcal{A}^+_{kA}(y)$ has a non-trivial $\mathrm{Sep}_\Gamma(\mathcal{A}_{kA}(y))$-invariant subgroup.*

Proof. Put $\mathrm{Sep}_\Gamma(\mathcal{A}_{kA}(x)) = S$. Then, by Lemma 3.6, $\Delta_S(A) = 1$ and we may assume that $\mathrm{Sep}_\Gamma(\mathcal{A}_{kA}(x)) = \Gamma$. As $S_\Gamma(\mathcal{A}_{kA}(x)) \geqslant \mathrm{Sep}_\Gamma(\mathcal{A}_{kA}(x))$, $S_\Gamma(\mathcal{A}_{kA}(x)) = \Gamma$. By Lemma 3.7, there is a non-trivial Γ-invariant subgroup B of A such that $\mathcal{A}^+_{kA}(x) \cap B$ is a dense subgroup in B. Then the theorem easily follows from Theorem 3.5 and Lemma 3.1(i). □

4. A control theorem for modules over group algebras of soluble groups of finite rank.

PROPOSITION 4.1. *Let G be a group with abelian normal torsion-free subgroup A of finite rank, and let B be a dense finitely generated subgroup of A. Let k be a field and let M be a kG-module which is not kA-torsion-free. Then*:

(i) *there is $x \in M \setminus \{0\}$ with the prime annihilator P_0 in kB such that the transcendence degree of the fraction field of the ring kB/P_0 is minimal and hence x has maximal annihilator in kB*;

(ii) if x satisfies (i) then $xkG = xkH \otimes_{kH} kG$, *where $H = \mathrm{Sep}_G(\mathcal{A}_{kA}(x))$.*

Proof. (i) This assertion is proved in [**12**, Theorem E].

(ii) By [**12**, Theorem E] there is a prime ideal P of kA such that $P \cap kB = P_0$ and $xkG = xkS \otimes_{kS} kG$, where $S = S_G(P)$. As $P \cap kB = P_0 = \mathcal{A}_{kB}(x)$, it is not difficult to show that $S = S_G(\mathcal{A}_{kA}(x))$. Thus, it is sufficient to show that $xkS = xkH \otimes_{kH} kS$, where $H = \mathrm{Sep}_G(\mathcal{A}_{kA}(x))$. So, we may assume that $S = G$.

Put $J = \mathcal{A}_{kG}(x)$; then it is sufficient to show that $J = (J \cap kH)kG$. Suppose that $J \neq (J \cap kH)kG$; then there is an element $q \in J$ which is not belonging to $(J \cap kH)kG$. Put $q = \sum_{i=1}^{n} \left(\sum_{j=1}^{k_i} \alpha_{ij} d_{ij} \right) t_i$, where $\alpha_{ij} \in kA$, $\{d_{ij}\}$ is a part of a right transversal to A in H and $\{t_i\}$ is a part of a right transversal to H in G. The element q can be chosen such that $m = \sum_{i=1}^{n} k_i$ is minimal with respect $q \in J$ and q is not contained in $(J \cap kH)kG$. We can also assume that $t_1 = e$ and $d_{11} = e$. Put $g_{ij} = d_{ij} t_i$ then for any g_{ij} and any $\beta \in I$ the element $q\beta^{g_{ij}}$ can be written in the form: $q\beta^{g_{ij}} = \sum_{l=1}^{n} \left(\sum_{r=1}^{k_l} \hat{\alpha}_{lr} d_{lr} \right) t_l$, where $\hat{\alpha}_{lr} = \alpha_{lr} \beta^{h_{lr}} \in kA$ and $h_{lr} = g_{ij} g_{lr}^{-1}$. Therefore, as $\alpha_{ij} g_{ij} \beta^{g_{ij}} = \beta \alpha_{ij} g_{ij} \in (J \cap kH)kG$, $b = q\beta^{g_{ij}} - \alpha_{ij} g_{ij} \beta^{g_{ij}} \in J$ for any $\beta \in I$. As the number of summands in b less than m, it follows from minimality of m that $b \in (J \cap kH)kG$ and hence, as $\alpha_{ij} g_{ij} \beta^{g_{ij}} \in (J \cap kH)kG$, $q\beta^{g_{ij}} \in (J \cap kH)kG$. Therefore, as $t_1 = e$, $\sum_{r=1}^{k_1} \hat{\alpha}_{1r} d_{1r} = \left(\sum_{r=1}^{k_1} \alpha_{1r} d_{1r} \right) \beta^{g_{ij}} \in J \cap kH$. Put $c = \sum_{r=1}^{k_1} \alpha_{1r} d_{1r}$; then $c\beta^{g_{ij}} \in J$ for any $\beta \in I$ and hence $cI^{g_{ij}} \subseteq J$. Thus, $I^{g_{ij}} \subseteq \mathcal{A}_{kA}(y)$, for any g_{ij}, where $y = xc$. Then, as $g_{11} = e$, it is not difficult to show that $\mathrm{Sp}(I)\cap\mathrm{Sp}(I^{g_{ij}})\neq\emptyset$ for any $g_{ij}\neq e$. This is a contradiction, because $g_{ij}\neq e$ is not contained in H and H separates I. □

THEOREM 4.2. *Let G be a soluble group of finite torsion-free rank and let A be a torsion-free abelian normal subgroup of G such that $\Delta_G(A)=1$. Let k be a field of characteristic zero and let M be a kG-module. If M is not kA-torsion-free then there is an element $a\in M\setminus\{0\}$ such that $akG=(akH)\oplus_{kH}kG$ and $r_0(H/C_H(akH))<r_0(G)$, where $H=\mathrm{Sep}_G(\mathscr{A}_{kA}(a))$.*

Proof. Let B be a finitely generated dense subgroup of A. By Proposition 4.1(i), there is an element $x\in M\setminus\{0\}$ such that $\mathscr{A}_{kB}(x)$ is a prime ideal of kB and the transcendence degree of the fraction field of the ring $kB/\mathscr{A}_{kB}(x)$ is minimal, and hence x has maximal annihilator in kB.

Put $K=\mathrm{Sep}_G(\mathscr{A}_{kA}(x))$. Then by Proposition 4.1(ii), $xkG=xkK\otimes_{kK}kG$. If $r_0(K)<r_0(G)$ then we may put $a=x$ and $H=K$.

Suppose that $r_0(K)=r_0(G)$. Evidently, $\mathscr{A}_{kA}(x)$ is a locally prime ideal of kA. Then, by Theorem 3.8, there is an element $a\in xKA\setminus\{0\}$ such that $\mathscr{A}^+_{kA}(a)$ has a non-trivial H-invariant subgroup D, where $H=\mathrm{Sep}_G(\mathscr{A}_{kA}(a))$. Therefore, $D\leq C_H(akH)$ and hence $r_0(H/C_H(akH))<r_0(G)$. As $a\in xKA$, $\mathscr{A}_{kB}(a)\geq\mathscr{A}_{kB}(x)$, and hence, as x has maximal annihilator in kB, $\mathscr{A}_{kB}(a)=\mathscr{A}_{kB}(x)$. Thus, the theorem follows from Proposition 4.1(ii). □

5. An application.

LEMMA 5.1. *Let G be a metabelian finitely generated group of finite Prüfer rank and let B be the derived subgroup of G. If the group G is not nilpotent-by-finite then there is a normal subgroup A of G such that $A\leq B$, $\Delta_G(A)=1$ and the quotient group G/A is nilpotent-by-finite.*

Proof. The proof is by induction on $r_0(B)$. Let T be the torsion subgroup of B. As G has the maximal condition for normal subgroups (see [5]) and Prufer rank of T is finite, T is finite. If $T=B$ then the group G is abelian-by-finite. Thus we may assume that $T\neq B$. Then there is $n\in\mathbb{N}$ such that $C=B^n$ is a torsion-free subgroup. If $\Delta_G(C)=1$ then we can put $A=C$. Thus we may assume that $\Delta_G(C)\neq 1$. Put $D=\langle d^g\mid g\in G\rangle$, where d is a non-identity element of $\Delta_G(C)$. It is easy to check that if the quotient group G/D is nilpotent-by-finite then so is G. Therefore, G/D is not nilpotent-by-finite and hence, by the induction hypothesis, there is a normal subgroup E of G such that $D\leq E\leq B$ and $\Delta_G(E/D)=1$. We will consider E as a $\mathbb{Z}\Gamma$-module, where $\Gamma=G/C_G(E)$. Since $E\leq B$, Γ is an abelian group. Evidently, the subgroup E may be chosen such that $(E/D)\otimes_{\mathbb{Z}}\mathbb{Q}$ is a simple $\mathbb{Q}\Gamma$-module. Since $d\in\Delta_G(C)$, $|G:C_\Gamma(D)|<\infty$ and hence, as $\Delta_\Gamma(E/D)=1$, there is an element $\gamma\in C_\Gamma(D)$ which is not contained in $C_\Gamma(E)$. Then, as the group Γ is abelian, the mapping φ given by $\varphi:x\to x(1-\gamma)$ is a non-zero $\mathbb{Z}\Gamma$-endomorphism of E such that $D\leq\operatorname{Ker}\varphi$. Hence, as $(E/D)\otimes_{\mathbb{Z}}\mathbb{Q}\Gamma$ is a simple $\mathbb{Q}\Gamma$-module, $\operatorname{Ker}\varphi=D$. Then $L=\varphi(E)\simeq E/D$ and hence $\Delta_\Gamma(L)=1$. Thus, L is a normal subgroup of G such that $L\leq B$ and $\Delta_G(L)=1$. So, passing to the quotient group G/L we can use the induction hypothesis. □

LEMMA 5.2. *Let S be a commutative ring acted upon by a group G, let M be an*

S-module and let F be a submodule of M. Suppose that there is a non-zero element $\alpha \in S$ *such that each element of* M/F *is annihilated by some product* $\alpha^{g_1} \dots \alpha^{g_k}$ *of conjugates of* α *by elements of G. Then for any non-zero ideal L of S each element of* $(ML \cap F)/FL$ *is annihilated by some product* $\alpha^{g_1} \dots \alpha^{g_k}$ *of conjugates of* α *by elements of G.*

Proof. Each element $a \in ML \cap F$ can be written in the form: $a = \sum_{i=1}^{n} a_i l_i$, where $a_i \in M$ and $l_i \in L$. Then there is an element $x = \alpha^{g_1} \dots \alpha^{g_k}$ where $g_i \in G$ such that $a_i x \in F$ for each i and hence $ax = \sum_{i=1}^{n} a_i x l_i \in FL$. □

LEMMA 5.3. *Let A be a torsion-free abelian minimax group acted upon by an abelian group* Γ *such that* $A \otimes_{\mathbb{Z}} \mathbb{Q}$ *is a simple* $\mathbb{Q}\Gamma$*-module. Let k be a field of characteristic zero and let* α *be a non-zero element of kA. Then there is a maximal ideal L of kA such that* $|A : L^+| < \infty$ *and L contains no conjugates of* α *by elements of* Γ.

Proof. Put $\alpha = \sum_{i=1}^{n} \alpha_i t_i$, where $\alpha_i \in k$ and $t_i \in A$, and let F be a subfield of k generated by α_i; then $\alpha \in FA$. Let $\mathscr{L}$ be the set of Γ-invariant maximal ideals M of FA with $|A : M^+| < \infty$; then, by [**17**, Theorem A], the intersection of ideals from $\mathscr{L}$ is zero. It easily implies that there is $M \in \mathscr{L}$ which contains no conjugates of α by elements of Γ. Then L may be chosen as a maximal ideal of kA which contains M. □

LEMMA 5.4. *Let G be a finitely generated metabelian group of finite Prüfer rank, let k be a field and let M be a simple kG-module. Let A be an abelian torsion-free normal subgroup of G such that A is contained in the derived subgroup of G and the quotient group G/A is polycyclic. Then the module M is not kA-torsion-free.*

Proof. By [**4**, Corollary 2.1], there are a free kA-submodule F of M and a non-zero element $\alpha \in kA$ such that each element of M/F is annihilated by some product $\alpha^{g_1} \dots \alpha^{g_k}$ of conjugates of α by elements of G. Let C be a normal subgroup of G such that $C \leq A$, the quotient group A/C is torsion-free and $C \otimes_{\mathbb{Z}} \mathbb{Q}$ is a simple $\mathbb{Q}G$-module. Then the element α may be written in the form $\alpha = \sum_{i=1}^{n} \alpha_i t_i$, where $\alpha_i \in kC$ and $\{t_1, \dots, t_n\}$ is a part of a transversal to C in A. Put $\beta = \prod_{i=1}^{n} \alpha_i$; as kC has no zero divisors, $\beta \neq 0$. By [**6**, Lemma 5.1], the subgroup C is minimax and hence, as the quotient group $\Gamma = G/C_G(A)$ is abelian there is, by Lemma 5.3, a maximal ideal L of kC such that $|C : L^+| < \infty$ and L contains no conjugates of β by elements of G. It implies that kAL contains no conjugates of α by elements of G.

Since $|C : L^+| < \infty$, it is not difficult to show that L contains a non-zero G-invariant ideal I. As the ideal I is G-invariant, it is not difficult to show that MI is a submodule of M and hence, as the module M is simple, either $MI = 0$ or $MI = M$. If $MI = 0$ then the lemma holds. Thus we may assume that $MI = M$ and hence $ML = M$. Then, by Lemma 5.2, each element of F/FL is annihilated by some product $\alpha^{g_1} \dots \alpha^{g_k}$ of conjugates of α by elements of G. As F is a free kA-module, $\oplus_i (kA/kAL)_i \simeq F/FL$ and hence some such product $\alpha^{g_1} \dots \alpha^{g_k}$ is contained in kAL. It is not difficult to note that the quotient ring

kA/kaL may be considered as a crossed product (see [**13**]) of a field kC/L and the torsion-free quotient group A/C. It is well known that such a crossed product has no zero divisors and hence, as kAL contains no conjugates of α by elements of G, $\alpha^{g_1}\dots\alpha^{g_k}$ may not be contained in kAL. This is a contradiction. □

THEOREM 5.5. *Let G be a finitely generated metabelian group of finite Prüfer rank, let k be a field of characteristic zero and let M be an irreducible kG-module such that* $C_G(M)=1$. *If the group G is not nilpotent-by-finite then there are a subgroup* $H\le G$ *and an irreducible kH-submodule* $U\le M$ *such that* $M=U\otimes_{kH}kG$ *and* $r_0(H)<r_0(G)$.

Proof. The proof is by induction on $r_0(G)$. By Lemma 5.1, there is an abelian normal torsion-free subgroup $A\le G$ such that $\Delta_G(A)=1$ and the quotient group G/A is nilpotent-by-finite. As the group G is finitely generated, the quotient group G/A is polycyclic. Then, by Lemma 5.4, M is not kA-torsion-free. By Theorem 4.2, there is an element $a\in M$ such that $M=U\otimes_{kH}kG$ and $r_0(H/C_H(U))<r_0(G)$, where $U=akA$ and $H=\mathrm{Sep}_G(\mathcal{A}_{kA}(a))$. Evidently, H contains the derived subgroup of G and hence, as the quotient group G/A is polycyclic, if $|G:H|=\infty$ then $r_0(H)<r_0(G)$. Thus we may assume that $|G:H|<\infty$. Since H contains the derived subgroup of G, H is a normal subgroup of G. Then H is a finitely generated subgroup. Suppose that the quotient group $H/C_H(U)$ is nilpotent-by-finite. Let $\{t_1,\dots,t_m\}$ be a right transversal to H in G. As $M=\oplus_{i=1}^m Ut_i$,

$C_G(M)\ge\bigcap_{i=1}^{m}(C_H(U))^{t_i}=C$ and therefore, as $C_G(M)=1$, $C=1$. Then, by Remak's theorem, $\prod_{i=1}^{m}(H/C_H(U)^{t_i})\le H$. It easily follows that the subgroup H is nilpotent-by-finite and hence, as $|G:H|<\infty$, so is G, and a contradiction ensues. Thus, the quotient group $H/C_H(U)$ is not nilpotent-by-finite and we may use the induction hypothesis. □

COROLLARY 5.6. *Let G be a finitely generated group of finite Prüfer rank, and let k be a field of characteristic zero. Suppose that G is an extension of an abelian group A by a cyclic group* $\langle g\rangle$. *If the group G is not nilpotent-by-finite then every faithful irreducible representation of G over k is induced from an irreducible representation of the group A over k.*

Proof. It is not difficult to note that the subgroup H in the proof of Theorem 5.4 contains A. As $r_0(H)<r_0(G)$, it implies that $A=H$. □

ACKNOWLEDGEMENT. I am deeply grateful to the referee for his very helpful comments and advice.

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