Ergod. Th. & Dynam. Sys. (1984), **4**, 527–539
Printed in Great Britain

A criterion for the positivity of the Liapunov characteristic exponent

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(*Received* 14 *October* 1983)

Abstract. We formulate sufficient conditions under which, for a finite subset of SL (2, ℝ), the maximal Liapunov exponent is positive. These conditions are based on the notion of compatible hyperbolicity. We then give an analytical formulation of such a condition and we apply this criterion to prove mixing properties of a particular transformation of the two-dimensional torus.

0. *Introduction*

Let X be a measure space with a probability measure μ and let $T: X \to X$ be a measure preserving transformation. Let $A: X \to \mathrm{SL}(2, \mathbb{R})$ be a measurable mapping. The *maximal Liapunov characteristic exponent* (m.L.c.e.) is by definition

$$\gamma^+(x) = \lim_{n \to +\infty} \ln \|A(T^{n-1}x) \cdots A(x)\|.$$

Oseledec's multiplicative ergodic theorem [**2**] asserts that $\gamma^+(x)$ exists almost everywhere (at least if $A(X)$ is bounded). The significance of the m.L.c.e. for the study of mixing properties of dynamical systems is now well established [**3**]. In this paper, we consider the case of a finite set $A(X) = \{A_1, \ldots, A_n\} \subset \mathrm{SL}(2, \mathbb{R})$ and formulate sufficient conditions under which the m.L.c.e. is positive. These conditions actually mean that, in some basis, all matrices from $A(X)$ have positive entries except for parabolic matrices which have only non-negative entries. We give an analytic formulation of such a condition. In particular, our criterion depends only very weakly on properties of $T: X \to X$. The criterion is an abstraction of methods used in proving positivity of the m.L.c.e. for some piecewise linear tranformations of the torus [**4**], [**1**], [**5**]. In § 3, we give another application of the criterion in the same spirit.

Our discussion is centred on the concept of compatible hyperbolicity of a set of matrices $\{H_1, \ldots, H_n\} \subset \mathfrak{sl}(2, \mathbb{R})$ which we study thoroughly in §§ 1, 2.

In proposition 1, we prove that the inverse of the exponental function in SL (2, ℝ) is linear up to a multiplication by a scalar. This is a crucial analytic tool in our work.

1. *Compatible hyperbolicity*

Let us consider the group SL (2, ℝ) of real 2×2 matrices with determinant equal to 1. $A \in \mathrm{SL}(2, \mathbb{R})$ is called a *hyperbolic matrix* if it has real eigenvalues different from

https://doi.org/10.1017/S0143385700002625 Published online by Cambridge University Press

1 and -1, an *elliptic matrix* if it has a pair of complex conjugate eigenvalues different from 1 and -1 and, finally, it is called a *parabolic matrix* if it has eigenvalues equal to either 1 or -1. Thus, we have that $A \in SL(2,\mathbb{R})$ is *hyperbolic* if $|\text{tr } A| > 2$, *elliptic* if $|\text{tr } A| < 2$ and *parabolic* if $|\text{tr } A| = 2$.

$\mathfrak{sl}(2,\mathbb{R})$ denotes the Lie algebra of $SL(2,\mathbb{R})$. It consists of real 2×2 matrices with zero trace. $H \in \mathfrak{sl}(2,\mathbb{R})$ is called *hyperbolic* if $e^{tH} \in SL(2,\mathbb{R})$ is hyperbolic for all real $t \neq 0$. Analogously, we define *elliptic* and *parabolic* elements of $\mathfrak{sl}(2,\mathbb{R})$.

The exponential function $\exp: \mathfrak{sl}(2,\mathbb{R}) \to SL(2,\mathbb{R})$ maps $\mathfrak{sl}(2,\mathbb{R})$ onto $\{A \in SL(2,\mathbb{R}) | \text{tr } A > -2 \text{ or } A = -I\}$. Moreover it is 1-1 on the subset of hyperbolic matrices. The following proposition states that the inverse function is linear up to multiplication by a scalar.

PROPOSITION 1. *Let*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = A \in SL(2,\mathbb{R}), \text{tr } A > -2$$

and

$$\begin{pmatrix} \dfrac{(a-d)}{2} & b \\ c & \dfrac{(d-a)}{2} \end{pmatrix} = H \in \mathfrak{sl}(2,\mathbb{R});$$

then there is $t > 0$ such that $e^{tH} = A$.

Proof. By straightforward computation, we have that the quadratic form $Q(x,y) = -cx^2 + (a-d)xy + by^2$ on $\mathbb{R}^2$ is invariant under the action of $A: \mathbb{R}^2 \to \mathbb{R}^2$.

Consider some non-zero quadratic form $\phi(x,y) = ex^2 + 2fxy + gy^2$.

We want to determine all one parameter subgroups of $SL(2,\mathbb{R})$ that preserve this quadratic form. For this purpose, let

$$\begin{pmatrix} p & q \\ r & -p \end{pmatrix} = K \in \mathfrak{sl}(2,\mathbb{R}),$$

and let $\mathbb{K}$ be the linear vector field in $\mathbb{R}^2$ defined by K. Taking the Lie derivative, we obtain

$$0 \equiv L_{\mathbb{K}}\phi(x,y) = 2(ep+fr)x^2 + 2(eq+gr)xy + 2(fq-gp)y^2.$$

Hence, we must have

$$\begin{aligned} ep + fr &= 0, \\ eq + gr &= 0, \\ fq - gp &= 0, \end{aligned}$$

since ϕ is preserved. For $e^2 + f^2 + g^2 > 0$ (i.e. $\{e, f, g\} \neq \{0\}$), these equations have an unique solution up to a multiplicative constant:

$$p = f, \quad q = g, \quad r = -e.$$

On the other hand, each $A \in SL(2,\mathbb{R})$ with $\text{tr } A > -2$ can be included in an unique one parameter subgroup of $SL(2,\mathbb{R})$, except for

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

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Also, for each $A \in \mathrm{SL}(2, \mathbb{R})$, except for I and $-I$, there is an invariant quadratic form (unique up to a constant). So we must have $e^{tH} = A$ for some $t \in \mathbb{R}$ and H defined above.

We have yet to prove that we can choose $t > 0$. For elliptic A, t is determined modulo the period; so clearly we can find $t > 0$. For hyperbolic A, t is uniquely determined and it is a continuous function of A nowhere equal to zero. Such matrices form an open connected subset of $\mathrm{SL}(2, \mathbb{R})$ (we consider only hyperbolic matrices with trace >2). So it is enough to check that t is positive for one diagonal matrix, which is obvious. □

Definition 2. A finite set $F = \{H_1, \ldots, H_n\} \subset \mathfrak{sl}(2, \mathbb{R})$ is called a *compatibly hyperbolic* (*compatibly non-elliptic*) family if every product

$$e^{t_k G_k} \cdots e^{t_1 G_1} \qquad \text{with } G_i \in F,\ t_i > 0,\ i = 1, \ldots, k$$

is a hyperbolic (non-elliptic) matrix. Let

$$\begin{pmatrix} p & q \\ r & -p \end{pmatrix} = H \in \mathfrak{sl}(2, \mathbb{R}).$$

We know that H is hyperbolic if $-\det H = p^2 + rq > 0$, elliptic if $p^2 + rq < 0$, and parabolic if $p^2 + rq = 0$. Hence, geometrically, elliptic matrices form the interior of the cone

$$S = \{H \in \mathfrak{sl}(2, \mathbb{R}) \mid -\det H \le 0\}.$$

For $\{H_1, \ldots, H_n\} \subset \mathfrak{sl}(2, \mathbb{R})$ we put

$$C(H_1, \ldots, H_n) = \{H \in \mathfrak{sl}(2, \mathbb{R}) \mid H = \lambda_1 H_1 + \cdots + \lambda_n H_n, \quad \lambda_i \ge 0, \quad i = 1, \ldots, n\}$$

i.e. $C(H_1, \ldots, H_n)$ is the cone spanned by $H_1, \ldots, H_n$. Note that S is centrally symmetric, $S = -S$, and $C(H_1, \ldots, H_n)$ is not, except for the cases when it is a linear subspace (the whole space, a plane or a line).

THEOREM 3. *Let $F = \{H_1, \ldots, H_n\} \subset \mathfrak{sl}(2, \mathbb{R})$. F is compatibly hyperbolic (non-elliptic) if and only if*

$$C(H_1, \ldots, H_n) \cap S = \{0\},$$

($C(H_1, \ldots, H_n) \cap \operatorname{int} S = \varnothing$) and $C(H_1, \ldots, H_n)$ is not a proper linear subspace (if $C(H_1, \ldots, H_n)$ is a plane then it must be tangent to S).

For the proof, we will need the following lemmas:

LEMMA 4. *If $F = \{H_1, \ldots, H_n\} \subset \mathfrak{sl}(2, \mathbb{R})$ is compatibly non-elliptic then*

$$C(H_1, \ldots, H_n) \cap \operatorname{int} S = \varnothing.$$

Proof. We have that $A(t) = e^{t\lambda_1 H_1} e^{t\lambda_2 H_2} \cdots e^{t\lambda_n H_n}$ for all $t > 0$ and fixed $\lambda_i \ge 0$ is a non-elliptic matrix in $\mathrm{SL}(2, \mathbb{R})$. Hence $dA(t)/dt|_{t=0}$ is certainly also non-elliptic i.e. it is outside int S. But $dA(t)/dt|_{t=0} = \lambda_1 H_1 + \cdots + \lambda_n H_n$. □

In the following lemma, we will interpret the conditions from theorem 3 in terms of the configuration of stable and unstable lines of $e^{t_1 H_1}, \ldots, e^{t_n H_n}$, $t_i > 0$. With every

$$\begin{pmatrix} p & q \\ r & -p \end{pmatrix} = H \in \mathfrak{sl}(2, \mathbb{R}),$$

we associate a quadratic form in $\mathbb{R}^2$ invariant for all e^{tH}, $t\in\mathbb{R}$,

$$Q_H(x,y) = -rx^2+2pxy+qy^2.$$

H is hyperbolic if Q_H is indefinite, elliptic if Q_H is definite and parabolic if Q_H is degenerate.

First, we must point out that the zero lines of Q_H are the eigendirections of H.

If H is hyperbolic then one, and the same, of the zero lines of Q_H is a stable line for all $e^{tH}\in SL(2,\mathbb{R})$, $t>0$. To describe which one, consider the space $\mathbb{R}P^1$ of all lines in $\mathbb{R}^2$ passing through the origin. e^{tH} defines in a natural way a mapping of $\mathbb{R}P^1$ into itself which we will also denote by e^{tH}. The fixed points of e^{tH}, $t>0$ in $\mathbb{R}P^1$ correspond to the stable and unstable lines of the linear mapping, we will denote them by s and u respectively. (They are hyperbolic fixed points – s is unstable and u is stable.)

Q_H is not defined on $\mathbb{R}P^1$ but the sign of Q_H is well defined on $\mathbb{R}P^1$. We choose the counterclockwise orientation on $\mathbb{R}P^1$. We claim that Q_H changes sign from $-$ to $+$ on u and from $+$ to $-$ on s, with respect to the chosen orientation.

The pattern must be the same for all hyperbolic matrices in $\mathfrak{sl}(2,\mathbb{R})$ since they form a connected set. So it is enough to check it only for one matrix, for instance the diagonal matrix

$$H=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$ □

LEMMA 5. *Let $C(H_1,H_2)\cap S=\{0\}$. Then $\lambda_1H_1+\lambda_2H_2$ is hyperbolic for all λ_1, λ_2, $\lambda_1^2+\lambda_2^2>0$, if and only if the fixed points of e^{tH_1}, u_1, s_1, and of e^{tH_2}, u_2, s_2 in $\mathbb{R}P^1$ alternate i.e. their order is u_1, u_2, s_1, s_2 or u_1, s_2, s_1, u_2 and are all different. $\lambda_1H_1+\lambda_2H_2$ is elliptic for some λ_1, λ_2 if and only if the fixed points of e^{tH_1} and e^{tH_2} appear in the order u_1, u_2, s_2, s_1 or u_2, u_1, s_1, s_2 and are all different.*

Proof. If the fixed points alternate and are in particular all different then the quadratic form $\lambda_1Q_{H_1}+\lambda_2Q_{H_2}$ has different signs at u_1, s_1 if $\lambda_2\neq 0$ and at u_2, s_2 if $\lambda_1\neq 0$. Hence the form is indefinite if $\lambda_1^2+\lambda_2^2>0$. This is equivalent to the hyperbolicity of $\lambda_1H_1+\lambda_2H_2$. If, on the contrary, the fixed points do not alternate, then without loss of generality, we can assume that the order of the fixed points is

$$u_1,\ u_2,\ s_2,\ s_1 \quad\text{or}\quad u_1,\ s_2,\ u_2,\ s_1,$$

where we do not exclude coincidence of some of the points. So when we take invariant lines of e^{tH_1} as coordinate axes then

$$Q_{H_1}(x,y)=fxy \qquad \text{with } f>0,$$

and

$$Q_{H_2}(x,y)=ax^2+bxy+cy^2 \qquad \text{where } ac\geq 0 \text{ and } b(a+c)<0.$$

The case $a\geq 0$, $c\geq 0$, $b<0$ corresponds to the ordering u_1, s_2, u_2, s_1 and the case $a\leq 0$, $c\leq 0$, $b>0$ to the ordering u_1, u_2, s_2, s_1. If $a\neq 0$ and $c\neq 0$ then all the fixed points are different.

Clearly, some linear combination $\lambda_1Q_{H_1}+\lambda_2Q_{H_2}$ is not indefinite and hence $\lambda_1H_1+\lambda_2H_2$ is not hyperbolic.

If $a>0$, $c>0$, $b<0$ then $\lambda_1H_1+\lambda_2H_2$ is elliptic for some $\lambda_1>0$, $\lambda_2>0$. □

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It can also be seen that, with the hypothesis of this lemma, $\lambda_1 H_1 + \lambda_2 H_2$ is parabolic only if $\lambda_1 H_1 + \lambda_2 H_2 = 0$. Thus u_i, s_i are distinct if $C(H_1, H_2)$ is not a line.

Proof of theorem 3. Let $u_1, \ldots, u_n$ and $s_1, \ldots, s_n$ denote the fixed points of $e^{tH_1}, \ldots, e^{tH_n}$ in $\mathbb{R}P^1$.

Sufficiency. By the condition for all $1 \le i, j \le n$, $C(H_i, H_j) \cap S = \{0\}$ and $C(H_i, H_j)$ is not a line. Hence, by lemma 5, $u_i \ne s_j$ and $u_j \ne s_i$. So

$$\{u_1, \ldots, u_n\} \cap \{s_1, \ldots, s_n\} = \varnothing$$

(we do not exclude the possibility that some of the points $u_1, \ldots, u_n$ or $s_1, \ldots, s_n$ coincide).

Let us consider a continuous deformation of $H_1, \ldots, H_n$ in the set of hyperbolic matrices.

$$H_i(t) = tH_1 + (1-t)H_i, \qquad 0 \le t \le 1.$$

We have $C(H_1(t), \ldots, H_n(t)) \cap S = \{0\}$ and $C(H_1(t), \ldots, H_n(t))$ is not a line or a plane for all $0 \le t \le 1$. Consequently,

$$\{u_1(t), \ldots, u_n(t)\} \cap \{s_1(t), \ldots, s_n(t)\} = \varnothing,$$

where $u_i(t)$, $s_i(t)$ are the fixed points of e^{tH_i} in $\mathbb{R}P^1$ corresponding to the unstable and stable lines respectively; $u_i(1) = u_1$, $s_i(1) = s_1$, $i = 1, \ldots, n$. By continuity we conclude that there must be an interval $I \subset \mathbb{R}P^1$ such that $u_i \in \operatorname{int} I$ and $s_i \notin I$, $i = 1, \ldots, n$. This interval clearly has the property $e^{tH_i} I \subset \operatorname{int} I$, $t > 0$, $i = 1, \ldots, n$. The same holds for any composition of the matrices e^{tH_i}, $t > 0$, $i = 1, \ldots, n$. But only hyperbolic matrices in SL $(2, \mathbb{R})$ have the property that they map some cone (interval in $\mathbb{R}P^1$) strictly into itself. This ends the proof of sufficiency in the hyperbolic version.

In the non elliptic case, we allow $C(H_1, \ldots, H_n)$ to be a line, a plane tangent to S or a cone tangent to S. Both a line and a plane tangent to S are subalgebras of $\mathfrak{sl}(2, \mathbb{R})$ corresponding to Lie subgroups of SL $(2, \mathbb{R})$. The latter is the subgroup of matrices which are triangular in a certain basis, i.e. a subgroup of matrices having a common invariant line. All elements of this subgroup are hyperbolic except for one parameter subgroups of parabolic matrices. So clearly in these cases we have compatible non-ellipticity.

The case of the proper cone tangent to S is similar to the hyperbolic case and we omit the details.

Necessity. In view of lemmas 4 and 5, we have to prove that $C(H_1, \ldots, H_n)$ cannot be a plane (if it is a plane then it must be tangent to S in the non-elliptic case).

Assume on the contrary that for example $C(H_1, H_2, H_3)$ is a plane. If the plane is tangent to S then $e^{t_1 H_1}$, $e^{t_2 H_2}$ and $e^{t_3 H_3}$ have a common invariant line which is a stable line for one of them and unstable for another. It follows easily that one of their compositions is parabolic. So assume $C(H_1, H_2, H_3)$ is not tangent to S. For the purpose of performing explicit computations, we will simplify H_1, H_2, H_3 by a change of variables. First of all we can make them all symmetric. This is so because by lemma 5 the eigenvectors of H_1 and H_2 alternate and so there is a linear

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transformation making both pairs orthogonal simultaneously. H_3 will become symmetric automatically and we can take care that it is diagonal. Using proposition 1 we can, without loss of generality, assume that for some $t_1>0$, $t_2>0$

$$e^{t_1H_1}=\begin{pmatrix} a_1 & 1\\ 1 & d_1\end{pmatrix},\quad e^{t_2H_2}=\begin{pmatrix} a_2 & -1\\ -1 & d_2\end{pmatrix}\quad\text{and}\quad e^{tH_3}=\begin{pmatrix} e^{vt} & 0\\ 0 & e^{-vt}\end{pmatrix}$$

where $v>0$, while

$$H_1=\begin{pmatrix} \frac{a_1-d_1}{2} & 1\\ 1 & \frac{d_1-a_1}{2}\end{pmatrix},\quad H_2=\begin{pmatrix} \frac{a_2-d_2}{2} & -1\\ -1 & \frac{d_2-a_2}{2}\end{pmatrix}\quad\text{and}\quad H_3=\begin{pmatrix} v & 0\\ 0 & -v\end{pmatrix}.$$

By assumption, there are $\lambda_1>0$, $\lambda_2>0$, $\lambda_3>0$ such that

$$\lambda_1H_1+\lambda_2H_2+\lambda_3H_3=0.$$

Hence $a_1-d_1+a_2-d_2<0$ i.e. $a_1+a_2<d_1+d_2$. But $d_i=2/a_i$, $i=1,2$ since $\det e^{t_iH_i}=1$. We conclude that $a_1a_2<2$ and $d_1d_2>2$. We have

$$\operatorname{tr} e^{t_1H_1}e^{tH_3}e^{t_2H_2}=e^{vt}(a_1a_2-1)+e^{-vt}(d_1d_2-1).$$

It is straightforward to show that there is a $t>0$ for which the trace above is less than 2 (and positive). Hence $\{H_1, H_2, H_3\}$ are not even compatibly hyperbolic. □

From the proof we derive in addition the following:

PROPOSITION 6. *A set $\{H_1,\dots,H_n\}\subset\mathfrak{sl}(2,\mathbb{R})$ is compatibly hyperbolic (non-elliptic) if and only if there is a basis in which all e^{tH_i}, $t>0$, $i=1,\dots,n$ have positive (non-negative) entries.*

Proof. If $\{H_1,\dots,H_n\}$ is compatibly hyperbolic then there is a cone in $\mathbb{R}^2$ which is mapped strictly into itself by all e^{tH_i}, $t>0$, $i=1,\dots,n$. If we choose the sides of the cone as the coordinate axes then all e^{tH_i}, $t>0$, $i=1,\dots,n$ will become matrices with positive entries.

The sufficiency of the condition is obvious. □

We will now express the conditions from theorem 3 in an analytic form. It is enough to do it only for a triple of matrices since $\{H_1,\dots,H_n\}$ is compatibly hyperbolic (non-elliptic) if every triple is. Let

$$H_i=\begin{pmatrix} p_i & q_i\\ r_i & -p_i\end{pmatrix},$$

$$Q_{H_i}(x,y)=-r_ix^2+2p_ixy+q_iy^2,\qquad i=1,2,3.$$

Now $C(H_1,H_2,H_3)\cap S=\{0\}$ and $C(H_1,H_2,H_3)$ is different from a proper subspace if and only if the quadratic form

$$\lambda_1Q_{H_1}+\lambda_2Q_{H_2}+\lambda_3Q_{H_3}$$

is indefinite for all $\lambda_1\geq 0$, $\lambda_2\geq 0$, $\lambda_3\geq 0$, $\lambda_1^2+\lambda_2^2+\lambda_3^2>0$. For the pair of hyperbolic matrices $\{H_i, H_j\}$ put $k_{ij}=\operatorname{tr} H_iH_j/\det H_i \det H_j$.

PROPOSITION 7. *$\{H_1, H_2, H_3\}$ is a compatibly hyperbolic family if and only if the quadratic form*

$$\Delta(\lambda_1, \lambda_2, \lambda_3) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + k_{12}\lambda_1\lambda_2 + k_{23}\lambda_2\lambda_3 + k_{13}\lambda_1\lambda_3$$

is positive for all $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, $\lambda_3 \geq 0$, $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 > 0$.

Proof. Straightforward computation. □

The condition from proposition 7 can be expressed explicitly in terms of the coefficients k_{12}, k_{23}, k_{13} but the formulation is so involved that it is of little if any interest. There are clearly simple numerical methods to determine the compatible hyperbolicity of $\{H_1, H_2, H_3\}$.
For two hyperbolic matrices $\{H_1, H_2\}$ the situation is simpler.

PROPOSITION 8. *$\{H_1, H_2\}$ is a compatibly hyperbolic (non-elliptic) pair if and only if $k_{12} > -2$ ($k_{12} \geq -2$).*

If our family of matrices F contains parabolic matrices then checking compatible non-ellipticity becomes simpler. First, it follows immediately from theorem 3 that a compatibly non-elliptic family of matrices can contain at most two non-colinear (equivalently: non-commuting) parabolic matrices.

PROPOSITION 9. *Let $F = \{H_1, \ldots, H_n\} \subset \mathfrak{sl}(2, \mathbb{R})$ be a family of hyperbolic matrices and let P_1, P_2 be two parabolic, non-commuting matrices. If $\operatorname{tr} P_1P_2 > 0$ and $\operatorname{tr} H_jP_i > 0$, $i = 1, 2$, $j = 1, \ldots, n$, then F is compatibly hyperbolic.*

Proof. For a parabolic matrix $P \in \mathfrak{sl}(2, \mathbb{R})$ consider the set $\{X \in \mathfrak{sl}(2, \mathbb{R}) | \operatorname{tr} XP > 0\}$. Geometrically it is a half-space on one side of the plane tangent to S and containing P. It is the half-space which does not contain the part of S in which P lies. In particular $\operatorname{tr} P_1P_2 > 0$ means that P_1 and P_2 lie in different parts of S. Hence

$$K = \{X \in \mathfrak{sl}(2, \mathbb{R}) | \operatorname{tr} XP_1 > 0 \text{ and } \operatorname{tr} XP_2 > 0\}$$

is a 'quarter' of the space that intersects S along two half-lines passing through P_1 and P_2. In particular K is convex and does not contain any plane or line. Hence, by theorem 3, the conditions are sufficient for hyperbolicity of F. □

The geometric considerations in the proof above and theorem 3 give us the following criterion:

PROPOSITION 10. *$\{H_1, \ldots, H_n\} \subset \mathfrak{sl}(2, \mathbb{R})$ is a compatibly hyperbolic family if and only if there are two parabolic matrices P_1, P_2 such that $\operatorname{tr} P_1P_2 > 0$ and $\operatorname{tr} H_jP_i > 0$, $j = 1, \ldots, n$, $i = 1, 2$.*

We can formulate one more criterion.

PROPOSITION 11. *Let $F = \{H_1, \ldots, H_n\} \subset \mathfrak{sl}(2, \mathbb{R})$ be a family of hyperbolic matrices and let P be a parabolic matrix. If $\operatorname{tr} H_iP \geq 0$, $i = 1, \ldots, n$, and $k_{ij} > -2$, $1 \leq i, j \leq n$, then F is compatibly hyperbolic.*

The proof is very much in the spirit of previous arguments and we omit it.

One can consider a discrete counterpart of compatible hyperbolicity for a finite set of matrices from $SL(2, \mathbb{R})$. Theorem 3 provides a sufficient condition for such a property but not a necessary condition. Indeed the following theorem holds:

THEOREM 12. *Let* $F=\{H_1,\ldots,H_n\}\subset s\ell(2,\mathbb{R})$ *be a family of hyperbolic matrices such that no unstable line of* e^{tH_i}, $t>0$, $i=1,\ldots,n$, *coincides with any of their stable lines then there is* $T>0$ *such that* $e^{t_1G_1}\cdots e^{t_kG_k}$, $G_i\in F$, $i=1,\ldots,k$, *is hyperbolic if* $t_i\geq T$, $i=1,\ldots,k$.

Proof. Let u_i, s_i be the fixed points of e^{tH_i}, $t>0$, in $\mathbb{R}P^1$ corresponding respectively to the unstable and stable lines. By the assumption there are closed intervals $I_1,\ldots,I_n$, $u_i\in \operatorname{int} I_i$, such that $s_j\notin\bigcup_{i=1}^n I_i$. So there is $T>0$ such that if $t\geq T$ then $e^{tH_j}I_j\subset \operatorname{int} I_j$, $1\leq i,j\leq n$. It follows that if $G_1=H_i$ then $e^{t_1G_1}\cdots e^{t_kG_k}I_i\subset \operatorname{int} I_i$. □

2. *Liapunov exponents*

We fix some norm (for instance the euclidian norm) in $\mathbb{R}^2$.

THEOREM 13. *Let* $F=\{H_1,\ldots,H_n\}\subset s\ell(2,\mathbb{R})$. *F is compatibly hyperbolic if and only if there are* $C>0$, $d>0$ *and a cone* $V\subset\mathbb{R}^2$ *such that*

$$\|e^{t_kG_k}\cdots e^{t_1G_1}v\|\geq C\,e^{Td}\|v\|,$$

where $G_i\in F$, $t_i>0$, $i=1,\ldots,k$, $T=t_1+\cdots+t_k$ *and* $v\in V$.

Proof. Sufficiency is obvious. For the proof of necessity we can, by proposition 6, without loss of generality, assume that all e^{tH_i}, $t>0$, $i=1,\ldots,n$ have positive entries. Hence, by proposition 1,

$$H_i=\begin{pmatrix}p_i & q_i\\ r_i & -p_i\end{pmatrix}$$

with $q_i>0$, $r_i>0$, $i=1,\ldots,n$. Let $q=\min_i q_i$, $r=\min_i r_i$, $p=\max_i |p_i|$. Consider a linear system of differential equations

$$\dot{u}=M(t)u,\qquad u\in\mathbb{R}^2, \tag{1}$$

where $M(t)$ is piecewise constant

$$M(t)=G_i\qquad \text{if } t_1+\cdots+t_{i-1}<t\leq t_1+\cdots+t_{i-1}+t_i,\quad i=1,\ldots,n.$$

The vector on the left hand side of the inequality is equal to $u(T)$ where $u(t)$, $0\leq t\leq T$ is a solution of (1) with the initial value $u(0)=v$.

We will find a Liapunov function for (1) in the positive octant (this is the cone V). Consider a quadratic function

$$f(x,y)=ax^2+by^2+2xy.$$

There are choices of $a>0$, $b>0$ and $g>0$ such that

$$\frac{df}{dt}(x,y)\geq gf(x,y)\qquad \text{for } x\geq 0,\ y\geq 0.$$

Indeed we have for $x\geq 0$, $y\geq 0$,

$$\frac{df}{dt}(x,y)\geq 2((r-ap)x^2+(q-bp)y^2+(aq+br)xy)\geq g(ax^2+by^2+2xy),$$

if a, b, g are sufficiently close to zero. As a consequence, we have

$$f(x(t),y(t))\geq e^{tg}f(x(0),y(0)),\qquad 0\geq t\geq T,$$

if $x(0)\ge 0$, $y(0)\ge 0$. On the other hand there is a constant $\gamma>0$ such that for $v=(x,y)$, $x\ge 0$, $y\ge 0$

$$\gamma^{-1}\|v\|\le\sqrt{f(x,y)}\le\gamma\|v\|.$$

Combining the estimates above we get the desired inequality. □

By similar arguments we can obtain:

THEOREM 14. *If $F=\{H_1,\dots,H_n\}\subset\mathfrak{sl}(2,\mathbb{R})$ is compatibly hyperbolic and $F\cup\{P_1,P_2\}$ is compatibly non-elliptic where P_1, P_2 are parabolic matrices then there are constants $C>0$, $d>0$ and a cone $V\subset\mathbb{R}^2$ (V is bounded by invariant lines of P_1, P_2 if they are different) such that*

$$\|e^{t_kG_k}\cdots e^{t_1G_1}v\|\ge C\,e^{Td}\|v\|,$$

where $G_i\in F\cup\{P_1,P_2\}$, $t_i>0$, $i=1,\dots,k$, $T=\sum_{G_i\in F}t_i$ and $v\in V$.

We are now ready to formulate our criterion for positivity of the maximal Liapunov characteristic exponent. Let $T:X\to X$ be a measure preserving transformation of the probability space (X,μ) and let $A:X\to \mathrm{SL}(2,\mathbb{R})$ be a measurable mapping. We assume that the values of A are non-elliptic matrices. Without loss of generality we can assume that traces of matrices in $A(X)$ are ≥ 2 (multiplication of some of the matrices by -1 does not affect Liapunov exponents).

Case I (*one parabolic matrix*). Suppose that $A(X)=\{A_1,\dots,A_n,B_1\}$ where A_i are hyperbolic, $i=1,\dots,n$, and B_1 is parabolic (i.e. $\operatorname{tr}B_1=2$).

If, for every $1\le i,j\le n$

$$2\operatorname{tr}A_iA_j-\operatorname{tr}A_i\operatorname{tr}A_j>-\sqrt{(\operatorname{tr}A_i)^2-4}\sqrt{(\operatorname{tr}A_j)^2-4},$$

and for every $1\le i\le n$

$$\operatorname{tr}(B_1-I)A_i\ge 0\quad\text{and}\quad \left(\bigcap_{n=0}^{\infty}T^{-n}(A^{-1}\{B_1\})\right)=0,$$

then the m.L.c.e. is positive almost everywhere.

Case II (*two parabolic matrices*). Suppose that $A(X)=\{A_1,\dots,A_n,B_1,B_2\}$ where A_i are hyperbolic, $i=1,\dots,n$, and B_1, B_2 are parabolic (i.e. $\operatorname{tr}B_1=\operatorname{tr}B_2=2$). If $\operatorname{tr}B_1B_2>2$ and for every $1\le i\le n$, $j=1,2$,

$$\operatorname{tr}(B_j-I)A_i>0\quad\text{and}\quad \left(\bigcap_{n=0}^{\infty}T^{-n}(A^{-1}\{B_1,B_2\})\right)=0,$$

then the m.L.c.e. is positive almost everywhere.

Proof. We have $A_i=e^{tH_i}$, $t>0$, with hyperbolic $H_i\in\mathfrak{sl}(2,\mathbb{R})$, $i=1,\dots,n$, and $B_j=e^{tP_j}$, $t>0$, with parabolic $P_j\in\mathfrak{sl}(2,\mathbb{R})$, $j=1,2$.

Using proposition 1, we express sufficient conditions for compatible hyperbolicity of $\{H_1,\dots,H_n\}$ using proposition 9 and proposition 11 and we get the conditions above. Now we are in a position to use theorem 14 which gives us immediately positivity of the m.L.c.e. □

3. *Application*

In this final section, we will deal with a particular kind of measure-preserving transformation on the 2-dimensional torus $\mathbb{T}^2$,

$$T(x_1, x_2) = (x_2, -x_1 + Cx_2 + f(x_2))$$

where f is a periodic function and C is an integer constant. We will use the square $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$ in $\mathbb{R}^2$ as the fundamental domain of $\mathbb{T}^2$. Let $S(x_1, x_2) = (x_2, x_1)$. T is S-reversible i.e. $S \circ T \circ S = T^{-1}$.

When $|C|$ is >2 and f is a smooth, C^1-small function, we have an Anosov diffeomorphism. It is also of interest that for $|C| \leq 2$ and an appropriate 'perturbation' f, the m.L.c.e. are positive in some part of $\mathbb{T}^2$, thus ensuring strong mixing properties of T (cf. [3], [4]). The case $C = 2$ corresponds to perturbations of the twist map and was treated in [4] and [5]. For $C = -2, -1, 1$, one can get results similar to those of [4] by essentially the same approach (this was done explicitly in [1]). We will obtain interesting dynamical behaviour for the case $C = 0$ using the criterion developed in previous sections (since it was proved in [1] that the criterion of [4] doesn't work in this case).

Thus, we study $T(x_1, x_2) = (x_2, -x_1 + f(x_2))$ with

$$f(t) = \begin{cases} -at - d & \text{on } [-\frac{1}{2}, 0], \\ \ \ at + d & \text{on } [0, \frac{1}{2}]. \end{cases}$$

So T is linear in B_+ and B_- where

$$B_\pm = \{(x, y) | 0 \leq \pm y \leq \tfrac{1}{2}, -\tfrac{1}{2} \leq x \leq \tfrac{1}{2}\}.$$

The matrix of T (or DT) in B_+ is D_1 and in B_- is D_2 where:

$$D_1 = \begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix}, \qquad D_2 = \begin{pmatrix} 0 & 1 \\ -1 & -a \end{pmatrix}.$$

We specify d to be such that $(-\frac{1}{4}, -\frac{1}{4})$ and $(\frac{1}{4}, \frac{1}{4})$ are fixed points for T i.e. $d = \frac{1}{4}a + \frac{1}{2}$.

For $|a| < 2$, our transformation is a rotation about the fixed point $(-\frac{1}{4}, -\frac{1}{4})$ in B_- and about the fixed point $(\frac{1}{4}, \frac{1}{4})$ in B_+. So we have two invariant 'elliptic islands' $I_\pm = \bigcap_{k=-\infty}^{\infty} T^k(B_\pm)$. These domains are ellipses if the rotation is irrational and polygons if it is rational. We shall prove the following:

THEOREM 15. *For $a = 2\cos \pi/n$, $n = 2, 3, \ldots$, T has positive m.L.c.e. almost everywhere in $\mathbb{T}^2 \backslash (I_+ \cup I_-)$. For the values of a described in this theorem, I_+ and I_- are polygons with $2n$ sides symmetric with respect to the diagonal (i.e. $S(I_\pm) = I_\pm$) in view of S-reversibility. Outside of them the interaction of different rotations produces strong mixing properties.*

Proof. We will consider the return map $\tilde{T}: B_+ \cap C_- \circlearrowleft$ where $C_\pm = T(B_\pm) = \{(x_1, x_2) | 0 \leq \pm x_1 \leq \frac{1}{2}\}$. Clearly it is enough to prove that $\tilde{T}$ has positive m.L.c.e.

$D\tilde{T}$ is equal to $D_2^j D_1^i$ where i is the number of times a point from $B_+ \cap C_-$ stays in B_+ before leaving it and j is the number of times it stays in B_- before returning to B_+. The crucial observation is that $1 \leq i, j \leq n-1$ and $2 \leq i + j \leq n$. The latter is

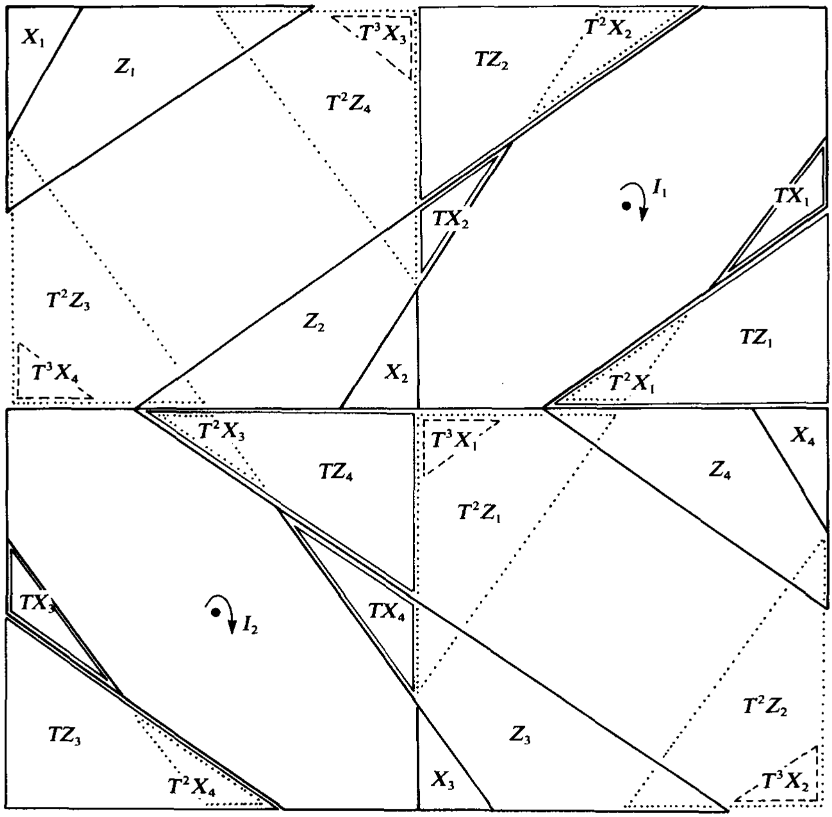

FIGURE 3-1. Case $n=4$. The domains I_i are the periodic elliptic islands, the zones Z_i stay only twice (i.e. Z_i and $T(Z_i)$) in the same part of the torus (i.e. where it's always the same differential) B_+ or B_- before going to the other half of $\mathbb{T}^2$. The X_i stay for 3 iterations (X_i, $T(X_i)$ and $T^2(X_i)$) in the same half-part of the torus. So the only allowed products are D_2D_1, $D_2D_1^2$, $D_2D_1^3$, $D_2^2D_1$, $D_2^2D_1^2$ and $D_2^3D_1$.

messy to prove but can be seen fairly easily geometrically as for example in the case $n=4$ shown in figure 3-1.

D_1^i and D_2^j can be expressed in the following form (as proved in [5])

$$D_1^i=\frac{(-1)^{i+1}}{\sin \pi/n}\begin{pmatrix}\sin (i-1)\pi/n & \sin i\pi/n \\ -\sin i\pi/n & -\sin (i+1)\pi/n\end{pmatrix}$$

$$D_2^j=\frac{1}{\sin \pi/n}\begin{pmatrix}-\sin (j-1)\pi/n & \sin j\pi/n \\ -\sin j\pi/n & \sin (j+1)\pi/n\end{pmatrix}$$

and thus

$$D_1^iD_2^j=\frac{(-1)^{i+1}}{\sin^2 \pi/n}\begin{pmatrix}d_{11} & d_{12} \\ d_{21} & d_{22}\end{pmatrix}$$

where

$$d_{11}=-\sin (j-1)\pi/n \sin (i-1)\pi/n-\sin i\pi/n \sin j\pi/n$$

$$d_{12}=-\sin (j-1)\pi/n \sin i\pi/n-\sin j\pi/n \sin (i+1)\pi/n$$

https://doi.org/10.1017/S0143385700002625 Published online by Cambridge University Press

$$d_{21} = -\sin j\pi/n \sin (i-1)\pi/n - \sin i\pi/n \sin (j+1)\pi/n$$
$$d_{22} = -\sin j\pi/n \sin i\pi/n - \sin (j+1)\pi/n \sin (i+1)\pi/n.$$

So we have

$$\begin{aligned} \operatorname{tr} D_1^i D_2^j &= \frac{(-1)^i}{\sin^2 \pi/n}[\sin (j-1)\pi/n \sin (i-1)\pi/n + 2 \sin i\pi/n \sin j\pi/n \\ &\quad + \sin (j+1)\pi/n \sin (i+1)\pi/n] \\ &= \frac{(-1)^i}{\sin^2 \pi/n}[(\sin i\pi/n \sin j\pi/n)(2 + 2\cos^2 \pi/n) \\ &\quad + 2 \cos j\pi/n \cos i\pi/n \sin^2 \pi/n]. \end{aligned}$$

After multiplying the matrices $D_2^j D_1^i$ by -1 when necessary, we see that all the traces are ≥ 2. The trace is equal to 2 for $i = n-1$, $j = 1$ and $i = 1$, $j = n-1$.

We put

$$B_1 = \begin{pmatrix} 1 & 4\cos \pi/n \\ 0 & 1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 1 & 0 \\ 4\cos \pi/n & 1 \end{pmatrix},$$

and let $\{A_1, \ldots, A_n\}$ be the set of all hyperbolic matrices $D_2^j D_1^i$ multiplied by -1 if necessary.

Now we apply the criterion from § 2 to $\tilde{T}: B_+ \cap C_- \circlearrowleft$ and $A: B_+ \cap C_- \to SL(2, \mathbb{R})$, $A(x) = D\tilde{T}_x$ (normalised to give trace ≥ 2). So we have to check that

$$\operatorname{tr} B_1 B_2 > 2, \tag{2}$$

$$\operatorname{tr} (B_k - I)A_i > 0, \tag{3}$$

and

$$\left(\bigcap_{n=0}^{\infty} \tilde{T}^{-n}(A^{-1}\{\pm B_1, \pm B_2\})\right) = 0. \tag{4}$$

Since

$$\begin{aligned} B_1 B_2 &= \begin{pmatrix} 1 & 4\cos \pi/n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 4\cos \pi/n & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 + 16\cos^2 \pi/n & 4\cos \pi/n \\ 4\cos \pi/n & 1 \end{pmatrix}, \end{aligned}$$

it's obvious that (2) is satisfied.

(3) means in our case that the matrices A_i have all positives entries. The property (4) is easily obtained from geometric considerations. Thus the m.L.c.e. are positive almost everywhere in $\mathbb{T}^2 \backslash (I_+ \cup I_-)$. □

Acknowledgments. We thank the Fonds National de la Recherche Scientifique of Belgium which made possible the stay of Maciej Wojtkowski at F.N.D.P. during which this work was begun, the Facultes Notre Dame de la Paix at Namur which supported the research of Eric Cornelis for this paper and finally the University of California, Berkeley, where Maciej Wojtkowski wrote part of this paper on leave from Warsaw University, Poland. We also want to thank the referees for their helpful remarks.

E. Cornelis is a Research Assistant of the National Fund for Scientific Research (Belgium).

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