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Abstract

A singularity in characteristic zero is said to be of *dense F-pure type* if its modulo p reduction is locally Frobenius split for infinitely many p. We prove that if $x \in X$ is an isolated log canonical singularity with $\mu(x \in X) \leqslant 2$ (where the invariant μ is as defined in Definition 1.4), then it is of dense F-pure type. As a corollary, we prove the equivalence of log canonicity and being of dense F-pure type in the case of three-dimensional isolated $\mathbb{Q}$-Gorenstein normal singularities.

Introduction

A singularity in characteristic zero is said to be of dense F-pure type if its modulo p reduction is locally Frobenius split for infinitely many p. The notion of 'strongly F-regular type' is a variant of 'dense F-pure type' and is defined similarly using the Frobenius morphism after reduction to characteristic $p > 0$ (see Definition 2.4 for the precise definition). Recently, it has turned out that these types have a strong connection to singularities associated with the minimal model program. In particular, Hara proved in [Har98b] that a normal $\mathbb{Q}$-Gorenstein singularity in characteristic zero is log terminal if and only if it is of strongly F-regular type. In this paper, as an analogous characterization for isolated log canonical singularities, we study the following conjecture.

Conjecture A_n. Let $x \in X$ be an n-dimensional normal $\mathbb{Q}$-Gorenstein singularity defined over an algebraically closed field k of characteristic zero, such that x is an isolated non-log-terminal point of X. Then $x \in X$ is log canonical if and only if it is of dense F-pure type.

Hara and Watanabe proved in [HW02] that normal $\mathbb{Q}$-Gorenstein singularities of dense F-pure type are log canonical. Unfortunately, the converse implication is wide open and only a few special cases are known. For example, the two-dimensional case follows from results of [MS91, Har98a], and the case of hypersurface singularities whose defining polynomials are very general was proved in [Her11]. This problem is now considered to be one of the most important problems on F-singularities. Making use of recent progress on the minimal model program, we shall prove Conjecture A_3.

Let $x \in X$ be an n-dimensional isolated log canonical singularity defined over an algebraically closed field k of characteristic zero. We suppose that $x \in X$ is not log terminal and that K_X is Cartier at x. Let $f : Y \to X$ be a resolution of singularities such that f is an isomorphism outside x and Supp $f^{-1}(x)$ is a simple normal crossings divisor on X. Then we can write

$$K_Y = f^*K_X + F - E,$$

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where E and F are effective divisors and have no common irreducible components. In [Fuj11c], the first author defined the invariant $\mu(x \in X)$ by

$$\mu = \mu(x \in X) = \min\{\dim W \mid W \text{ is a stratum of } E\}$$

and showed that this invariant plays an important role in the study of $x \in X$. Using the method in [Fuj11c] (which is based on the minimal model program), we can check that any minimal stratum W of E is a projective resolution of a μ-dimensional projective variety V with only rational singularities, such that K_V is linearly trivial. Also, by running a minimal model program with scaling (see [BCHM10] for the minimal model program with scaling), we show that $H^\mu(V, \mathcal{O}_V)$ can be viewed as the socle of the top local cohomology module $H^n_x(\mathcal{O}_X)$ of $x \in X$.

Here we introduce the following conjecture.

Conjecture B_d. *Let Z be a d-dimensional projective variety over an algebraically closed field of characteristic zero with only rational singularities, such that K_Z is linearly trivial. Then the action induced by the Frobenius morphism on the cohomology group $H^d(Z_p, \mathcal{O}_{Z_p})$ of its modulo p reduction Z_p is bijective for infinitely many p.*

Conjecture B_d is open in general, but from a combination of the results of [Ogu81, BZ09, JR03] it follows that Conjecture B_d is true if $d \leqslant 2$.

Now we suppose that Conjecture B_μ is true. Applying Conjecture B_μ to V, we see that the Frobenius action on the cohomology group $H^\mu(V_p, \mathcal{O}_{V_p})$ of the modulo p reduction V_p of V is bijective for infinitely many p. On the other hand, by Matlis duality, F-purity of the modulo p reduction $x_p \in X_p$ of $x \in X$ is equivalent to injectivity of the Frobenius action on $H^n_{x_p}(\mathcal{O}_{X_p})$. This injectivity can be checked by the injectivity of the Frobenius action on its socle $H^\mu(V_p, \mathcal{O}_{V_p})$. Thus, summing up the above, we conclude that $x \in X$ is of dense F-pure type.

A similar argument works in a more general setting, and our main result is stated as follows.

Main Theorem (Theorem 3.4). *Let $x \in X$ be a log canonical singularity defined over an algebraically closed field k of characteristic zero, such that x is an isolated non-log-terminal point of X. If Conjecture B_μ holds true with $\mu = \mu(x \in X)$, then $x \in X$ is of dense F-pure type. In particular, if $\mu(x \in X) \leqslant 2$, then $x \in X$ is of dense F-pure type.*

As a corollary of the above theorem, we show that Conjecture A_{n+1} is equivalent to Conjecture B_n (Corollary 3.9). Since Conjecture B_2 is known to be true, Conjecture A_3 holds true; that is, log canonicity is equivalent to being of dense F-pure type in the case of three-dimensional isolated normal $\mathbb{Q}$-Gorenstein singularities (Corollary 3.10).

1. Preliminaries on log canonical singularities

In this section, we work over an algebraically closed field of characteristic zero. We start with the definition of singularities of pairs. Let X be a normal variety and let D be an effective $\mathbb{Q}$-divisor on X such that $K_X + D$ is $\mathbb{Q}$-Cartier.

Definition 1.1. Let $\pi : \widetilde{X} \to X$ be a birational morphism from a normal variety $\widetilde{X}$. Then we can write

$$K_{\widetilde{X}} = \pi^*(K_X + D) + \sum_E a(E, X, D)E,$$

where E runs through all the distinct prime divisors on $\widetilde{X}$ and $a(E, X, D)$ is a rational number. We say that the pair (X, D) is *canonical* (respectively, *plt* or *log canonical*) if $a(E, X, D) \geqslant 0$

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(respectively, if $a(E,X,D)>-1$ or $a(E,X,D)\geqslant -1$) for every exceptional divisor E over X. If $D=0$, we simply say that X has only canonical (respectively, log terminal or log canonical) singularities. We say that (X,D) is *dlt* if (X,D) is log canonical and there exists a log resolution $\pi:\widetilde{X}\to X$ such that $a(E,X,D)>-1$ for every π-exceptional divisor E on $\widetilde{X}$. Here, saying that $\pi:\widetilde{X}\to X$ is a *log resolution* of (X,D) means that π is a proper birational morphism, $\widetilde{X}$ is a smooth variety, $\mathrm{Exc}(\pi)$ is a divisor and $\mathrm{Exc}(\pi)\cup \mathrm{Supp}\,\pi_*^{-1}D$ is a simple normal crossings divisor.

DEFINITION 1.2. A subvariety W of X is said to be a *log canonical center* for the log canonical pair (X,D) if there exist a proper birational morphism $\pi:\widetilde{X}\to X$ from a normal variety $\widetilde{X}$ and a prime divisor E on $\widetilde{X}$ with $a(E,X,D)=-1$ such that $\pi(E)=W$. Then W is denoted by $c_X(E)$.

Remark 1.3. Let (X,D) be a dlt pair. There then exists a log resolution $f:Y\to X$ such that f induces an isomorphism over the generic point of any log canonical center of (X,D) and $a(E,X,D)>-1$ for every f-exceptional divisor E. This is an immediate consequence of the divisorial log terminal theorem in [Sza95].

From now on, let X be a normal $\mathbb{Q}$-Gorenstein algebraic variety and $x\in X$ a germ. The *index* of X at x is the smallest positive integer r such that rK_X is Cartier at x.

DEFINITION 1.4. Let $x\in X$ be a log canonical singularity such that x is a log canonical center. First, we assume that the index of X at x is one. Take a projective birational morphism $f:Y\to X$ from a smooth variety Y such that $\mathrm{Supp}\, f^{-1}(x)$ and $\mathrm{Exc}(f)$ are simple normal crossings divisors. Then we can write

$$K_Y=f^*K_X+F-E,$$

where E and F are effective divisors on Y that have no common irreducible components. By assumption, E is a reduced simple normal crossings divisor on Y. We define $\mu(x\in X)$ by

$$\mu(x\in X)=\min\{\dim W \mid W \text{ is a stratum of } E \text{ and } f(W)=x\}.$$

Here we say that a subvariety W is a *stratum* of $E=\sum_{i\in I}E_i$ if there exists a subset $\{i_1,\ldots,i_k\}\subseteq I$ such that W is an irreducible component of the intersection $E_{i_1}\cap\cdots\cap E_{i_k}$. This definition is independent of the choice of the resolution f.

In general, we take an index-one cover $\rho:X'\to X$ with $x'=\rho^{-1}(x)$ and define $\mu(x\in X)$ by

$$\mu(x\in X)=\mu(x'\in X').$$

Since the index-one cover is unique up to étale isomorphisms, the above definition of $\mu(x\in X)$ is well-defined.

In Appendix A we will give a brief overview of properties of the invariant μ and some related topics for the reader's convenience.

Remark 1.5. (i) The first author showed in [Fuj11c, Theorem 5.5] that the invariant μ coincides with Ishii's Hodge-theoretic invariant (see [Ish85] and [Fuj11c, 5.1] for the definition). See also [Fuj12, Ish12].

(ii) By the main result of [Fuj01], the index of $x\in X$ is bounded if $\mu(x\in X)\leqslant 2$. The reader is referred to [Fuj01] for the precise values of indices.

In order to prove the main result of this paper, we use the notion of dlt *blow-ups*, which was first introduced by Christopher Hacon.

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Lemma 1.6 (cf. [Fuj11c, Lemma 2.9] and [Fuj11a, § 4]). *Let X be a log canonical variety of index one. Suppose that X is quasi-projective, x is an isolated non-log-terminal point of X, and X is canonical outside x. Then there exists a projective birational morphism $g : Z \to X$ such that $K_Z + D = g^*K_X$ where D is a reduced divisor on Z, the pair (Z, D) is a $\mathbb{Q}$-factorial dlt pair and g is a small morphism outside x.*

Lemma 1.7. *In Lemma 1.6, Z has only canonical singularities.*

Proof. If $a(E, Z, D) > -1$, then $a(E, Z, D) \geqslant 0$ because $K_Z + D$ is Cartier. Since K_Z is $\mathbb{Q}$-Cartier and D is an effective divisor on Z, one has $a(E, Z, 0) \geqslant 0$. If $a(E, Z, D) = -1$, then we may assume that Z is a smooth variety and that D is a reduced simple normal crossings divisor on Z by shrinking Z around the log canonical center $c_Z(E)$. In this case, $a(E, Z, 0) \geqslant 0$. Thus, Z has only canonical singularities. □

2. Preliminaries on F-pure singularities

In this section, we briefly review the definition of an F-pure singularity and its properties that we will need later.

Definition 2.1 [HH89, HR76]. Let $x \in X$ be a point of an F-finite integral scheme X of characteristic $p > 0$.

(i) $x \in X$ is said to be *F-pure* if the Frobenius map

$$F : \mathcal{O}_{X,x} \to F_*\mathcal{O}_{X,x}, \quad a \mapsto a^p$$

splits as an $\mathcal{O}_{X,x}$-module homomorphism.

(ii) $x \in X$ is said to be *strongly F-regular* if for every non-zero $c \in \mathcal{O}_{X,x}$ there exists an integer $e \geqslant 1$ such that

$$cF^e : \mathcal{O}_{X,x} \to F^e_*\mathcal{O}_{X,x}, \quad a \mapsto ca^{p^e}$$

splits as an $\mathcal{O}_{X,x}$-module homomorphism.

Remark 2.2. Strong F-regularity implies F-purity.

The following criterion for F-purity is well known to experts, but we include it here for the reader's convenience.

Lemma 2.3 (cf. [HR76]). *Let $x \in X$ be a closed point with index one of an n-dimensional F-finite integral scheme X. Then $x \in X$ is F-pure if and only if $F(z) \neq 0$, where F is the natural Frobenius action on $H^n_x(\mathcal{O}_X)$ and z is a generator of the socle $(0 : \mathfrak{m}_x)_{H^n_x(\mathcal{O}_X)}$.*

Proof. First, note that $H^n_x(\mathcal{O}_X)$ is isomorphic to the injective hull of the residue field $\mathcal{O}_{X,x}/\mathfrak{m}_x$, because $\mathcal{O}_{X,x}$ is quasi-Gorenstein. By definition, $x \in X$ is F-pure if and only if

$$F^\vee : \mathrm{Hom}_{\mathcal{O}_{X,x}}(F_*\mathcal{O}_{X,x}, \mathcal{O}_{X,x}) \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}, \mathcal{O}_{X,x}) = \mathcal{O}_{X,x}$$

is surjective. $F^\vee$ is the Matlis dual of the natural Frobenius action F on $H^n_x(\mathcal{O}_X)$, so surjectivity of $F^\vee$ is equivalent to injectivity of F. Since $H^n_x(\mathcal{O}_X)$ is an essential extension of the socle $(0 : \mathfrak{m}_x)_{H^n_x(\mathcal{O}_X)}$, F is injective if and only if $F|_{(0:\mathfrak{m}_x)_{H^n_x(\mathcal{O}_X)}}$ is injective. Finally, the latter condition is equivalent to saying that $F(z) \neq 0$, because the socle $(0 : \mathfrak{m}_x)_{H^n_x(\mathcal{O}_X)}$ is a one-dimensional $\mathcal{O}_{X,x}/\mathfrak{m}_x$-vector space. □

Next, we define the notions of F-purity and strong F-regularity in characteristic zero, using reduction from characteristic zero to positive characteristic.

Definition 2.4. Let $x \in X$ be a point of a scheme of finite type over a field k of characteristic zero. Choosing a suitable finitely generated $\mathbb{Z}$-subalgebra $A \subseteq k$, we can construct a (non-closed) point x_A of a scheme X_A of finite type over A such that $(X_A, x_A) \times_{\operatorname{Spec} A} k \cong (X, x)$. By the generic freeness, we may assume that X_A and x_A are flat over $\operatorname{Spec} A$. We refer to $x_A \in X_A$ as a *model* of $x \in X$ over A. Given a closed point $s \in \operatorname{Spec} A$, we denote by $x_s \in X_s$ the fiber of $x \in X$ over s. Then X_s is a scheme defined over the residue field $\kappa(s)$ of s, which is a finite field. The reader is referred to [HH99, ch. 2] and [MS11, § 3.2] for more details on reduction from characteristic zero to characteristic p.

(i) $x \in X$ is said to be of *strongly F-regular type* if there exists a model of $x \in X$ over a finitely generated $\mathbb{Z}$-subalgebra A of k and a dense open subset $S \subseteq \operatorname{Spec} A$ such that $x_s \in X_s$ is strongly F-regular for all closed points $s \in S$.

(ii) $x \in X$ is said to be of *dense F-pure type* if there exists a model of $x \in X$ over a finitely generated $\mathbb{Z}$-subalgebra A of k and a dense subset of closed points $S \subseteq \operatorname{Spec} A$ such that $x_s \in X_s$ is F-pure for all $s \in S$.

Remark 2.5. The definitions of strongly F-regular type and dense F-pure type are independent of the choice of model.

Theorem 2.6 [Har98b, Theorem 5.2]. *Let $x \in X$ be a normal $\mathbb{Q}$-Gorenstein singularity defined over a field of characteristic zero. Then $x \in X$ is log terminal if and only if it is of strongly F-regular type.*

In this paper, we will discuss an analogous statement for log canonical singularities. Specifically, we will consider the following conjecture.

Conjecture A_n. Let $x \in X$ be an n-dimensional normal $\mathbb{Q}$-Gorenstein singularity defined over an algebraically closed field k of characteristic zero, such that x is an isolated non-log-terminal point of X. Then $x \in X$ is log canonical if and only if it is of dense F-pure type.

Remark 2.7. Conjecture A_n is known to be true when $n = 2$ (see [Har98a, MS91, Wat88]) or when $x \in X$ is a hypersurface singularity whose defining polynomial is very general (see [Her11]). The reader is referred to [Tak11, Remark 2.6] for further details.

Remark 2.8. There are several generalizations of Conjecture A_n.

One is to consider a pair consisting of a normal variety X and an effective $\mathbb{Q}$-divisor Δ on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Since Hara and Watanabe generalized the notion of F-purity to such pairs [HW02], one can ask whether (X, Δ) is log canonical if and only if it is of dense F-pure type. It was shown in [HW02] that if the pair is of dense F-pure type, then it is log canonical. One of the difficulties in establishing this generalized conjecture is illustrated in Remark 3.6.

Another generalization is to allow non-normal singularities. The notion of semi-log canonical singularities is a generalization of log canonical singularities to a non-normal setting. Since F-purity is defined for any F-finite reduced scheme, one might ask whether $x \in X$ is semi-log canonical if and only if it is of dense F-pure type. If the aforementioned conjecture for pairs holds, then this generalized conjecture for non-normal singularities follows from the work [MS12] of Miller and Schwede.

We can also consider the case where X is normal but not $\mathbb{Q}$-Gorenstein. In [dFH09], de Fernex and Hacon generalized the notion of log canonical singularities to the case where X is not necessarily $\mathbb{Q}$-Gorenstein. One could then ask whether $x \in X$ is log canonical in the sense of de Fernex and Hacon if and only if it is of dense F-pure type.

Definition 2.9. Let X be an F-finite scheme of characteristic $p>0$. If $X=\operatorname{Spec} R$ is affine, we denote by $R[F]$ the ring

$$R[F]=\frac{R\{F\}}{\langle r^pF-Fr \mid r\in R\rangle},$$

which is obtained from R by adjoining a non-commutative variable F subject to the relation $r^pF=Fr$ for all $r\in R$. For a general scheme X, we denote by $\mathcal{O}_X[F]$ the sheaf of rings obtained by gluing the respective rings $\mathcal{O}_X(U_i)[F]$ over an affine open cover $X=\bigcup_i U_i$.

Example 2.10. (i) Let $f:Y\to X$ be a morphism of schemes over an F-finite affine scheme Z. Then for all $i\geqslant 0$, $H^i(X,\mathcal{O}_X)$ and $H^i(Y,\mathcal{O}_Y)$ each have a natural $\mathcal{O}_Z[F]$-module structure and f induces an $\mathcal{O}_Z[F]$-module homomorphism $f_*:H^i(X,\mathcal{O}_X)\to H^i(Y,\mathcal{O}_Y)$.

(ii) Let Y be a closed subscheme of a scheme X over an F-finite affine scheme Z. Then for all $i\geqslant 0$, we have the natural exact sequence of $\mathcal{O}_Z[F]$-modules

$$\cdots\to H^i_Y(X,\mathcal{O}_X)\to H^i(X,\mathcal{O}_X)\to H^i(X\backslash Y,\mathcal{O}_X)\to H^{i+1}_Y(X,\mathcal{O}_X)\to\cdots.$$

(iii) Let X be a scheme over an F-finite affine scheme Z, and let $Y_1,Y_2\subseteq X$ be closed subschemes. Let Y denote the scheme-theoretic union of Y_1 and Y_2. Then for all $i\geqslant 0$, the Mayer–Vietoris exact sequence

$$\cdots\to H^i(Y,\mathcal{O}_Y)\to H^i(Y_1,\mathcal{O}_{Y_1})\oplus H^i(Y_2,\mathcal{O}_{Y_2})\to H^i(Y_1\cap Y_2,\mathcal{O}_{Y_1\cap Y_2})$$
$$\to H^{i+1}(Y,\mathcal{O}_Y)\to\cdots$$

becomes an exact sequence of $\mathcal{O}_Z[F]$-modules.

Proof. The proof follows immediately from the fact that every cohomology module in Example 2.10 can be computed from the Čech complex. □

The following proposition is key to proving the main result of this paper.

Proposition 2.11. *Let $x\in X$ be an n-dimensional normal singularity with index one defined over an algebraically closed field k of characteristic zero. Let $g:Z\to X$ be a projective birational morphism and let D be a reduced $\mathbb{Q}$-Cartier divisor on Z satisfying the following properties:*

(i) *Z has only rational singularities;*

(ii) *$K_Z+D\sim_g 0$;*

(iii) *$g|_{Z\backslash D}:Z\backslash D\to X\backslash\{x\}$ is an isomorphism;*

(iv) $\operatorname{Supp} D=\operatorname{Supp} g^{-1}(x)$.

Then $x\in X$ is of dense F-pure type if and only if given any model of D over a finitely generated $\mathbb{Z}$-subalgebra A of k, there exists a dense subset $S\subseteq\operatorname{Spec} A$ such that the action of Frobenius on $H^{n-1}(D_s,\mathcal{O}_{D_s})$ is bijective for every closed point $s\in S$.

Proof. Without loss of generality, we may assume that X is affine. Suppose that we are given a model of $(x\in X,Z,D,g)$ over a finitely generated $\mathbb{Z}$-subalgebra A of k.

First, we will show that, by enlarging A if necessary, we can view $H^{n-1}(Z_s,\mathcal{O}_{Z_s})$ as an $\mathcal{O}_{X_s}[F]$-submodule of $H^n_{x_s}(\mathcal{O}_{X_s})$ for all closed points $s\in\operatorname{Spec} A$. Since $f|_{Z\backslash D}:Z\backslash D\to X\backslash\{x\}$ is an isomorphism, we have the natural isomorphisms

$$H^{n-1}(Z\backslash D,\mathcal{O}_Z)\cong H^{n-1}(X\backslash\{x\},\mathcal{O}_X)\cong H^n_x(\mathcal{O}_X).$$

On the other hand, we have the natural exact sequence

$$H^{n-1}_D(Z,\mathcal{O}_Z)\to H^{n-1}(Z,\mathcal{O}_Z)\to H^{n-1}(Z\backslash D,\mathcal{O}_Z),$$

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and $H^{n-1}_D(Z, \mathcal{O}_Z) = 0$ by the dual form of the Grauert–Riemenschneider vanishing theorem (see, for example, [Fuj09, Lemma 4.19 and Remark 4.20]). Hence we can view $H^{n-1}(Z, \mathcal{O}_Z)$ as an $\mathcal{O}_X$-submodule of $H^n_x(\mathcal{O}_X)$. By (i) and (ii) of Example 2.10, after possibly enlarging A, we may assume that $H^{n-1}(Z_s, \mathcal{O}_{Z_s})$ is an $\mathcal{O}_{X_s}[F]$-submodule of $H^n_{x_s}(\mathcal{O}_{X_s})$ for all closed points $s \in \operatorname{Spec} A$.

Next, we will show that one may assume that

$$H^{n-1}(Z_s, \mathcal{O}_{Z_s}) \cong H^{n-1}(D_s, \mathcal{O}_{D_s})$$

as an $\mathcal{O}_{X_s}[F]$-module homomorphism for all closed points $s \in \operatorname{Spec} A$. The short exact sequence $0 \to \mathcal{O}_Z(-D) \to \mathcal{O}_Z \to \mathcal{O}_D \to 0$ induces the exact sequence

$$H^{n-1}(Z, \mathcal{O}_Z(-D)) \to H^{n-1}(Z, \mathcal{O}_Z) \to H^{n-1}(D, \mathcal{O}_D) \to H^n(Z, \mathcal{O}_Z(-D)) = 0$$

of $\mathcal{O}_X$-modules. It follows from the Grauert–Riemenschneider vanishing theorem that

$$H^{n-1}(Z, \mathcal{O}_Z(-D)) \cong H^{n-1}(Z, \mathcal{O}_Z(K_Z)) = 0,$$

so we have an $\mathcal{O}_X$-module isomorphism $H^{n-1}(Z, \mathcal{O}_Z) \cong H^{n-1}(D, \mathcal{O}_D)$. By Example 2.10(i), after possibly enlarging A, we may assume that $H^{n-1}(Z_s, \mathcal{O}_{Z_s}) \cong H^{n-1}(D_s, \mathcal{O}_{D_s})$ as an $\mathcal{O}_{X_s}[F]$-module homomorphism for all closed points $s \in \operatorname{Spec} A$.

Finally, we will check that $H^{n-1}(D_s, \mathcal{O}_{D_s})$ is the socle of the $\mathcal{O}_{X_s,x_s}[F]$-module of $H^n_{x_s}(\mathcal{O}_{X_s})$. Since

$$\mathfrak{m}_x \cdot H^{n-1}(Z, \mathcal{O}_Z) = H^0(Z, \mathcal{O}_Z(-D)) \cdot H^{n-1}(Z, \mathcal{O}_Z) \subseteq H^{n-1}(Z, \mathcal{O}_Z(-D)) = 0,$$

$H^{n-1}(D, \mathcal{O}_D) \cong H^{n-1}(Z, \mathcal{O}_Z)$ is contained in the socle of $H^n_x(\mathcal{O}_X)$. Let ω_D be the dualizing sheaf of D. Then we obtain $\omega_D \simeq \mathcal{O}_Z(K_Z + D) \otimes \mathcal{O}_D$ since $K_Z + D$ is Cartier. Therefore, $\omega_D \simeq \mathcal{O}_D$ because $K_Z + D \sim_g 0$ and $g(D) = x$. By Serre duality (which holds for top cohomology groups even if the variety is not Cohen–Macaulay), one has $\dim_k H^{n-1}(D, \mathcal{O}_D) = 1$ because $H^0(D, \omega_D) = H^0(D, \mathcal{O}_D) \cong k$. The socle of $H^n_x(\mathcal{O}_X)$ is the one-dimensional k-vector space, so it coincides with $H^{n-1}(D, \mathcal{O}_D)$.

By the above argument, the bijectivity of the Frobenius action on $H^{n-1}(D_s, \mathcal{O}_{D_s})$ means that the restriction of the Frobenius action on $H^n_{x_s}(\mathcal{O}_{X_s})$ to its socle is injective. This condition is equivalent to saying that X_s is F-pure by Lemma 2.3. Thus, $x_s \in X_s$ is of dense F-pure type if and only if there exists a dense subset of closed points $S \subseteq \operatorname{Spec} A$ such that the Frobenius action on $H^{n-1}(D_s, \mathcal{O}_{D_s})$ is bijective for all $s \in S$. □

3. Main result

In order to state our main result, we present the following conjecture.

Conjecture B_n. Let V be an n-dimensional projective variety over an algebraically closed field k of characteristic zero with only rational singularities, such that K_V is linearly trivial. Given a model of V over a finitely generated $\mathbb{Z}$-subalgebra A of k, there exists a dense subset of closed points $S \subseteq \operatorname{Spec} A$ such that the natural Frobenius action on $H^n(V_s, \mathcal{O}_{V_s})$ is bijective for every $s \in S$.

Remark 3.1. (i) An affirmative answer to [MS11, Conjecture 1.1] implies an affirmative answer to Conjecture B_n. Indeed, take a resolution of singularities $\pi : \widetilde{V} \to V$. Since V has only rational singularities, π induces the isomorphism $H^n(V, \mathcal{O}_V) \cong H^n(\widetilde{V}, \mathcal{O}_{\widetilde{V}})$. Suppose we are given a model of π over a finitely generated $\mathbb{Z}$-subalgebra A of k. If [MS11, Conjecture 1.1] holds true, then there exists a dense subset of closed points $S \subseteq \operatorname{Spec} A$ such that the Frobenius action on $H^n(\widetilde{V}_s, \mathcal{O}_{\widetilde{V}_s})$ is bijective for every $s \in S$. Since, by Example 2.10(i), we may assume that $H^n(V_s, \mathcal{O}_{V_s}) \cong H^n(\widetilde{V}_s, \mathcal{O}_{\widetilde{V}_s})$ as $\kappa(s)[F]$-modules for all $s \in S$, we obtain the assertion.

(ii) Let W be an n-dimensional smooth projective variety defined over a perfect field of characteristic $p > 0$. If W is ordinary (in the sense of Bloch and Kato), then the natural Frobenius action on $H^n(W, \mathcal{O}_W)$ is bijective; see [MS11, Remark 5.1]. If W is an abelian variety or a curve, then the converse implication also holds; see [MS11, Examples 5.4 and 5.5].

Lemma 3.2. *Conjecture* B_{n+1} *implies Conjecture* B_n.

Proof. Let V be an n-dimensional projective variety over an algebraically closed field k of characteristic zero with only rational singularities, such that K_V is linearly trivial. Let C be an elliptic curve over k, and write $W = V \times C$. Suppose we are given a model of (V, C, W) over a finitely generated $\mathbb{Z}$-subalgebra A of k. Applying Conjecture B_{n+1} to W, we can take a dense subset of closed points $S \subseteq \operatorname{Spec} A$ such that the Frobenius action on

$$H^{n+1}(W_s, \mathcal{O}_{W_s}) = H^n(V_s, \mathcal{O}_{V_s}) \otimes H^1(C_s, \mathcal{O}_{C_s})$$

is bijective for every $s \in S$. This implies that the Frobenius action on $H^n(V_s, \mathcal{O}_{V_s})$ is bijective for every $s \in S$. $\square$

Lemma 3.3. *Conjecture* B_n *holds true if* $n \leqslant 2$.

Proof. By an argument similar to the proof of [MS11, Proposition 5.3], we may assume that $k = \overline{\mathbb{Q}}$ without loss of generality. By Lemma 3.2, it suffices to consider the case where $n = 2$.

Let $\pi : \widetilde{X} \to X$ be a minimal resolution, where $\widetilde{X}$ is an abelian surface or a K3 surface. Suppose we are given a model of π over a finitely generated $\mathbb{Z}$-subalgebra A of k. Then there exists a dense subset of closed points $S \subseteq \operatorname{Spec} A$ such that the Frobenius action on $H^2(\widetilde{X}_s, \mathcal{O}_{\widetilde{X}_s})$ is bijective for every $s \in S$ (the abelian surface case follows from a result of [Ogu81], and the K3 surface case follows from a result of [BZ09] or [JR03]). Since X has only rational singularities, by Example 2.10(i) we may assume that $H^2(X_s, \mathcal{O}_{X_s}) \cong H^2(\widetilde{X}_s, \mathcal{O}_{\widetilde{X}_s})$ as $\kappa(s)[F]$-modules for all $s \in S$. Thus, we obtain the assertion. $\square$

Our main result is stated as follows.

Theorem 3.4. *Let $x \in X$ be a log canonical singularity defined over an algebraically closed field k of characteristic zero, such that x is an isolated non-log-terminal point of X. If Conjecture B_μ holds true with $\mu = \mu(x \in X)$, then $x \in X$ is of dense F-pure type. In particular, if $\mu(x \in X) \leqslant 2$, then $x \in X$ is of dense F-pure type.*

We need the following proposition for the proof of Theorem 3.4.

Proposition 3.5. *Let $x \in X$ be an n-dimensional log canonical singularity defined over an algebraically closed field k of characteristic zero. Suppose that the index of X at x is one and that x is an isolated non-log-terminal point of X. Let $g : (Z, D) \to X$ be a dlt blow-up as in Lemma 1.6. Then there exists a birational model $\widetilde{g} : (\widetilde{Z}, \widetilde{D}) \to X$ of g which satisfies the following properties:*

(i) *$\widetilde{Z}$ has only canonical singularities;*

(ii) *$K_{\widetilde{Z}} + \widetilde{D}$ is linearly trivial over X;*

(iii) *$\widetilde{g}|_{\widetilde{Z} \setminus \widetilde{D}} : \widetilde{Z} \setminus \widetilde{D} \to X \setminus \{x\}$ is an isomorphism;*

(iv) $\operatorname{Supp} \widetilde{D} = \operatorname{Supp} \widetilde{g}^{-1}(x)$;

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(v) *given models of D and $\widetilde{D}$ over a finitely generated $\mathbb{Z}$-subalgebra A of k, upon enlarging A if necessary, we may assume that*

$$H^{n-1}(D_s, \mathcal{O}_{D_s}) \cong H^{n-1}(\widetilde{D}_s, \mathcal{O}_{\widetilde{D}_s})$$

as $\kappa(s)[F]$-modules for all closed points $s \in \operatorname{Spec} A$.

Remark 3.6. Let (X, Δ) be a log canonical pair such that $K_X + \Delta$ is Cartier at $x \in X$. If x is an isolated non-Kawamata-log-terminal point of the pair (X, Δ), then we see that $x \notin \operatorname{Supp} \Delta$. Therefore it is not clear how to formulate Proposition 3.5 for log canonical *pairs*. We need the assumption that x is an isolated non-log-terminal point of X in order to prove property (iii) in Proposition 3.5 (see also property (iii) in Proposition 2.11).

Proof of Proposition 3.5. We may assume that X is affine and K_X is Cartier. We run a K_Z-minimal model program over X with scaling (see [BCHM10] for the minimal model program with scaling). Then we obtain a sequence of divisorial contractions and flips

$$\begin{array}{ccccccccc} Z = Z_0 & \overset{\phi_0}{\dashrightarrow} & Z_1 & \overset{\phi_1}{\dashrightarrow} & \cdots & \overset{\phi_{k-2}}{\dashrightarrow} & Z_{k-1} & \overset{\phi_{k-1}}{\dashrightarrow} & Z_k = Z' \\ \cup & & \cup & & & & \cup & & \cup \\ D = D_0 & \dashrightarrow & D_1 & \dashrightarrow & \cdots & \dashrightarrow & D_{k-1} & \dashrightarrow & D_k = D' \end{array}$$

such that $K_{Z'}$ is nef over X. Suppose we are given a model of the above sequence over a finitely generated $\mathbb{Z}$-subalgebra A of k.

Claim 1. *Assume that $\phi_i : Z_i \dashrightarrow Z_{i+1}$ is a flip. By enlarging A if necessary, we may assume that*

$$H^j(D_{i_s}, \mathcal{O}_{D_{i_s}}) \cong H^j(D_{i+1,s}, \mathcal{O}_{D_{i+1,s}})$$

as $\mathcal{O}_{X_s}[F]$-modules for all j and all closed points $s \in \operatorname{Spec} A$.

Proof of Claim 1. Consider the following flipping diagram.

$$\begin{array}{ccc} Z_i & \overset{\phi_i}{\dashrightarrow} & Z_{i+1} \\ & {\scriptstyle\psi_i}\searrow \quad \swarrow{\scriptstyle\psi_{i+1}} & \\ & W_i & \end{array}$$

By enlarging A if necessary, we may assume that a model of the above diagram over A is given. Note that $K_{Z_i} + D_i \sim_{\psi_i} 0$ and $K_{Z_{i+1}} + D_{i+1} \sim_{\psi_{i+1}} 0$. Then we have the following exact sequences:

$$0 \to \mathcal{O}_{Z_i}(K_{Z_i}) \to \mathcal{O}_{Z_i} \to \mathcal{O}_{D_i} \to 0,$$
$$0 \to \mathcal{O}_{Z_{i+1}}(K_{Z_{i+1}}) \to \mathcal{O}_{Z_{i+1}} \to \mathcal{O}_{D_{i+1}} \to 0.$$

We put $C_i = \psi_i(D_i) = \psi_{i+1}(D_i) \subseteq W_i$. Since Z_i, Z_{i+1} and W_i each have only rational singularities, by the Grauert–Riemenschneider vanishing theorem one has

$$\mathbf{R}\psi_{i*}\mathcal{O}_{D_i} \cong \mathcal{O}_{C_i} \cong \mathbf{R}\psi_{i*}\mathcal{O}_{D_{i+1}}$$

in the derived category of coherent sheaves on C_i. Therefore, ψ_i and ψ_{i+1} induce the isomorphisms

$$H^j(D_i, \mathcal{O}_{D_i}) \overset{\psi_{i*}}{\cong} H^j(C_i, \mathcal{O}_{C_i}) \overset{\psi_{i+1*}}{\cong} H^j(D_{i+1}, \mathcal{O}_{D_{i+1}})$$

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for all j. By Example 2.10(i), after possibly enlarging A, we may assume that

$$H^j(D_{i_s}, \mathcal{O}_{D_{i_s}}) \cong H^j(D_{i+1,s}, \mathcal{O}_{D_{i+1,s}})$$

as $\mathcal{O}_{X_s}[F]$-modules for all closed points $s \in \operatorname{Spec} A$. $\square$

Claim 2. *Assume that $\phi_i : Z_i \dashrightarrow Z_{i+1}$ is a divisorial contraction. By enlarging A if necessary, we may assume that*

$$H^j(D_{i_s}, \mathcal{O}_{D_{i_s}}) \cong H^j(D_{i+1,s}, \mathcal{O}_{D_{i+1,s}})$$

as $\mathcal{O}_{X_s}[F]$-modules for all j and all closed points $s \in \operatorname{Spec} A$.

Proof of Claim 2. Let E be the ϕ_i-exceptional prime divisor on Z_i. First, we will check that $\phi_i(D_i) = D_{i+1}$. This is obvious when E is not an irreducible component of D_i, so we focus on the case where E is an irreducible component of D_i. Since $K_{Z_i} + D_i$ and $K_{Z_{i+1}} + D_{i+1}$ are both linearly trivial over X, we have

$$K_{Z_i} + D_i = \phi_i^*(K_{Z_{i+1}} + D_{i+1}).$$

Hence $\phi_i(E)$ is a log canonical center of the pair (Z_{i+1}, D_{i+1}). Each Z_i has only canonical singularities, because Z has only canonical singularities by Lemma 1.7 and we run a K_Z-minimal model program. Thus, $\phi_i(E)$ has to be contained in D_{i+1}, which implies that $\phi_i(D_i) = D_{i+1}$.

By an argument analogous to the proof of Claim 1 (that is, using the Grauert–Riemenschneider vanishing theorem), we have $\mathbf{R}\phi_{i*}\mathcal{O}_{D_i} \cong \mathcal{O}_{D_{i+1}}$ in the derived category of coherent sheaves on D_{i+1}. Therefore ϕ_i induces the isomorphism

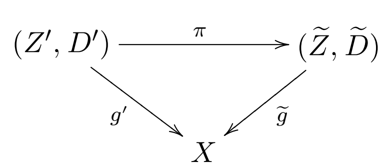
$$H^j(D_i, \mathcal{O}_{D_i}) \overset{\phi_{i_*}}{\cong} H^j(D_{i+1}, \mathcal{O}_{D_{i+1}})$$

for all j. By Example 2.10(i), after possibly enlarging A, we may assume that

$$H^j(D_{i_s}, \mathcal{O}_{D_{i_s}}) \cong H^j(D_{i+1,s}, \mathcal{O}_{D_{i+1,s}})$$

as $\mathcal{O}_{X_s}[F]$-modules for all $s \in \operatorname{Spec} A$. $\square$

Let $g' : (Z', D') \to X$ be the output of the minimal model program. By the base point free theorem, we obtain the diagram



such that $\widetilde{Z}$ is the canonical model of Z' over X and

$$K_{Z'} + D' = \pi^*(K_{\widetilde{Z}} + \widetilde{D}).$$

Upon enlarging A if necessary, we may assume that a model of the above diagram over A is given. By an argument similar to the proof of Claim 2, we can check that $\pi(D') = \widetilde{D}$ and $\mathbf{R}\pi_*\mathcal{O}_{D'} \cong \mathcal{O}_{\widetilde{D}}$ in the derived category of coherent sheaves on $\widetilde{D}$. Thus, π induces the isomorphism

$$H^{n-1}(D', \mathcal{O}_{D'}) \overset{\pi_*}{\cong} H^{n-1}(\widetilde{D}, \mathcal{O}_{\widetilde{D}}).$$

By Example 2.10(i), after possibly enlarging A, we may assume that

$$H^{n-1}(D'_s, \mathcal{O}_{D'_s}) \cong H^{n-1}(\widetilde{D}_s, \mathcal{O}_{\widetilde{D}_s})$$

as $\mathcal{O}_{X_s}[F]$-modules for all closed points $s \in \operatorname{Spec} A$.

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Summing up the above arguments, we know that $\widetilde{g}:(\widetilde{Z},\widetilde{D})\to X$ has the following properties:

(i) $\widetilde{Z}$ has only canonical singularities;

(ii) $K_{\widetilde{Z}}+\widetilde{D}\sim_{\widetilde{g}}0$;

(iii) $K_{\widetilde{Z}}$ is $\widetilde{g}$-ample;

(iv) $H^{n-1}(D_s,\mathcal{O}_{D_s})\cong H^{n-1}(\widetilde{D}_s,\mathcal{O}_{\widetilde{D}_s})$ for all closed points $s\in\operatorname{Spec}A$.

Since $-\widetilde{D}$ is $\widetilde{g}$-ample by (ii) and (iii), one has $\operatorname{Supp}\widetilde{D}=\operatorname{Supp}\widetilde{g}^{-1}(x)$. Therefore, it remains to show that $\widetilde{g}$ is an isomorphism outside x. Note that $X\backslash\{x\}$ has only canonical singularities. Then we can write

$$K_{\widetilde{Z}\backslash\widetilde{D}}=\widetilde{g}^{*}K_{X\backslash\{x\}}+F,$$

where F is a $\widetilde{g}$-exceptional effective $\mathbb{Q}$-divisor on $\widetilde{Z}\backslash\widetilde{D}$. Since $K_{\widetilde{Z}\backslash\widetilde{D}}$ is $\widetilde{g}$-ample, one has $F=0$. Again, by the $\widetilde{g}$-ampleness of $K_{\widetilde{Z}\backslash\widetilde{D}}$, the birational morphism $\widetilde{g}:\widetilde{X}\backslash\widetilde{D}\to X\backslash\{x\}$ has to be finite, i.e. it is an isomorphism. □

Remark 3.7. Let $f:Y\to X$ be any resolution as in Definition 1.4. By the uniqueness of the relative canonical model, we have

$$\widetilde{Z}\cong\operatorname{Proj}\bigoplus_{m\geqslant 0}f_*\mathcal{O}_Y(mK_Y)$$

over X. Unfortunately, by this construction, it is not clear how to relate the cohomology group $H^{n-1}(D_s,\mathcal{O}_{D_s})$ to $H^{n-1}(\widetilde{D}_s,\mathcal{O}_{\widetilde{D}_s})$. Moreover, the relationship between $\widetilde{D}$ and a minimal stratum of E in Definition 1.4 is also unclear. Therefore, we take a dlt blow-up and run a minimal model program with scaling to construct $\widetilde{Z}$.

Now we start the proof of Theorem 3.4.

Proof of Theorem 3.4. Since F-purity and log canonicity are preserved under index-one covers (see [Wat91] for F-purity and [KM98, Proposition 5.20] for log canonicity), we may assume that the index of X at x is one. We can also assume that X is affine and K_X is Cartier.

By Lemma 1.6, there exist a birational projective morphism $g:Z\to X$ and a reduced divisor D on Z such that $K_Z+D=g^*K_X$, (Z,D) is a $\mathbb{Q}$-factorial dlt pair and g is a small morphism outside x. By Remark 1.3, there exists a projective birational morphism $h:Y\to Z$ from a smooth variety Y with the following properties:

(i) $\operatorname{Exc}(h)$ and $\operatorname{Exc}(h)\cup\operatorname{Supp}h_*^{-1}D$ are simple normal crossings divisors on Y;

(ii) h is an isomorphism over the generic point of any log canonical center of the pair (Z,D);

(iii) $a(E,Z,D)>-1$ for every h-exceptional divisor E.

Then we can write

$$K_Y=h^*(K_Z+D)+F-E,$$

where E and F are effective divisors on Y which have no common irreducible components. By the construction of h, E is a reduced simple normal crossings divisor on Y and $E=h_*^{-1}D$. It follows from [Fuj09, Corollary 4.15] or [Fuj11c, Corollary 2.5] that $\mathbf{R}h_*\mathcal{O}_E\cong\mathcal{O}_D$ in the derived category of coherent sheaves on D. Therefore, we have the isomorphism

$$H^i(E,\mathcal{O}_E)\overset{h_*}{\cong}H^i(D,\mathcal{O}_D)$$

for every i. Suppose we are given models of D and E over a finitely generated $\mathbb{Z}$-subalgebra A of k. By Example 2.10(i), after possibly enlarging A, we may assume that

$$H^i(E_s, \mathcal{O}_{E_s}) \cong H^i(D_s, \mathcal{O}_{D_s})$$

as $\mathcal{O}_{X_s}[F]$-modules for all closed points $s \in \operatorname{Spec} A$.

Let W be a minimal stratum of a simple normal crossings variety E. By an argument similar to that in [Fuj11c, 4.11] (see also Appendix A), one has $\dim W = \mu$. Since $K_Z + D$ is linearly trivial over X and D is a g-exceptional divisor on Z, by the adjunction formula one has $K_D \sim 0$. We also note that D is sdlt (see [Fuj00, Definition 1.1] for the definition of sdlt varieties). Applying [Fuj11c, Remark 5.3] to $h : E \to D$, we obtain the following assertion.

Claim. *Suppose that models of W and E over A are given. Then, after possibly enlarging A, we may assume that*

$$H^{n-1}(E_s, \mathcal{O}_{E_s}) \cong H^{\mu}(W_s, \mathcal{O}_{W_s})$$

as $\mathcal{O}_{X_s}[F]$-modules for all closed points $s \in \operatorname{Spec} A$.

Proof. It follows from [Fuj11c, Theorem 5.2 and Remark 5.3] that

$$H^{\mu}(W, \mathcal{O}_W) \cong \cdots \cong H^{n-1}(E, \mathcal{O}_E),$$

where each isomorphism is the connecting homomorphism of a suitable Mayer–Vietoris exact sequence (see also [Fuj12]). Then, by Example 2.10(iii), after possibly enlarging A we may assume that

$$H^{n-1}(E_s, \mathcal{O}_{E_s}) \cong \cdots \cong H^{\mu}(W_s, \mathcal{O}_{W_s})$$

as $\mathcal{O}_{X_s}[F]$-modules for all closed points $s \in \operatorname{Spec} A$. □

Let $V = h(W) \subseteq D$. Then V is a minimal log canonical center of the pair (Z, D). On the other hand, by the adjunction formula for dlt pairs, we obtain $K_V = (K_Z + D)|_V \sim 0$. Thus V has only Gorenstein rational singularities. Since $h : W \to V$ is birational by the construction of h, one has the isomorphism

$$H^{\mu}(W, \mathcal{O}_W) \overset{h_*}{\cong} H^{\mu}(V, \mathcal{O}_V).$$

Suppose that models of W and V are given over A. By Example 2.10(i), after possibly enlarging A, we may assume that

$$H^{\mu}(W_s, \mathcal{O}_{W_s}) \cong H^{\mu}(V_s, \mathcal{O}_{V_s})$$

as $\mathcal{O}_{X_s}[F]$-modules for all closed points $s \in \operatorname{Spec} A$.

Now we combine the above arguments with Proposition 3.5 (using the same notation as in Proposition 3.5). Suppose we are given models of $\widetilde{D}$ and V over a finitely generated $\mathbb{Z}$-subalgebra A of k. Then, after possibly enlarging A, we may assume that

$$H^{\mu}(V_s, \mathcal{O}_{V_s}) \cong H^{n-1}(\widetilde{D}_s, \mathcal{O}_{\widetilde{D}_s})$$

as $\mathcal{O}_{X_s}[F]$-modules for all closed points $s \in \operatorname{Spec} A$. It follows from an application of Conjecture B_{μ} to V that there exists a dense subset of closed points $S \subseteq \operatorname{Spec} A$ such that the natural Frobenius action on $H^{\mu}(V_s, \mathcal{O}_{V_s})$ is bijective for all $s \in S$. Then the Frobenius action on $H^{n-1}(\widetilde{D}_s, \mathcal{O}_{\widetilde{D}_s})$ is also bijective for all closed points $s \in S$, which implies, by Proposition 2.11, that $x \in X$ is of dense F-pure type. □

The following remark was pointed out by János Kollár.

Remark 3.8. In Kollár's framework discussed in [Kol], V is called the *source* of the log canonical center $x \in X$. The key fact that $H^{\mu}(V, \mathcal{O}_V)$ is isomorphic to the socle of $H^n_x(\mathcal{O}_X)$ can be proved by his method as well.

COROLLARY 3.9. *Conjecture* A_{n+1} *is equivalent to Conjecture* B_n.

Proof. First we will show that Conjecture B_n implies Conjecture A_{n+1}. Let $x \in X$ be an $(n+1)$-dimensional normal $\mathbb{Q}$-Gorenstein singularity defined over an algebraically closed field k of characteristic zero, such that x is an isolated non-log-terminal point of X. If $x \in X$ is of dense F-pure type, then by [HW02, Theorem 3.9] it is log canonical. Conversely, suppose that $x \in X$ is a log canonical singularity. Since $\mu := \mu(x \in X) \leqslant \dim X - 1 = n$, by Lemma 3.2 we have that Conjecture B_μ holds true. It then follows from Theorem 3.4 that $x \in X$ is of dense F-pure type.

Next, we will prove that Conjecture A_{n+1} implies Conjecture B_n. Let V be an n-dimensional projective variety over an algebraically closed field k of characteristic zero with only rational singularities, such that $K_V \sim 0$. Take any ample Cartier divisor D on V and consider its section ring $R = R(V, D) = \bigoplus_{m \geqslant 0} H^0(V, \mathcal{O}_V(mD))$. By [SS10, Proposition 5.4], the affine cone $\operatorname{Spec} R$ of V has only quasi-Gorenstein log canonical singularities and its vertex is an isolated non-log-terminal point of $\operatorname{Spec} R$. It then follows from Conjecture A_{n+1} that given a model of (V, D, R) over a finitely generated $\mathbb{Z}$-subalgebra A of k, there exists a dense subset of closed points $S \subseteq \operatorname{Spec} A$ such that $\operatorname{Spec} R_s$ is F-pure for all $s \in S$. Note that, upon replacing S by a smaller dense subset if necessary, we may assume that $R_s = R(V_s, D_s)$ for all $s \in S$. Since $\operatorname{Spec} R_s$ is F-pure, the natural Frobenius action on the local cohomology module $H^{n+1}_{\mathfrak{m}_{R_s}}(R_s)$ is injective, where $\mathfrak{m}_{R_s} = \bigoplus_{m \geqslant 1} H^0(V_s, \mathcal{O}_{V_s}(mD_s))$ is the unique homogeneous maximal ideal of R_s. Then the Frobenius action on $H^n(V_s, \mathcal{O}_{V_s})$ is also injective, because $H^n(V_s, \mathcal{O}_{V_s})$ is the degree-zero part of $H^{n+1}_{\mathfrak{m}_{R_s}}(R_s)$. □

Since Conjecture B_2 is known to be true (see Lemma 3.3), Conjecture A_3 also holds true.

COROLLARY 3.10. *Let $x \in X$ be a three-dimensional normal $\mathbb{Q}$-Gorenstein singularity defined over an algebraically closed field of characteristic zero, such that x is an isolated non-log-terminal point of X. Then $x \in X$ is log canonical if and only if it is of dense F-pure type.*

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Appendix A. A quick review of [Fuj01, Fuj11c]

In this appendix, we give a brief review of the invariant μ and related topics from [Fuj01] and [Fuj11c] for the reader's convenience. Since [Fuj01] was written, the minimal model program has developed substantially (cf. [BCHM10]). In [Fuj11c], only isolated log canonical singularities were treated. Here, we survey the basic properties of μ and some related results in the framework of [Fuj11c]. For details, see [Fuj01, Fuj11c].

Let X be a quasi-projective log canonical variety defined over an algebraically closed field k of characteristic zero with index one. Assume that $x \in X$ is a log canonical center. Let $f : Y \to X$

be a projective birational morphism from a smooth variety Y such that

$$K_Y = f^*K_X + F - E,$$

where E and F are effective Cartier divisors on Y that have no common irreducible components. We further assume that $f^{-1}(x)$ and $\mathrm{Supp}(E+F)$ are simple normal crossings divisors on Y. Let $E = \sum_{i \in I} E_i$ be the decomposition into irreducible components. Note that E is a reduced simple normal crossings divisor on Y. We put

$$J = \{i \in I \mid f(E_i) = x\} \subset I$$

and

$$G = \sum_{i \in J} E_i.$$

Then, by [Fuj11b, Proposition 8.2], we obtain

$$f_*\mathcal{O}_G \cong \kappa(x).$$

In particular, G is connected. We apply a $(K_Y + E)$-minimal model program with scaling over X (cf. [BCHM10] and [Fuj11a, § 4]). Then we obtain a projective birational morphism

$$f' : Y' \to X$$

such that (Y', E') is a $\mathbb{Q}$-factorial dlt pair and $K_{Y'} + E' = f'^*K_X$ where E' is the pushforward of E on Y'; it is a dlt blow-up of X (see Lemma 1.6). Note that each step of the minimal model program is an isomorphism at the generic point of any log canonical center of (Y, E), because

$$K_Y + E = f^*K_X + F.$$

Therefore, we obtain that

$$\mu(x \in X) = \min\{\dim W \mid W \text{ is a log canonical center of } (Y', E') \text{ with } f'(W) = x\}.$$

By the proof of [Fuj11b, Theorem 10.5(iv)], we have

$$f'_*\mathcal{O}_{G'} \cong \kappa(x)$$

where G' is the pushforward of G on Y'. In particular, G' is connected. By applying [Fuj11c, Proposition 3.3] to each irreducible component of G', we can check that if W is a minimal log canonical center of (Y', E') with $f'(W) = x$, then $\dim W = \mu(x \in X)$. By this observation, every minimal stratum of E which is mapped to x by f is $\mu(x \in X)$-dimensional, and $\mu(x \in X)$ is independent of the choice of the resolution f (cf. [Fuj11c, 4.11]), that is, $\mu(x \in X)$ is well-defined.

Let W_1 and W_2 be any minimal log canonical centers of (Y', E') such that $f'(W_1) = f'(W_2) = x$. Then we can check that W_1 is birationally equivalent to W_2 (cf. [Fuj11c, Proposition 3.3]). Therefore, all the minimal strata of E mapped to x by f are birationally equivalent to each other. More precisely, we can take a common resolution

$$\begin{array}{ccc} & W & \\ {}^{\alpha_1}\swarrow & & \searrow^{\alpha_2} \\ W_1 & \dashleftarrow\dashrightarrow & W_2 \end{array}$$

such that $\alpha_1^*K_{W_1} = \alpha_2^*K_{W_2}$ (see [Fuj11c, Proposition 3.3]).

By the adjunction formula for dlt pairs (see [Fuj07, Proposition 3.9.2]), we can check that

$$K_W = (K_{Y'} + E')|_W \sim 0$$

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and that W has only canonical Gorenstein singularities if W is a minimal log canonical center of (Y', E') with $f'(W) = x$.

Let V be any minimal stratum of E. Then we can prove that

$$H^{\mu}(V, \mathcal{O}_V) \overset{\delta}{\cong} H^{n-1}(E, \mathcal{O}_E)$$

when $f(E) = x$ or, equivalently, when $x \in X$ is an isolated non-log-terminal point, where $\mu = \mu(x \in X)$ and $n = \dim X$. The isomorphism δ is a composition of connecting homomorphisms of suitable Mayer–Vietoris exact sequences. For the details, see [Fuj11c, § 5] (and also [Fuj12]). Although we assume that the base field is $\mathbb{C}$ and use the theory of mixed Hodge structures in [Fuj11c], the above isomorphism holds over an arbitrary algebraically closed field k of characteristic zero by the Lefschetz principle.

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