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DERIVED FUNCTORS AND HILBERT POLYNOMIALS OVER REGULAR LOCAL RINGS

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Abstract Let $(A, \mathfrak{m})$ be a regular local ring of dimension $d \geq 1$, I an $\mathfrak{m}$-primary ideal. Let N be a nonzero finitely generated A-module. Consider the functions

$$t^I(N, n) = \sum_{i=0}^{d} \ell(\operatorname{Tor}_i^A(N, A/I^n)) \text{ and } e^I(N, n) = \sum_{i=0}^{d} \ell(\operatorname{Ext}_A^i(N, A/I^n))$$

of polynomial type and let their degrees be $t^I(N)$ and $e^I(N)$. We prove that $t^I(N) = e^I(N) = \max\{\dim N, d-1\}$. A crucial ingredient in the proof is that $D^b(A)_f$, the bounded derived category of A with finite length cohomology, has no proper thick subcategories.

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1. Introduction

In this paper, all rings considered are commutative, Noetherian, local with unity and all modules considered will be finitely generated. Let $(A, \mathfrak{m})$ be a local ring of dimension $d \geq 1$, I an $\mathfrak{m}$-primary ideal in A and let L be an A-module. If T is an A-module of finite length then we denote by $\ell(T)$ its length. The Hilbert–Samuel polynomial $n \mapsto \ell(L/I^nL)$ of L with respect to I is well-studied. It is known that it is of polynomial type and of degree $\dim L$. Considerably less is known of the function $n \mapsto \ell(\operatorname{Tor}_i^A(L, A/I^n))$ for $i \geq 1$. It is known that this function is of polynomial type and of degree $\leq d-1$. There are some results which show under certain conditions the maximal degree is attained, see [2], [4] and [7]. However this function can also be identically zero, see [7, Remark 20]. Similarly not much is known of the function $n \mapsto \ell(\operatorname{Ext}_A^i(L, A/I^n))$ for $i \geq 1$. It is known that this function is of polynomial type and of degree $\leq d-1$. There are some results which show under certain conditions the maximal degree is attained, see [1], [3]. Even less is known of the functions $n \mapsto \ell(\mathrm{Tor}_i^A(L, M/I^n M))$ and $n \mapsto \ell(\mathrm{Ext}^i_A(L, M/I^n M))$ where M is an A-module.

Perhaps the first case to consider for these functions is when A is regular. In this case, projdim N is finite for any A-module N. Surprisingly, we found out that the functions

$$t^I_M(N, n) = \sum_{i=0}^{d} \ell(\mathrm{Tor}_i^A(N, M/I^n M)) \text{ and } e^I_M(N, n) = \sum_{i=0}^{d} \ell(\mathrm{Ext}^i_A(N, A/I^n))$$

are *easier* to tackle. One can then work with $K^b(\mathrm{proj}\, A)$, the homotopy category of bounded complexes of projective A-modules, which is the bounded derived category of A. More generally, let $(A, \mathfrak{m})$ be a local ring (not necessarily regular). Let $\mathbf{X}_\bullet \colon \mathbf{X}_\bullet^{-1} \to \mathbf{X}_\bullet^0 \to \mathbf{X}_\bullet^1$ be a complex of A-modules. In [9, Proposition 3], it is shown that if $\ell(H^0(\mathbf{X}_\bullet \otimes M/I^n M))$ has finite length for all $n \geq 1$ then the function $n \to \ell(H^0(\mathbf{X}_\bullet \otimes M/I^n M))$ is of polynomial type. The precise degree of this polynomial is difficult to determine (a general upper bound for the degree is given in [9, Proposition 3]).

1.1. *In this paper, we prove a surprising result. Let $(A, \mathfrak{m})$ be a local ring and let $K^b(\mathrm{proj}\, A)$ be the homotopy category of bounded complexes of projective A-modules, Let $K^b_f(\mathrm{proj}\, A)$ denote the homotopy category of bounded complexes of projective A-modules with finite length cohomology. Let $\mathbf{X}_\bullet \in K^b_f(\mathrm{proj}\, A)$. We note that for any A-module M and an ideal I we have $\ell(H^i(\mathbf{X}_\bullet \otimes M/I^n M))$ which has finite length for all $n \geq 1$ and for all $i \in \mathbb{Z}$. The main point of this paper is that it is better to look at the function*

$$\psi^{M,I}_{\mathbf{X}_\bullet}(n) = \sum_{i \in \mathbb{Z}} \ell(H^i(\mathbf{X}_\bullet \otimes M/I^n M)), \quad \textit{for } n \geq 1.$$

We know that $\psi^{M,I}_{\mathbf{X}_\bullet}(n)$ is of polynomial type say of degree $r^M_I(\mathbf{X}_\bullet)$. The main result of this paper is

Theorem 1.2. [with hypotheses as in 1.1]. *Assume $M \neq 0$ and $I \neq A$. Then, there exists a nonnegative integer r^M_I depending only on I and M such if $\mathbf{X}_\bullet \in K^b_f(\mathrm{proj}\, A)$ is nonzero then $r^M_I(\mathbf{X}_\bullet) = r^M_I$.*

The essential reason why this happens is because $K^b_f(\mathrm{proj}\, A)$ has *no* proper thick subcategories.

1.3. *Thus, to determine $r^M_I(X)$, it suffices to compute it for a single nonzero complex $\mathbf{X}_\bullet$ in $K^b_f(\mathrm{proj}\, A)$. As a consequence of Theorem 1.2, we show*

Theorem 1.4. [with hypotheses as in Theorem 1.2]. *If $\dim M > 0$ and I is $\mathfrak{m}$-primary then $r^M_I = \dim M - 1$.*

1.5. *Let A be a Cohen–Macaulay local ring. Let $I \neq A$ be an ideal of A and let M be a nonzero A-module. If L is a nonzero module of finite length and finite projective dimension, set $t^I_M(L, n)$ and $e^I_M(L, n)$ as before. Also let $t^I_M(L)$ and $e^I_M(L)$ denote the degree of the corresponding functions of polynomial type. We show*

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Corollary 1.6. (with hypotheses as in 1.5). *Let L_1, L_2 be two nonzero modules of finite length and finite projective dimension. Then*

$$t^I_M(L_1) = t^I_M(L_2) = e^I_M(L_1) = e^I_M(L_2).$$

1.7. *We now consider the case when $\dim M > 0$ and I is $\mathfrak{m}$-primary. Let $\mathbf{X}_\bullet \in K^b(\operatorname{proj} A)$. Then by [9, Proposition 3], it follows that $\psi^{M,I}_{\mathbf{X}_\bullet}(n)$ is of degree*

$$s^M_I(\mathbf{X}_\bullet) \leq \max\{\dim H^*(\mathbf{X}_\bullet \otimes M), \dim M - 1\}.$$

Furthermore if $\dim H^*(\mathbf{X}_\bullet \otimes M) \geq \dim M$ then $s^M_I(\mathbf{X}_\bullet) = \dim H^*(\mathbf{X}_\bullet \otimes M)$. We prove

Theorem 1.8. (with hypotheses as in 1.7) *We have*

$$s^M_I(\mathbf{X}_\bullet) = \max\{\dim H^*(\mathbf{X}_\bullet \otimes M), \dim M - 1\}.$$

1.9. *Let $I \neq A$ be an $\mathfrak{m}$-primary ideal of A and let M be a A-module with $\dim M > 0$. If L is a nonzero module of finite projective dimension, set $t^I_M(L, n)$ and $e^I_M(L, n)$ as before. Also let $t^I_M(L)$ and $e^I_M(L)$ denote the degree of the corresponding functions of polynomial type. As an application of Theorem 1.8, we have*

Corollary 1.10. (with hypotheses as in 1.9). *We have*

$$t^I_M(L) = e^I_M(L) = \max\{\dim M \otimes L, \dim M - 1\}.$$

As an application of this corollary (with $N = L$ and $M = A$), we get the result stated in the abstract.

We now describe in brief the contents of this paper. In § 2, we discuss a few preliminary results. In § 3, we prove Theorem 1.2 and Corollary 1.6. In § 4, we give a proof of Theorem 1.4. In § 5, we give a proof of Theorem 1.8. Finally, in § 6, we give a proof of Corollary 1.10.

2. Preliminaries

In this section, we discuss a few preliminary results that we need. We use [6] for notation on triangulated categories. However, we will assume that if $\mathcal{C}$ is a triangulated category then $\operatorname{Hom}_{\mathcal{C}}(X, Y)$ is a set for any objects X, Y of $\mathcal{C}$.

2.1. *Let $\mathcal{C}$ be an essentially small triangulated category with shift operator Σ and let $\operatorname{Iso}(\mathcal{C})$ be the set of isomorphism classes of objects in $\mathcal{C}$. By a weak triangle function on $\mathcal{C}$, we mean a function $\xi\colon \operatorname{Iso}(\mathcal{C}) \to \mathbb{Z}$ suchthat*

(1) *$\xi(X) \geq 0$ for all $X \in \mathcal{C}$.*
(2) *$\xi(0) = 0$.*
(3) *$\xi(X \oplus Y) = \xi(X) + \xi(Y)$ for all $X, Y \in \mathcal{C}$.*
(4) *$\xi(\Sigma X) = \xi(X)$ for all $X \in \mathcal{C}$.*
(5) *If $X \to Y \to Z \to \Sigma X$ is a triangle in $\mathcal{C}$ then $\xi(Z) \leq \xi(X) + \xi(Y)$.*

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2.2. *Set*

$$\ker \xi = \{X \mid \xi(X) = 0\}.$$

The following result (essentially an observation) is a crucial ingredient in our proof of Theorem 1.2.

Lemma 2.3. (with hypotheses as above). $\ker \xi$ *is a thick subcategory of* $\mathcal{C}$.

Proof. We have

(1) $0 \in \ker \xi$.
(2) If $X \cong Y$ and $X \in \ker \xi$. Then note $\xi(Y) = \xi(X) = 0$. So $Y \in \ker \xi$.
(3) If $X \in \ker \xi$ then note $\xi(\Sigma X) = \xi(X) = 0$. So $\Sigma X \in \ker \xi$. Similarly $\Sigma^{-1} X \in \ker \xi$.
(4) If $X \to Y \to Z \to \Sigma X$ is a triangle in $\mathcal{C}$ with $X, Y \in \ker \xi$. Then note

$$0 \leq \xi(Z) \leq \xi(X) + \xi(Y) = 0 + 0 = 0.$$

So $Z \in \ker \xi$.
(5) If $X \oplus Y \in \ker \xi$ then $\xi(X) + \xi(Y) = \xi(X \oplus Y) = 0$. As $\xi(X), \xi(Y)$ are nonnegative, it follows that $\xi(X) = \xi(Y) = 0$. Thus $X, Y \in \ker \xi$.

It follows that $\ker \xi$ is a thick subcategory of $\mathcal{C}$. □

2.4. *Let* A *be a ring. Let* $K^b(\operatorname{proj} A)$ *be the homotopy category of bounded complexes of projective complexes. We index complexes cohomologically,*

$$\mathbf{X}_\bullet \colon \cdots \to \mathbf{X}_\bullet^{n-1} \to \mathbf{X}_\bullet^{n} \to \mathbf{X}_\bullet^{n+1} \to \cdots.$$

We note that $\mathbf{X}_\bullet = 0$ *in* $K^b(\operatorname{proj} A)$ *if and only if* $H^*(\mathbf{X}_\bullet) = 0$. *If* $\mathbf{X}_\bullet = 0$ *in* $K^b(\operatorname{proj} A)$ *then note that* $H^*(X \otimes N) = 0$ *for any* A*-module* N.

2.5. *Let* $K^b_f(\operatorname{proj} A)$ *denote the homotopy category of bounded complexes of projective complexes with finite length cohomology. We note that if* $\mathbf{X}_\bullet \in K^b_f(\operatorname{proj} A)$ *and* N *is an* A*-module then* $H^*(\mathbf{X}_\bullet \otimes N)$ *also has finite length. To see this if* P *is a prime ideal in* A *with* $P \neq \mathfrak{m}$ *then*

$$H^*(\mathbf{X}_\bullet \otimes_A N)_P = H^*(\mathbf{X}_{\bullet P} \otimes_{A_P} N_P) = 0 \quad \textit{as } \mathbf{X}_{\bullet P} = 0 \textit{ in } K^b(\operatorname{proj} A_P).$$

Lemma 2.6. *Let* $\mathbf{X}_\bullet \in K^b(\operatorname{proj} A)$ *be nonzero. Let* $N \neq 0$. *Then* $H^*(\mathbf{X}_\bullet \otimes N) \neq 0$.

Proof. We may assume $\mathbf{X}_\bullet$ is a minimal complex. Furthermore (after a shift), we may assume that $\mathbf{X}_\bullet^0 \neq 0$ and $\mathbf{X}_\bullet^i = 0$ for $i \geq 1$. Let $H^0(\mathbf{X}_\bullet) = E \neq 0$ since $\mathbf{X}_\bullet$ is minimal. It is straight forward to check that $H^0(\mathbf{X}_\bullet \otimes N) = E \otimes N \neq 0$. The result follows. □

2.7. *Suppose for an A-module M and an ideal I we have $\ell(H^i(\mathbf{X}_\bullet \otimes M/I^nM))$ has finite length for all $n \geq 1$ and for all $i \in \mathbb{Z}$. Consider the function*

$$\psi^{M,I}_{\mathbf{X}_\bullet}(n) = \sum_{i \in \mathbb{Z}} \ell(H^i(\mathbf{X}_\bullet \otimes M/I^nM)), \quad \text{for } n \geq 1.$$

By [9, Proposition 3] we know that $\psi^{M,I}_{\mathbf{X}_\bullet}(n)$ is of polynomial type say of degree $r^M_I(X)$ and

$$r_I(M) \leq \dim M.$$

2.8. *Let I be an $\mathfrak{m}$-primary ideal in A and let M be an A-module. An element $x \in I$ is said to be M-superficial with respect to I if there exists c such that $(I^{n+1}M : x) \cap I^cM = I^nM$ for all $n \gg 0$. Superficial elements exist when $k = A/\mathfrak{m}$ is infinite, (see [8, p. 7] for the case when $M = A$; the same proof generalizes).*

2.9. *If* $\operatorname{grade}(I, M) > 0$ *and x is M-superficial with respect to I then x is M-regular. This fact is well-known. We give a proof due to lack of a suitable reference. Let $(I^{n+1}M : x) \cap I^cM = I^nM$ for all $n \gg 0$. Let $u \in I$ be M-regular. If $xm = 0$ then $xu^cm = 0$. So $u^cm \in I^n$ for all $n \gg 0$. Thus $u^cm = 0$ and so $m = 0$.*

2.10. *A sequence $\mathbf{x} = x_1, \ldots, x_r \in M$ is said to be an M-superficial sequence if x_i is $M/(x_1, \ldots, x_{i-1})M$-superficial for $i = 1, \ldots, r$. If* $\operatorname{grade}(I, M) \geq r$ *then it follows from 2.9 that $\boldsymbol{x}$ is an A-regular sequence.*

3. Proof of Theorem 1.2 and Corollary 1.6

In this section, we give proofs of Theorem 1.2 and Corollary 1.6. We first give

Proof of Theorem 1.2. By 2.6, it follows that the function $\psi^{M,I}_{\mathbf{X}_\bullet}(n) \neq 0$ for all $n \geq 1$. Thus $r^M_I(\mathbf{X}_\bullet) \geq 0$ for all $\mathbf{X}_\bullet \neq 0$. Also by 2.7, $r^M_I(\mathbf{X}_\bullet) \leq \dim A$ for any $\mathbf{X}_\bullet \in K^b_f(\operatorname{proj} A)$. Let

$$c = \max\{r^M_I(\mathbf{X}_\bullet) \mid \mathbf{X}_\bullet \neq 0\}.$$

For $\mathbf{Y}_\bullet \in K^b(\operatorname{proj} A)_f$ define

$$\eta(\mathbf{Y}_\bullet) = \lim_{n \to \infty} \frac{c!}{n^c} \psi^{M,I}_{\mathbf{Y}_\bullet}(n).$$

Clearly $\eta(\mathbf{Y}_\bullet) \in \mathbb{Z}_{\geq 0}$. Furthermore if $\mathbf{Y}_\bullet \cong \mathbf{Z}_\bullet$ then clearly $\eta(\mathbf{Y}_\bullet) = \eta(\mathbf{Z}_\bullet)$. Thus, we have a function $\eta\colon \operatorname{Iso}(K^b_f(\operatorname{proj} A)) \to \mathbb{Z}$ where $\operatorname{Iso}(K^b_f(\operatorname{proj} A))$ denotes the set of isomorphism classes of objects in $K^b_f(\operatorname{proj} A)$.

Claim: η is a weak triangle function on $K^b_f(\operatorname{proj} A)$.

Assume the claim for the time-being. By 2.3, $\ker \eta$ is a thick subcategory of $K^b_f(\operatorname{proj} A)$. Let $\mathbf{X}_\bullet$ be such that $r^M_I(\mathbf{X}_\bullet) = c$. Then $\eta(\mathbf{X}_\bullet) > 0$. So $\mathbf{X}_\bullet \notin \ker \eta$. Thus $\ker \eta \neq K^b(\operatorname{proj} A)$. By [5, Lemma 1.2], it follows that $\ker \eta = 0$. Thus $r^M_I(\mathbf{Y}_\bullet) = c$ for any $\mathbf{Y}_\bullet \neq 0$ in $K^b_f(\operatorname{proj} A)$.

It remains to prove the claim. The first four properties of definition in 2.1 are trivial to verify. Let $\mathbf{X}_\bullet \xrightarrow{f} \mathbf{Y}_\bullet \to \mathbf{Z}_\bullet \to \mathbf{X}_\bullet[1]$ be a triangle in $K^b_f(\operatorname{proj} A)$. Then $\mathbf{Z}_\bullet \cong \operatorname{cone}(f)$ and we have an exact sequence in $C^b(\operatorname{proj} A)$

$$0 \to \mathbf{Y}_\bullet \to \operatorname{cone}(f) \to \mathbf{X}_\bullet[1] \to 0.$$

As $\mathbf{X}^i_\bullet$ are free A-modules we have an exact sequence for all $n \geq 1$,

$$0 \to \mathbf{Y}_\bullet \otimes M/I^n M \to \operatorname{cone}(f) \otimes M/I^n M \to \mathbf{X}_\bullet[1] \otimes M/I^n M \to 0.$$

Taking homology we have

$$\psi^{M,I}_{\mathbf{Z}_\bullet}(n) \leq \psi^{M,I}_{\mathbf{Y}_\bullet}(n) + \psi^{M,I}_{\mathbf{X}_\bullet[1]}(n)$$

for all $n \geq 1$. It follows that

$$\eta(\mathbf{Z}_\bullet) \leq \eta(\mathbf{Y}_\bullet) + \eta(\mathbf{X}_\bullet[1]) = \eta(\mathbf{Y}_\bullet) + \eta(\mathbf{X}_\bullet).$$

Thus, η is a weak triangle function on $K^b_f(\operatorname{proj} A)$. □

Next we give

Proof of Corollary 1.6. By Theorem 1.2, we have that there exists c with $r^M_I(\mathbf{X}_\bullet) = c$ for any nonzero $\mathbf{X}_\bullet \in K^b_f(\operatorname{proj} A)$. Let L be a nonzero finite length A-module with finite projective dimension. Let $\mathbf{Y}_\bullet$ be a minimal projective resolution of L. Then $\mathbf{Y}_\bullet \in K^b_f(\operatorname{proj} A)$ and is nonzero. It follows that $r^M_I(\mathbf{Y}_\bullet) = c$. Observe that $r^M_I(\mathbf{Y}_\bullet) = t^I_M(L)$. Set $\mathbf{Y}^*_\bullet = \operatorname{Hom}_A(\mathbf{Y}_\bullet, A)$. Note that $\mathbf{Y}^*_\bullet \in K^b_f(A)$ and is nonzero. Also observe

$$\operatorname{Ext}^*_A(L, M/I^n M) = H^*(\operatorname{Hom}_A(\mathbf{Y}_\bullet, M/I^n M) \cong H^*(\mathbf{Y}^*_\bullet \otimes_A M/I^n M).$$

Therefore

$$e^I_M(L) = r^M_I(\mathbf{Y}^*_\bullet) = c.$$

The result follows. □

4. Proof of Theorem 1.4

In this section, we assume $(A, \mathfrak{m})$ is local ring, M is an A-module with $\dim M > 0$ and I is an $\mathfrak{m}$-primary ideal. In this section, we give a proof of Theorem 1.4. We first discuss the invariant $r^M_I(A)$ under base change.

4.1. *Base change:*
(1) *We first consider a flat base change $A \to B$ where $(B, \mathfrak{n})$ is local and $\mathfrak{n} = \mathfrak{m}B$. We claim that $r^M_I(A) = r^{M\otimes_A B}_{IB}(B)$.*

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In this case, we first observe that if E is an A-module of finite length then $\ell_B(E \otimes_A B) = \ell_A(E)$. Also if $\mathbf{X}_\bullet$ is a bounded complex of A-modules with finite length cohomology then $\mathbf{X}_\bullet \otimes_A B$ is a bounded complex of B-modules with finite length cohomology and $\ell_B(H^*(\mathbf{X}_\bullet \otimes B)) = \ell_A(H^*(\mathbf{X}_\bullet))$. If $\mathbf{Y}_\bullet \in K^b_f(\operatorname{proj} A)$ then $\mathbf{Y}_\bullet \otimes_A B \in K^b_f(\operatorname{proj} B)$. Let $\mathbf{Y}_\bullet \in K^b_f(\operatorname{proj} A)$ be nonzero. Set

$$\psi^{M,I}_{\mathbf{Y}_\bullet, A}(n) = \sum_{i \in \mathbb{Z}} \ell_A(H^i(\mathbf{Y}_\bullet \otimes M/I^n M)), \quad \text{for } n \geq 1.$$

Then

$$\begin{aligned} \psi^{M \otimes_A B, IB}_{\mathbf{Y}_\bullet \otimes_A B, B}(n) &= \sum_{i \in \mathbb{Z}} \ell_B(H^i(\mathbf{Y}_\bullet \otimes_A B \otimes_B (M/I^n M \otimes_A B)) \\ &= \sum_{i \in \mathbb{Z}} \ell_B(H^i((\mathbf{Y}_\bullet \otimes_A M/I^n M) \otimes_A B)) \\ &= \psi^{M,I}_{\mathbf{Y}_\bullet, A}(n). \end{aligned}$$

It follows that degree of the function $\psi^{M,I}_{\mathbf{Y}_\bullet, A}(n)$ is equal to degree of $\psi^{M \otimes_A B, IB}_{\mathbf{Y}_\bullet \otimes_A B, B}(n)$. The result follows.

(2) If $(Q, \mathfrak{n}) \to (A, \mathfrak{m})$ is a surjective ring homomorphism and if J is any $\mathfrak{n}$-primary ideal in Q with $JA = I$ then $r^M_I(A) = r^M_J(Q)$. To see this, if $\mathbf{Y}_\bullet \in K^b_f(\operatorname{proj} Q)$ then $\mathbf{Y}_\bullet \otimes_Q A \in K^b_f(\operatorname{proj} A)$. Let $\mathbf{Y}_\bullet \in K^b_f(\operatorname{proj} Q)$ be nonzero. Set

$$\psi^{M,J}_{\mathbf{Y}_\bullet, Q}(n) = \sum_{i \in \mathbb{Z}} \ell_Q(H^i(\mathbf{Y}_\bullet \otimes_Q M/J^n M), \quad \text{for } n \geq 1.$$

Then

$$\begin{aligned} \psi^{M,I}_{\mathbf{Y}_\bullet \otimes_Q A, A}(n) &= \sum_{i \in \mathbb{Z}} \ell_A(H^i(\mathbf{Y}_\bullet \otimes_Q A \otimes_A M/I^n M) \\ &= \sum_{i \in \mathbb{Z}} \ell_Q(H^i((\mathbf{Y}_\bullet \otimes_Q M/J^n M) \\ &= \psi^{M,J}_{\mathbf{Y}_\bullet, Q}(n). \end{aligned}$$

The result follows.

(3) If $\mathfrak{q} \subseteq \operatorname{ann}_A M$ then note that M can be considered as a $C = A/\mathfrak{q}$-module. Set $J = (I + \mathfrak{q}/\mathfrak{q})$. Note J is primary to the maximal ideal of C. Then $r^M_I = r^M_J$. The proof of this assertion is similar to (2).

We now give

Proof of Theorem 1.4. By 1.7, we have $r^M_I \leq \dim M - 1$. We first do the following base-changes:

(1) If the residue field of A is finite then we set $B = A[X]_{\mathfrak{m}A[X]}$ then $(B, \mathfrak{n})$ is a flat extension of A with $\mathfrak{m}B = \mathfrak{n}$ and the residue field of B is $k(X)$ is infinite. So we replace M by $M \otimes_A B$ and I by IB (see 4.1(1)).

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(2) We then complete A (see 4.1(1)).
(3) By (1), (2) we assume A is complete with an infinite residue field. Let A be a quotient of a regular local ring Q. Then, we can replace A by Q (see 4.1(2)).
(4) By (3), we can assume A is regular local with infinite residue field. We note $a = \operatorname{grade}(\operatorname{ann} M) = \operatorname{height} \operatorname{ann} M$. Choose $y_1, \ldots, y_a \in \operatorname{ann} M$ an A-regular sequence. By 4.1(3), we can replace A with $A/(y_1, \ldots, y_a)$.

Thus, we can assume A is Cohen–Macaulay with infinite residue field and $\dim A = \dim M > 0$. Let $d = \dim A$ and let $\mathbf{x} = x_1, \ldots, x_d$ be a maximal $M \oplus A$-superficial sequence with respect to I. Then as $\mathbf{x}$ is an A-superficial sequence with respect to I it is an A-regular sequence, see 2.10. Let $\mathbf{K}_\bullet$ be the Koszul complex on $\mathbf{x}$. Then $\mathbf{K}_\bullet \in K^b_f(\operatorname{proj} A)$. We also note that as x_1 is M-superficial with respect to I there exists c and n_0 such that $(I^n M \colon x_1) \cap I^c M = I^{n-1}M$ for all $n \geq n_0$.

Set

$$\psi^{M,I}_{\mathbf{K}_\bullet, A}(n) = \sum_{i \in \mathbb{Z}} \ell_A(H^i(\mathbf{K}_\bullet \otimes M/I^n M)), \quad \text{for } n \geq 1$$

and let r be its degree. By 2.7, $r \leq d - 1$. We note that

$$H^d(\mathbf{K}_\bullet \otimes M/I^n M) = \frac{I^n M \colon \mathbf{x}}{I^n M} \supseteq \frac{(I^n M \colon \mathbf{x}) \cap I^c M}{I^n M} = \frac{I^{n-1} M}{I^n M} \ (\text{for } n \geq n_0).$$

So $\psi^{M,I}_{\mathbf{K}_\bullet, A}(n) \geq \ell(I^{n-1}M/I^n M)$ for all $n \geq n_0$. So $r \geq d-1$. Thus $r = d-1$. By Theorem 1.2, it follows that $r^M_I = r = d - 1$. □

5. Proof of Theorem 1.8

In this section, we give a proof of Theorem 1.8. We need the following well-known result. Suppose $\dim E > 0$. Then, there exists $x \in \mathfrak{m}$ such that $(0 \colon_E x)$ has finite length and $\dim E/xE = \dim E - 1$.

We now give

Proof of Theorem 1.8. By 1.7, it suffices to consider the case when $\dim H^*(\mathbf{X}_\bullet \otimes M) \leq \dim M - 1$.

We first consider the case when $\dim H^*(\mathbf{X}_\bullet \otimes M) = 0$. We prove the result by inducting on $\dim H^*(\mathbf{X}_\bullet)$. If $\dim H^*(\mathbf{X}_\bullet) = 0$ then the result follows from Theorem 1.4. If $\dim H^*(\mathbf{X}_\bullet) > 0$ then choose x such that map $H^*(\mathbf{X}_\bullet) \xrightarrow{x} H^*(\mathbf{X}_\bullet)$ has finite length kernel and $\dim H^*(\mathbf{X}_\bullet)/xH^*(\mathbf{X}_\bullet) = \dim H^*(\mathbf{X}_\bullet) - 1$. Consider the triangle $\mathbf{X}_\bullet \xrightarrow{x} \mathbf{X}_\bullet \to \mathbf{Y}_\bullet \to \mathbf{X}_\bullet[1]$. By taking long exact sequence of homology, we get an exact sequence

$$0 \to H^*(\mathbf{X}_\bullet)/xH^*(\mathbf{X}_\bullet) \to H^*(\mathbf{Y}_\bullet) \to (0 \colon_{H^*(\mathbf{X}_\bullet)} x)[1] \to 0.$$

It follows that $\dim H^*(\mathbf{Y}_\bullet) = \dim H^*(\mathbf{X}_\bullet) - 1$. Furthermore note

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$\mathbf{Z}_\bullet = \text{cone}(x, \mathbf{X}_\bullet) \cong \mathbf{Y}_\bullet$. We have an exact sequence

$$0 \to \mathbf{X}_\bullet \to \mathbf{Z}_\bullet \to \mathbf{X}_\bullet[1] \to 0.$$

As $\mathbf{X}_\bullet^i$ is free for all i we have an exact sequence for all $n \geq 0$

$$0 \to \mathbf{X}_\bullet \otimes M/I^nM \to \mathbf{Z}_\bullet \otimes M/I^nM \to \mathbf{X}_\bullet[1] \otimes M/I^nM \to 0,$$

and

$$0 \to \mathbf{X}_\bullet \otimes M \to \mathbf{Z}_\bullet \otimes M \to \mathbf{X}_\bullet[1] \otimes M \to 0.$$

By considering later short exact sequence of complexes, we get by looking at long exact sequence in homology that $\dim H^*(\mathbf{Z}_\bullet \otimes M) = 0$. So by induction hypothesis $s_I^M(\mathbf{Y}_\bullet) = \dim M - 1$. By considering all $n \geq 1$ and summing all i, we get

$$\psi^{M,I}_{\mathbf{Y}_\bullet}(n) \leq 2\psi^{M,I}_{\mathbf{X}_\bullet}(n).$$

It follows that $s_I^M(\mathbf{X}_\bullet) \geq s_I^M(\mathbf{Y}_\bullet) = \dim M - 1$. But $s_I^M(\mathbf{X}_\bullet) \leq \dim M - 1$. The result follows.

We now assume $0 < a = \dim H^*(\mathbf{X}_\bullet \otimes M) \leq \dim M - 1$ and the result is proved for complexes $\mathbf{Z}_\bullet$ with $\dim H^*(\mathbf{Z}_\bullet \otimes M) = a - 1$. Choose x such that map $H^*(\mathbf{X}_\bullet \otimes M) \xrightarrow{x} H^*(\mathbf{X}_\bullet \otimes M)$ has finite length kernel and $\dim H^*(\mathbf{X}_\bullet \otimes M)/xH^*(\mathbf{X}_\bullet \otimes M) = \dim H^*(\mathbf{X}_\bullet \otimes M) - 1$. Consider the triangle $\mathbf{X}_\bullet \xrightarrow{x} \mathbf{X}_\bullet \to \mathbf{Y}_\bullet \to \mathbf{X}_\bullet[1]$. Note $\mathbf{Z}_\bullet = \text{cone}(x, \mathbf{X}_\bullet) \cong \mathbf{Y}_\bullet$. We have an exact sequence

$$0 \to \mathbf{X}_\bullet \to \mathbf{Z}_\bullet \to \mathbf{X}_\bullet[1] \to 0.$$

As $\mathbf{X}_\bullet^i$ is free for all i, we have an exact sequence for all $n \geq 0$

$$0 \to \mathbf{X}_\bullet \otimes M/I^nM \to \mathbf{Z}_\bullet \otimes M/I^nM \to \mathbf{X}_\bullet[1] \otimes M/I^nM \to 0,$$

and

$$0 \to \mathbf{X}_\bullet \otimes M \to \mathbf{Z}_\bullet \otimes M \to \mathbf{X}_\bullet[1] \otimes M \to 0.$$

By considering the latter short exact sequence of complexes, we get by looking at long exact sequence in homology we get an exact sequence

$$0 \to H^*(\mathbf{X}_\bullet \otimes M)/x^*H^*(\mathbf{X}_\bullet \otimes M) \to H^*(\mathbf{Z}_\bullet \otimes M) \to (0 :_{H^*(\mathbf{X}_\bullet \otimes M)} x)[1] \to 0.$$

Therefore $\dim H^*(\mathbf{Z}_\bullet \otimes M) = \dim H^*(\mathbf{X}_\bullet \otimes M) - 1$. So by induction hypothesis $s_I^M(\mathbf{Y}_\bullet) = \dim M - 1$. By considering all $n \geq 1$ and summing all i, we get

$$\psi^{M,I}_{\mathbf{Y}_\bullet}(n) \leq 2\psi^{M,I}_{\mathbf{X}_\bullet}(n)$$

It follows that $s_I^M(\mathbf{X}_\bullet) \geq s_I^M(\mathbf{Y}_\bullet) = \dim M - 1$. But $s_I^M(\mathbf{X}_\bullet) \leq \dim M - 1$. The result follows. □

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6. Proof of Corollary 1.10

In this section, we give a proof of Corollary 1.10. We need the following result:

Lemma 6.1. *Let A be a Cohen–Macaulay local ring and let L be a nonzero A-module of finite projective dimension. Then*

$$\dim M \otimes L = \dim \operatorname{Ext}_A^*(L, M).$$

Proof. It is clear that $\operatorname{Supp}(M \otimes L) = \operatorname{Supp} M \cap \operatorname{Supp} L$. Thus, it follows that $\operatorname{Supp} \operatorname{Ext}_A^*(L, M) \subseteq \operatorname{Supp}(M \otimes L)$. Conversely let $P \in \operatorname{Supp} M \otimes L$. We localize at P. So it suffices to prove $\operatorname{Ext}^*(L, M) \neq 0$. By taking a minimal resolution of L, it clear that if $c = \operatorname{projdim} L$ then $\operatorname{Ext}_A^c(L, M) \neq 0$. The result follows. □

We now give

Proof of Corollary 1.10. Let $\mathbf{X}_\bullet$ be a minimal projective resolution of L. Then $t_M^I(L, n) = \ell(H^*(\mathbf{X}_\bullet \otimes M/I^n M))$. By 1.8, it follows that

$$t_M^I(L) = \max\{\dim H^*(\mathbf{X}_\bullet \otimes M), \dim M - 1\}.$$

The result follows as $\dim H^*(\mathbf{X}_\bullet \otimes M) = \dim M \otimes L$.

Set $\mathbf{X}_\bullet^* = \operatorname{Hom}_A(\mathbf{X}_\bullet, A)$. Observe

$$\operatorname{Ext}_A^*(L, M/I^n M) = H^*(\operatorname{Hom}_A(\mathbf{X}_\bullet, M/I^n M)) \cong H^*(\mathbf{X}_\bullet^* \otimes_A M/I^n M).$$

So

$$e_M^I(L) = \max\{\dim H^*(\mathbf{X}_\bullet^* \otimes M), \dim M - 1\}.$$

Notice $H^*(\mathbf{X}_\bullet^* \otimes M) = \operatorname{Ext}_A^*(L, M)$. The results follows from Lemma 6.1. □

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References

(1) A. Crabbe, D. Katz, J. Striuli and E. Theodorescu, *Hilbert-Samuel polynomials for the contravariant extension functor*, *Nagoya Math. J.* **198** (2010), 1–22.

(2) S. Iyengar and T. J. Puthenpurakal, *Hilbert-Samuel functions of modules over Cohen-Macaulay rings*, *Proc. Amer. Math. Soc.* **135**(3) (2007), 637–648.

(3) G. Kadu and T. J. Puthenpurakal, *Bass and Betti numbers of A/I^n*, *J. Algebra* **573** (2021), 620–640.

(4) D. Katz and E. Theodorescu, *On the degree of Hilbert polynomials associated to the torsion functor*, *Proc. Amer. Math. Soc.* **135**(10) (2007), 3073–3082.

(5) A. Neeman, *The chromatic tower for $D(R)$*, With an appendix by Marcel Bökstedt, *Topology* **31**(3) (1992), 519–532.

(6) A. Neeman, *Triangulated categories*, Annals of Mathematics Studies, 148, (Princeton University Press, Princeton, NJ, 2001).

https://doi.org/10.1017/S0013091524000646 Published online by Cambridge University Press

(7) T. J. Puthenpurakal, *Hilbert coefficients of a Cohen-Macaulay module*, *J. Algebra* **264**(1) (2003), 82–97.

(8) J. D. Sally, *Numbers of Generators of Ideals in Local rings* (Marcel Dekker, Inc., New York-Basel, 1978).

(9) E. Theodorescu, *Derived functors and Hilbert polynomials*, *Math. Proc. Cambridge Philos. Soc.* **132**(1) (2002), 75–88.