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WEAK MIXING MANIFOLD HOMEOMORPHISMS PRESERVING AN INFINITE MEASURE

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Introduction. Let $\mathscr{H} = \mathscr{H}(M, \mu)$ denote the group of all homeomorphisms of a σ-compact manifold which preserve a σ-finite, nonatomic, locally positive and locally finite measure μ. In two recent papers [**4**, **5**] the possible ergodicity of a homeomorphism h in $\mathscr{H}$ was shown to be related to the homeomorphism h^* induced by h on the ends of M. An end of a manifold is, roughly speaking, a distinct way of going to infinity. Those papers demonstrated in particular that $\mathscr{H}(M, \mu)$ always contains an ergodic homeomorphism, paralleling the similar result of Oxtoby and Ulam [**11**] for compact manifolds with finite measures. Unfortunately the techniques used in [**4**] and [**5**] rely on the fact that a skyscraper construction with an ergodic base transformation is ergodic, a result which cannot be extended to finer properties than ergodicity.

In this paper we use different techniques, but still related to the ends of M, to establish sufficient conditions that $\mathscr{H}$ contains homeomorphisms that are weak mixing (by which we mean ergodic Cartesian square). Actually our results apply equally well to any "typical property" $\mathscr{V}$, that is, to any conjugate-invariant property which constitutes a dense G_δ subset of the group $\mathscr{G} = \mathscr{G}(M, \mu)$ of all automorphisms of the infinite σ-finite Lebesgue space (M, μ) with respect to the coarse topology. This sufficient condition (Theorem 2) on (M, μ) is that there is a homeomorphism h in $\mathscr{H}(M, \mu)$ such that h^* is topologically weak mixing on the ends of M with infinite measure. A very special case is the manifold R^n, $n \geqq 2$, (which has a single end) with Lebesgue measure, since the identity on this singleton end space is trivially topologically weak mixing. That case was treated separately in [**2**] from a different point of view. However manifolds which have more than one but finitely many ends are never covered by our condition, since topological weak mixing is not possible on a non-singleton finite space. Indeed an important open question is whether such a manifold, for example the infinite cylinder (two ends) can support a weakly mixing homeomorphism. To show that our condition is not vacuous we give an example (the disk with a deleted Cantor set) of a manifold with uncountably many ends which supports a homeomorphism inducing topological weak mixing on the ends.

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The main new technique introduced in this paper is an extension (Theorem 1) of a purely measure theoretic "conjugacy approximation" theorem of Choksi and Kakutani [**8**, Theorem 6]. They showed that given two ergodic automorphisms τ and θ of an infinite σ-finite Lebesgue space one can always find a conjugate θ' of θ which agrees pointwise with τ on a given set of finite measure. We show that if τ additionally satisfies a certain mixing type property, then θ' can be also made to agree with τ setwise on a finite collection of infinite measured sets. In our application to manifolds these infinite measured sets correspond to the ends of the manifold.

The paper is organized as follows. Section 1 presents the definitions of the terms mentioned in this introduction and states precisely our main result (Theorem 2). Section 2 gives the proof of our extension (Theorem 1) of the Choksi-Kakutani result. In Section 3 we prove Theorem 2, and in Section 4 we give an example of a manifold with uncountably many ends which supports a weak mixing homeomorphism preserving an infinite σ-finite measure.

1. Definitions and statement of results. An end of the manifold M is a map e which assigns to every compact subset K of M a nonempty connected unbounded component $e(K)$ of $M - K$. The only restriction on the map is the monotonicity condition: $K_1 \subset K_2$ implies $e(K_2) \subset e(K_1)$. The set of all ends is denoted by E. When M is compactified by adjoining E, the relative topology on E is given by typical neighborhoods $N_K(e_0)$ of the following form. For any e_0 in E and any compact subset K of M,

$$N_K(e_0) = \{e \text{ in } E : e(K) = e_0(K)\}.$$

For any compact set K, let $\mathscr{P}_K$ be the finite partition of E obtained by letting e vary over E in $N_K(e)$. Let $\mathscr{Q}_K$ be the finite algebra generated by $\mathscr{P}_K$ and observe that

$$\mathscr{Q} = \bigcup_{K \text{compact}} \mathscr{Q}_K$$

is the algebra of clopen subsets of E. An end e in E is said to be of finite measure if $\mu(e(K)) < \infty$ for some compact subset K; otherwise e is said to have infinite measure. Let $\hat{E}$ denote the set of ends of infinite measure. Observe that $\hat{E}$ is a closed and therefore compact subset of E. Every homeomorphism h in $\mathscr{H}(M, \mu)$ induces a homeomorphism $h^*: E \to E$ such that

$$(h^*(e))(K) = h(e(h^{-1}(K)))$$

for every end e in E and compact subset K of M. In particular h^* leaves $\hat{E}$ invariant so we may consider the restriction $\hat{h}$ of h^* to $\hat{E}$. We will classify $\hat{h}$ according to its topological dynamics using the following definitions.

Definition. A homeomorphism σ of a compact space onto itself is called *topologically weak mixing* if for any two nonempty open sets U, V, the set

$$\{n:\sigma^{-n}U \cap V \neq \emptyset\}$$

is thick, i.e., contains arbitrarily long intervals. If the above condition is satisfied for all clopen sets U and V (a weaker condition) we will say that σ is *componentwise weak mixing*.

To state our main result we consider the embedding of $\mathscr{H}(M, \mu)$ in the space $\mathscr{G} = \mathscr{G}(M, \mu)$ of all bimeasurable μ-preserving automorphisms of an infinite σ-finite Lebesgue space. We endow $\mathscr{G}$ with the coarse topology, under which a sequence of automorphisms g_n in $\mathscr{G}$ converges to a limit g if and only if $\mu(g_nB \,\Delta\, gB) \to 0$ for every finite measured subset B of M.

THEOREM 2. *Suppose the space $\mathscr{H}(M, \mu)$ contains a homeomorphism h which induces a componentwise weak mixing homeomorphism on the ends of M with infinite measure. Then for any conjugate-invariant subset $\mathscr{V} \subset \mathscr{G}(M, \mu)$ which is dense and G_δ in the coarse topology, $\mathscr{V} \cap \mathscr{H}(M, \mu)$ is nonempty.*

2. Conjugacy theorem. This section is devoted to proving a generalization of the conjugacy theorem of Choksi and Kakutani (see below). This material is entirely measure theoretic, so we consider M only as an infinite σ-finite Lebesgue space, forgetting the manifold structure. Recall that $\mathscr{G} = \mathscr{G}(M, \mu)$ is the group of all μ-preserving bijections of M.

THEOREM 1 (Conjugacy Theorem). *Let $\tau, \theta \in \mathscr{G}$ with θ ergodic. Let $M = E_0 \cup E_1 \cup \ldots \cup E_n$ be a measurable partition with $0 < \mu(E_0) < \infty$ and $\mu(E_i) = \infty$ for $i = 1, \ldots, n$. Assume that*

1) *There are no non-null τ-invariant subsets of E_0, and*

2) *The $n \times n$ 0 − 1 matrix $T = T(\tau, E_1, E_2, \ldots, E_n)$, defined by $t_{ij} = 1$ if $\mu(\tau E_i \cap E_j) = \infty$ and $t_{ij} = 0$ if $\mu(\tau E_i \cap E_j) \neq \infty$, is primitive. (This means that T^N has all positive entries, for some positive integer N.)*

Then there exists $\pi \in \mathscr{G}$ such that $\theta' = \pi^{-1}\theta\pi$ satisfies

1′) *$\theta'(x) = \tau(x)$ for μ-a.e. x in E_0, and*

2′) *$\theta'(E_i) = \tau(E_i)$ for $i = 0, \ldots, n$.*

In other words, there is a conjugate of any ergodic transformation which agrees with τ pointwise on the finite measure set E_0 and agrees with τ setwise on each of the infinite measured sets. The Choksi-Kakutani Theorem [**8**, Theorem 6] established the first part, that 1) implies 1′), under the weaker assumption that θ is antiperiodic rather than ergodic. A finite measure version of the Choksi-Kakutani Theorem was proved by Alpern [**1**, Theorem 4], giving pointwise agreement on E_0, assuming that

$$\mu(E_0 \cup \tau E_0) < \mu(M) < \infty.$$

To prove Theorem 1 we will need two constructions used by Choksi and Kakutani, which we state below as Lemmas.

LEMMA 1 [**8**, Theorem 6, Step I]. *Let $\tau \in \mathcal{G}$ and a finite measured set $B \subset M$ be given. Assume that there are no non-null τ-invariant subsets of B. Then there are disjoint sets $B_{k,i}$ $k \in \mathbf{N}$, $1 \leqq i \leqq k + 1$, such that*

$$\tau B_{k,i} = B_{k,i+1} \quad 1 \leqq i \leqq k \quad \textit{and}$$

$$B = \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{k} B_{k,i}.$$

LEMMA 2 [**8**, Theorem 6, Step II]. *Let $\theta \in \mathcal{G}$ be antiperiodic and incompressible and let d_m, $m = 1, 2, \ldots,$ be nonnegative numbers with finite sum. Then there exist disjoint sets $F_{m,i}$, $m \in \mathbf{N}$, $1 \leqq i \leqq m + 1$, such that*

$$\mu(F_{m,i}) = d_m \quad \textit{for all } m, i \quad \textit{and}$$

$$\theta(F_{m,i}) = F_{m,i+1} \quad \textit{for } i \leqq m.$$

Proof of Conjugacy Theorem (Theorem 1). We first reduce the theorem to the case where the additional assumption * holds.

$$* \quad \mu(\tau E_i \cap E_j) = 0 \quad \text{if } \mu(\tau E_i \cap E_j) < \infty,$$

i.e., if $t_{ij} = 0$.

Suppose the theorem holds under assumption * and let τ, E_i satisfy the hypotheses of the theorem (but not necessarily *). For pairs $i, j \geqq 1$ with $t_{ij} = 0$ define

$$W_{ij} = \tau E_i \cap E_j.$$

Observe that the sets W_{ij} have finite measure. For all $i, j \geqq 1$ with $t_{ij} = 0$ choose measurable sets Z_{ij} satisfying $Z_{ij} \subset E_i$, $\mu(Z_{ij}) = \mu(W_{ij})$ and with $\tau(E_0)$, all the W_{ij}, and all the Z_{ij} disjoint. This is clearly possible since

$$\mu(E_i) = \infty \quad \text{for } i = 1, \ldots, n.$$

Let $\alpha \in \mathcal{G}$ transpose the pairs of sets $\tau^{-1}W_{ij}$ and Z_{ij}, for all $i, j \geqq 1$ with $t_{ij} = 0$ and be the identity off these sets. Define

$$Z = \bigcup_{t_{ij}=0} Z_{ij}, \quad \widetilde{E}_0 = E_0 \cup Z \quad \text{and}$$

$$\widetilde{E}_i = E_i - Z \quad \text{for } i = 1, \ldots, n.$$

Then the automorphism $\tilde{\tau} \in \mathcal{G}$ defined by $\tilde{\tau} = \tau\alpha$, together with the partition $\widetilde{E}_i$, $i = 0, \ldots, n$, satisfy the hypotheses of the theorem and also condition *. According to our assumption that the theorem holds with additional hypotheses *, there is an automorphism $\theta' = \pi^{-1}\theta\pi$ which satisfies 1′) and 2′) with respect to $\tilde{\tau}$ and the partition $\widetilde{E}_i$. But it is easily seen that θ' also satisfies 1′) and 2′) with respect to τ and the partition E_i, $i = 0, \ldots, n$.

So without loss of generality, we may assume that condition * holds.

Main part of proof. Let $B = E_0$ and apply Lemma 1 to produce disjoint sets $B_{k,i}$, $k \in \mathbf{N}$, $1 \leqq i \leqq k + 1$ such that

$$\tau B_{k,i} = B_{k,i+1}, \quad 1 \leqq i \leqq k, \quad \text{and}$$

$$B = \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{k} B_{k,i}.$$

Define

$$d_m = 0 \quad \text{for } m = 1, \ldots, N \quad \text{and}$$

$$d_{N+k} = \mu(B_{k,i}) \quad \text{for } k \in \mathbf{N},$$

where N is the least positive integer with $T^N > 0$, where T is the primitive matrix of transitions. Apply Lemma 2 to the $\theta \in G$ given in the theorem and the numbers d_m just defined. This yields a family of disjoint sets $F_{m,i}$, $m = N + 1, N + 2, \ldots, 1 \leqq i \leqq m + 1$, with

$$\mu(B_{k,i}) = \mu(F_{N+k,i}) \quad \text{for all } k, i \quad \text{and}$$

$$\theta F_{m,i} = F_{m,i+1}.$$

Let

$$F = \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{k} F_{N+k,i}.$$

We now define a μ-preserving invertible transformation

$$\hat{\pi}: B \cup \tau B \rightarrow F \cup \theta F$$

in such a way that

$$\hat{\pi}^{-1}\theta\hat{\pi}x = \tau x$$

for all x in $B = E_0$. Later we will extend $\hat{\pi}$ to an automorphism $\pi \in \mathcal{G}$ so as to satisfy condition 2′. Define $\hat{\pi}$ so that

$$\hat{\pi}B_{k,1} = F_{N+k,1} \quad \text{for all } k = 1, 2, \ldots.$$

Extend $\hat{\pi}$ to $B \cup \tau B$ by the following formula: If $x \in B_{k,i}$ for some i, let

$$\hat{\pi}x = \theta^{i-1}\hat{\pi}\tau^{1-i}x.$$

Since θ is ergodic, the forward and backward θ-orbits of every x in $\sim(F \cup \theta F)$ will both eventually hit $F \cup \theta F$. The forward orbit will first hit $F \cup \theta F$ at some point of $F - \theta F$, and the backward orbit, at some point of $\theta F - F$. Let $r = r(x)$ denote the length of the orbit of x in $\sim(F \cup \theta F)$ and let

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$$k = k(x) \in \{1, \ldots, r\}$$

denote the position of x in this orbit. Thus

$$\theta^{-k}(x) \in \theta F - F, \theta^{r-k+1}(x) \in F - \theta F \quad \text{and}$$

$$\theta^{l}(x) \in \sim(F \cup \theta F) \quad \text{for } -k < l < r - k + 1.$$

Partition $\theta F - F$ into n sets

$$D_i = \hat{\pi}((\tau B - B) \cap E_i) = (\theta F - F) \cap \hat{\pi} E_i, \quad i = 1, \ldots, n.$$

Similarly partition $F - \theta F$ into n sets

$$A_j = \hat{\pi}((B - \tau B) \cap \tau E_j) = (F - \theta F) \cap \hat{\pi}\tau E_j, \quad j = 1, \ldots, n.$$

Hence for some i, j with $1 \leqq i, j \leqq n$,

$$\theta^{-k}(x) \in D_i \quad \text{and} \quad \theta^{r-k+1}x \in A_j.$$

Call the set of all such points x, $S(i, j, r, k)$. It is clear that

$$\sim(F \cup \theta F) = \bigcup_{i=1}^{n} \bigcup_{j=1}^{n} \bigcup_{r=N}^{\infty} \bigcup_{k=1}^{r} S(i, j, r, k)$$

and

$$\theta S(i, j, r, k) = S(i, j, r, k + 1) \quad \text{for } k < r.$$

The reason that the orbit length r is at least N is that we chose the height of the columns $F_{N+k,i}$ to be N more than the corresponding columns of $B_{k,i}$.

The isomorphism $\hat{\pi}^{-1}$ will be extended to $\sim(F \cup \theta F)$ by assigning labels $l \in \{1, \ldots, n\}$ to the sets $S(i, j, r, k)$. Let $L(i, j, r, k)$ be the label assigned to $S(i, j, r, k)$. If

$$L(i, j, r, k) = l \text{ and } L(i, j, r, k + 1) = m$$

we will map $\pi^{-1}(S(i, j, r, k + 1))$ into

$$E_{l,m} = \tau E_l \cap E_m.$$

The labeling process involves the notion of a universal word

$$u = u_1 u_2 \ldots u_W$$

for the primitive matrix T. The word u is universal for T if

$$t_{u_p u_{p+1}} = 1 \quad \text{for } p = 1, \ldots, W - 1, \quad t_{u_W u_1} = 1$$

and for any pair i, j with $t_{ij} = 1$ there is a p with $u_p = i$ and $u_{p+1} = j$. It is easy to construct a universal word for any primitive matrix (irreducibility is sufficient).

To define $L(i, j, r, k)$ fix i, j and r, and denote

$$l_k = L(i, j, r, k) \quad \text{for } k = 1, \dots, r.$$

Denote $l_0 = i$ and $l_{r+1} = j$. For (orbit lengths) r with $N \leqq r < 2N + W$ simply choose the l_k so that

$$t_{l_k l_{k+1}} = 1 \quad \text{for } k = 0, \dots, r.$$

This is possible because $r \geqq N$ and $T^N > 0$. For all other r there is an integer a with

$$2N + aW \leqq r < 2N + (a + 1)W,$$

where W is the length of a universal word u. Choose $l_1, \dots, l_N$ so that

$$t_{l_k l_{k+1}} = 1 \quad \text{for } k = 0, \dots, N$$

where $l_{N+1} = u_1$, the first entry of u. Let

$$l_{bN+m} = u_m \quad \text{for } 1 \leqq b \leqq a \quad \text{and } m = 1, \dots, N,$$

so that l_N is followed by repetitions of the word u. Finally, define $l_{N+aW+1}, \dots, l_r$ so that

$$t_{l_k l_{k+1}} = 1 \quad \text{for } k = N + aW, \dots, r$$

where

$$l_{N+aW} = u_W \quad \text{and} \quad l_{r+1} = j.$$

The sequence $l_0, \dots, l_{r+1}$ looks like:

$$i \rightsquigarrow [u_1 \dots u_W][u_1 \dots u_W] \dots [u_1 \dots u_W] \rightsquigarrow j$$

Observe that for large values of r, most of the labels come from the repetitions of the word u.

For pairs l, m in $\{1, \dots, n\}$ with $t_{lm} = 1$, define $R(l, m)$ as the union of all sets $S(i, j, r, k)$ with

$$L(i, j, r, k - 1) = l \text{ and } L(i, j, r, k) = m.$$

(Recall that $L(i, j, r, 0) = i$ and $L(i, j, r, k + 1) = j$.) Since the pair l, m appears in that order somewhere in the universal word u, the labelling process ensures that

$$\mu(R(l, m)) = \infty.$$

The set

$$E_{l,m} = \tau E_l \cap E_m$$

also has infinite measure, because $t_{lm} = 1$. Consequently we may define an invertible μ-preserving transformation

$$\hat{\pi}_{lm}: E_{l,m} \to R(l, m).$$

If we piece together ~~the~~ maps $\hat{\pi}$ and $\hat{\pi}_{lm}$, with $t_{lm} = 1$, we obtain the automorphism $\pi \in \mathscr{G}$ required by the theorem.

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3. Proof of theorem 2. Our proof of Theorem 2 is based on viewing the space of homeomorphisms $\mathscr{H} = \mathscr{H}(M, \mu)$ as a subset of the automorphisms $\mathscr{G} = \mathscr{G}(M, \mu)$. We "extend" the compact-open topology to $\mathscr{G}(M, \mu)$ in such a way that the relative topology on $\mathscr{H}(M, \mu)$ is the usual compact-open topology. Specifically we define, for any automorphism g in $\mathscr{G}$, compact subset K of M, and positive ϵ, the sub-basic open neighborhood

$$\mathscr{C}(g, K, \epsilon) = \{f \text{ in } \mathscr{G}: d(f(x), g(x)) < \epsilon \text{ for } \mu\text{-a.e. } x \text{ in } K\},$$

where d is the metric on M (σ-compact implies metrizable). The compact-open topology on $\mathscr{G}$ thus defined, is finer than the coarse topology. The relative topology on $\mathscr{H}$ is the usual, topologically complete, compact-open topology. It is shown in our previous paper [**4**, Lemma 0] that we may restrict the compact sets K to sets called "special compact sets", since every compact set is contained in a special compact set. A compact set K is called special if it is a connected manifold with boundary, of the same dimension as M, such that $M - K$ has no bounded components and the boundary of K has μ measure zero. The significance of the special compact sets is that they are needed in the following result.

THEOREM A (Proposition 3, [**4**], see also [**10**], [**13**], or [**3**]). *Let K be a special compact subset of M, $\delta > 0$, and g in $\mathscr{G}(M, \mu)$ satisfying*

(i) $g(K) = K$

(ii) $d(x, g(x)) < \delta$ *for* μ-a.e. x *in* K

(iii) $g(P(K)) = P(K)$ *for every set of ends* P *in* $\mathscr{P}_K$.

Then any coarse topology neighborhood of g contains a compactly supported homeomorphism $\bar{h}$ in $\mathscr{H}(M, \mu)$ which satisfies (i), (ii) *and* (iii) (*with $\bar{h}$ replacing g*).

A proof of Theorem 2 can be based on the following proposition which yields an automorphism $g = h^{-1}f$ satisfying the hypotheses of Theorem A.

PROPOSITION 1. *Let h be an ergodic homeomorphism in $\mathscr{H}(M, \mu)$ whose induced end-homeomorphism $\sigma = h^*$ is componentwise weak mixing on the ends of infinite measure $\hat{E}$. Let K be a compact subset of M such that $M - K$ has no bounded components (in particular, K can be any special compact set). Let θ be any ergodic automorphism in $\mathscr{G}(M, \mu)$. Then there is an automorphism f in $\mathscr{G}(M, \mu)$ which is conjugate to θ and satisfies the following conditions*:

(i) $f(K) = h(K)$

(ii) $f(x) = h(x)$ *for* μ-a.e. x *in* K

(iii) $f(P(K)) = h(P(K))$ *for all* P *in* $\mathscr{P}_K$.

Proof. Let $\mathscr{P}_K = \{F_1, \dots, F_m, I_1, \dots, I_n\}$ where

$$\mu(F_r(K)) < \infty, \quad r = 1, \dots, m, \quad \text{and}$$

$$\mu(I_i(K)) = \infty, \quad i = 1, \dots, n.$$

Thus

$$M = K \cup F_1(K) \cup \dots \cup F_m(K) \cup I_1(K) \cup \dots \cup I_n(K)$$

is the partition of M into K and the connected components of $M - K$. Apply Theorem 1 with

$$E_0 = K \cup F_1(K) \cup \dots \cup F_m(K),$$

$$E_i = I_i(K), \; i = 1, \dots, n, \quad \tau = h, \quad \text{and } \theta = \theta.$$

Hypothesis 1) of Theorem 1 is satisfied by the assumed ergodicity of h.

We now demonstrate how condition 2) of Theorem 1 follows from the assumption that σ is componentwise weak mixing on the ends of infinite measure. To this end we first show that

$$t_{ij} = 1 \quad \text{if } \sigma \hat{I}_i \cap \hat{I}_j \neq \emptyset,$$

where $\hat{I}_r = I_r \cap \hat{E}$. So suppose there is an end e of infinite measure with e in I_i and σe in I_j. It follows that

$$\begin{aligned} h(I_i(K)) \cap I_j(K) &\supset h(e(K)) \cap \sigma e(K) \\ &= h(e(K)) \cap h(e(h^{-1}(K))) \\ &= h[e(K) \cap e(h^{-1}(K))] \\ &\supset h[e(K \cup h^{-1}(K))]. \end{aligned}$$

Consequently

$$\begin{aligned} \mu[h(I_i(K)) \cap I_j(K)] &\geqq \mu[h(e(K \cup h^{-1}K))] \\ &= \mu[e(K \cup h^{-1}K)] = \infty, \end{aligned}$$

because e is an end of infinite measure. More generally we have that for any natural number p,

$$t_{ij}^p = 1 \quad \text{if } \sigma^p \hat{I}_i \cap \hat{I}_j \neq \emptyset.$$

Since the sets I_i are clopen in E, the sets $\hat{I}_i$ are clopen in the relative topology on $\hat{E}$. By assumption the restriction of σ to $\hat{E}$ is componentwise weak mixing, so by the definition it follows that each set

$$S_{ij} = \{p : t_{ij}^p = 1\}$$

is thick. The finite intersection

$$S = \bigcap_{i,j=1}^{n} S_{ij}$$

is therefore nonempty. For any positive integer N in S,

$$t_{ij}^{N} = 1 \quad \text{for all } i, j = 1, \dots, n.$$

So we have demonstrated that condition 2) of Theorem 1 is satisfied, and now the conjugate $f = \theta'$ of θ produced by Theorem 1 satisfies the requirements of this proposition.

THEOREM 2. *Suppose the space $\mathscr{H}(M, \mu)$ contains a homeomorphism whose induced end-homeomorphism σ is topologically weak mixing on the ends $\hat{E}$ of M of infinite measure. Then for any conjugate-invariant subset $\mathscr{V}$ of $\mathscr{G}(M, \mu)$ which is dense and G_δ in the coarse topology,*

$$\mathscr{V} \cap \mathscr{H}(M, \mu) \neq \emptyset.$$

Proof. Observe that the space

$$\mathscr{H}_\sigma = \{h \text{ in } \mathscr{H}(M, \mu) : h^* = \sigma\}$$

is a closed subset of $\mathscr{H}(M, \mu)$ in the compact-open topology, hence it is topologically complete. Using a Baire category argument we will show that $\mathscr{V} \cap \mathscr{H}_\sigma$ is a dense G_δ subset of $\mathscr{H}_\sigma$ in the compact-open topology.

It has been shown by Sachdeva **[12]** and by Choksi and Kakutani **[8]** that the ergodic automorphisms $\mathscr{E}$ in $\mathscr{G}(M, \mu)$ constitute a dense G_δ set in the coarse topology. Since $\mathscr{V}$ is also a dense G_δ set the intersection $\mathscr{V} \cap \mathscr{E}$ is nonempty. Therefore $\mathscr{V}$ contains an ergodic automorphism θ and consequently its entire conjugacy class. Write

$$\mathscr{V} = \bigcap_{m=1}^{\infty} \mathscr{V}_m$$

where each $\mathscr{V}_m$ is coarse topology open and contains the conjugacy class of θ. The theorem will follow by a Baire category argument if we can establish that for each m the set $\mathscr{V}_m \cap \mathscr{H}_\sigma$ is a dense open subset of $\mathscr{H}_\sigma$ in the compact-open topology.

The set $\mathscr{V}_m \cap \mathscr{H}_\sigma$ is open because the compact-open topology is finer than the coarse topology. To prove denseness we must show that

$$\mathscr{V}_m \cap \mathscr{H}_\sigma \cap \mathscr{C}(h, K, \epsilon) \neq \emptyset,$$

where $\mathscr{C}(h, K, \epsilon)$ is a compact-open basic neighborhood of some homeomorphism in h in $\mathscr{H}_\sigma$. We now make use of a result established in a previous paper to show that we may assume that h is ergodic. Corollary 1 of **[5]** says that if the restriction of σ to $\hat{E}$ is transitive (topologically ergodic) then the ergodic homeomorphisms are dense in $\mathscr{H}_\sigma$ with respect to the compact-open topology. Since componentwise weak mixing implies transitivity, we may use that result to assume without loss of generality that the compact-open basic open set $\mathscr{C}$ is centered at an ergodic homeomorphism h. As mentioned above, we may also assume that K is a special compact set. Now apply Proposition 1 to this h and K and the ergodic automorphism θ found in $\mathscr{V}_m$. Let f be the conjugate of θ given by Proposition 1, and observe that consequently f belongs to $\mathscr{V}_m$. Let $\delta = \omega(\epsilon)$ where ω is the uniform modulus of continuity of h on K.

Thus the automorphism g defined by $g = h^{-1}f$ belongs to the coarse topology open set $h^{-1}\mathscr{V}_m$ and satisfies the following (actually $g(x) = x$ on K):

(i) $g(K) = K$

(ii) $d(x, g(x)) < \delta$ for μ-a.e. x in K

(iii) $g(P(K)) = P(K)$ for all P in $\mathscr{P}_K$.

Applying Theorem A, we may approximate the automorphism g by a compactly supported homeomorphism $\bar{h}$ which belongs to the coarse open set $h^{-1}\mathscr{V}_m$ and satisfies

$$\bar{h}(K) = K \quad \text{and} \quad d(x, \bar{h}(x)) < \delta \quad \text{for all } x \text{ in } K.$$

We claim that the homeomorphism $h\bar{h}$ belongs to

$$\mathscr{V}_m \cap \mathscr{H}_\sigma \cap \mathscr{C}(h, K, \epsilon)$$

proving that set to be nonempty, thus completing the proof. Clearly $h\bar{h}$ belongs to $\mathscr{V}_m$ because $\bar{h}$ belongs to $h^{-1}\mathscr{V}_m$. Next observe that since $\bar{h}$ has compact support, its induced end-homeomorphism $\bar{h}^*$ is the identity. The $*$-operation is a group homomorphism so

$$(h\bar{h})^* = h^* \bar{h}^* = h^* = \sigma$$

and $h\bar{h}$ belongs to $\mathscr{H}_\sigma$. Finally, for all x in K we have

$$d(h(x), h\bar{h}(x)) < \epsilon,$$

so $h\bar{h}$ belongs to $\mathscr{C}(h, K, \epsilon)$, completing the proof.

The above proof of Theorem 2 used the fact (established in [**5**]) that ergodicity is generic in $\mathscr{H}_\sigma$ when σ is transitive on E. We now outline a modification of the arguments given in this section which gives a proof of Theorem 2 which is independent of that fact. We begin by observing that two hypotheses of Proposition 1 can be weakened. First observe that the ergodic automorphism h need not be a homeomorphism. Secondly, h need not be ergodic on M since the weaker condition, that there are no non-null h-invariant subsets of the set

$$E_0 = K \cup F_1(K) \cup \ldots \cup F_m(K)$$

is sufficient to achieve condition 1) of Theorem 1.

These observations lead to the following alternate proof of Theorem 2. Let $\mathscr{C} = \mathscr{C}(h, K, \epsilon)$ be the compact-open neighborhood of an h in $\mathscr{H}_\sigma$ given in the previous proof of Theorem 2. While we no longer assume that h is ergodic, a simple perturbation argument [**4**, Lemma 6] enables us to assume that none of the sets K or $e(K)$, e in E, is h-invariant. That is, sets $\mathscr{C}(h, K, \epsilon)$ with this property form a sub-basic family. Let $\delta = \omega(\epsilon)$ be the uniform modulus of continuity of h on K. We now approximate the given homeomorphism h by an automorphism h' satisfying the (weakened) hypotheses of Proposition 1, using the following result which is step 1 of [**4**, Proposition 2].

LEMMA 3. *Let h be a homeomorphism in $\mathscr{H}_\sigma(M, \mu)$ and let K be a special compact subset of M such that none of the sets K or $e(K)$, e in E, are h-invariant. Then for any positive number δ there is an automorphism h' in $\mathscr{G}(M, \mu)$ such that $h'(e(K)) = h(e(K))$ for every end e in E, there are no non-null h'-invariant subsets of*

$$K \cup \bigcup_{e \in E - \hat{E}} e(K) \quad (= K \cup F_1(K) \cup \ldots \cup F_m(K)),$$

and

$$d(h^{-1}(y), (h')^{-1}(y)) < \delta \quad \text{for } \mu\text{-a.e. } y \text{ in } h(K).$$

Now we proceed as before. First apply Proposition 1 to h' (instead of h), obtaining an automorphism f in $\mathscr{G}(M, \mu)$ which agrees with h' pointwise on K and setwise on each set $e(K)$, and belongs to the coarse topology open set $\mathscr{V}_m$. Let $g = h^{-1}f$ and observe that $g(K) = K$, $g(e(K)) = e(K)$ for all ends e in E, and

$$d(g(x), x) = d(h^{-1}f(x), x) = d(h^{-1}(y), f^{-1}(y)) < \delta$$

for μ-a.e. $x = f^{-1}y$ in K. So applying Theorem A to g we get a compactly supported homeomorphism $\bar{h}$ such that $h\bar{h}$ belongs, as before, to the required set

$$\mathscr{V}_m \cap \mathscr{H}_\sigma \cap \mathscr{C}(h, K, \epsilon).$$

4. Example. To see that the conditions for Theorem 2 are not vacuous we give an example of a σ-compact manifold (M, μ) which supports a homeomorphism h in $\mathscr{H}(M, \mu)$ that induces a topologically weak mixing homeomorphism on the ends E of M.

The manifold is given by $M = D - C$ where D is the unit disk

$$\{(x, y): x^2 + y^2 \leqq 1\}$$

and C is the standard Cantor ternary set lying on the line $I = [-1/2, 1/2]$ along the x-axis. The Cantor set may be identified with the set E of ends of M. Let μ be any infinite σ-finite non-atomic Borel measure on M which is locally positive and locally finite and for which all the ends have infinite measure. We will give some explicit constructions of such a measure later. Let σ be any homeomorphism of C onto itself which is topologically weak mixing, for example the two sided shift when C is viewed as the countable product of a two-symbol set. Antoine [**6**] proved that any homeomorphism of C can be extended to a homeomorphism of D. Let $g:D \to D$ be a homeomorphism which extends σ. The restriction f of g to M

is a homeomorphism of the manifold M which induces the homeomorphism σ on the ends C. Unfortunately the homeomorphism f of M need not preserve the measure μ. This can be remedied as follows. Observe that the Borel measure μf^{-1} defined by

$$\mu f^{-1}(A) = \mu(f^{-1}(A))$$

for Borel subsets A of M is, like μ, a good (non-atomic, locally positive and locally finite) measure. Since all ends have infinite measure with respect to μ (by assumption) they all have infinite measure with respect to μf^{-1}. It has been recently proved by Berlanga and Epstein [7] that whenever two good Borel measures on a σ-compact manifold have the same set of infinite measured ends, there is an end-preserving homeomorphism of the manifold which takes one measure into the other. Applying this result to the measures μf^{-1} and μ, we obtain an end-preserving homeomorphism $r:M \rightarrow M$ such that

$$(\mu f^{-1})r^{-1} = \mu.$$

The μ-preserving homeomorphism h of M described in the previous paragraph can now be defined by $h = r \cdot f$. The construction of r ensures that h preserves μ and that h induces the homeomorphism

$$h^* = (rf)^* = r^*f^* = \sigma$$

on the (infinite measured) end set C.

We now outline the construction of a good σ-finite Borel measure μ on M such that all ends (points of C) have infinite measure. Let $I(0)$ and $I(1)$ denote the left and right thirds of the interval $I = [-1/2, 1/2]$ on the x-axis. For $i_k = 0, 1$ and $n \geqq 1$ let $I(i_1, \ldots, i_n, 0)$ and $I(i_1, \ldots, i_n, 1)$ denote the left and right thirds of $I(i_1, \ldots, i_n)$, respectively. Let m_1 and m_2 denote respectively one and two dimensional Lebesgue measure. For each Borel subset A of M define $\mu(A)$ by the formula

$$\mu(A) = m_2(A) + \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n} 3^n m_1(A \cap I(i_1, \ldots, i_n)).$$

Another construction of a suitable measure μ goes as follows. Let $R(i_1, \ldots, i_n)$ be the closed rectangular 3^{-n}-neighborhood of $I(i_1, \ldots, i_n)$ in D and let

$$K_n = D - \bigcup_{i_1,\ldots,i_n} \operatorname{int} R(i_1, \ldots, i_n).$$

Then the sets $L_n = K_{n+1} - K_n$ consist of 2^n congruent components each with measure

$$a_n = m_2(L_n)/2^n.$$

Set

$$\mu(A) = m_2(A) + \sum_n (1/a_n)m_2(A \cap L_n).$$

Finally, we note that there is nothing special in our example about dimension two. We could have taken our manifold to be $M = D^n - C$ where D^n is the unit n-dimensional ball and C is a Cantor set. The Cantor set however cannot be wild. For $n > 2$ Antoine's result can be replaced by the extension theorems of Keldys **[9]** or Oxtoby **[10]** for certain suitably chosen Cantor subsets of D^n.

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