Canad. Math. Bull. 2024, pp. 1–15
http://dx.doi.org/10.4153/S0008439524000559

Simple weight modules over the $N = 1$ Heisenberg–Virasoro superalgebra

Chao Lan and Dong Liu

Abstract. This paper presents a classification of all simple Harish-Chandra modules for the $N = 1$ Heisenberg–Virasoro superalgebra, which turn out to be highest weight modules, lowest weight modules, and evaluation modules of the intermediate series (all weight spaces are one dimensional). Moreover, a characterization of the tensor product of highest weight modules with intermediate series modules is obtained.

1 Introduction

Harish-Chandra modules play important roles in the representation theory of Lie algebras and Lie superalgebras. They are characterized by certain algebraic properties that make them amenable to the study of the asymptotic behavior of the corresponding Lie group or Lie supergroup. In various Lie superalgebras such as the $N = 1$ Ramond algebra in [7] and the Witt superalgebra in [6, 23], the classifications of such modules have been completed through the use of the A cover theory introduced in [5]. These Lie superalgebras are all $\mathbb{Z}$-graded. However, the task of classifying all simple jet modules for $\frac{1}{2}\mathbb{Z}$-graded Lie superalgebras is more complicated, as demonstrated in [8] for the case of the $N = 1$ Neveu-Schwarz algebra.

This paper appears to be focused on the study of the representation theory of the $N = 1$ Heisenberg–Virasoro superalgebra $\mathfrak{g}$ (a kind of superconformal current algebra), which is a Lie superalgebra that arises in the context of mathematical physics and theoretical physics [12]. It is a supersymmetric extension of the twisted Heisenberg–Virasoro algebra. This algebra can be realized from Balinsky–Novikov superalgebras, which construct local translation invariant Lie superalgebras of vector-valued functions on the line, as described in [20]. The Verma modules, Whittaker modules, and smooth modules over $\mathfrak{g}$ are studied in several papers [1, 2, 15]. Drawing on the research conducted on the $N = 1$ Neveu-Schwarz algebra in [8] and the Ovsienko-Roger superalgebra in [10], this paper classifies all simple Harish-Chandra modules over $\mathfrak{g}$ using methods developed in [14]. Note that our methods here are appropriate to the $\mathbb{Z}$-graded and $\frac{1}{2}\mathbb{Z}$-graded superconformal current algebras. Such

Received by the editors June 24, 2024; revised August 3, 2024; accepted August 15, 2024.
This work was supported by the National Natural Science Foundation of China (Grant No. 12071405).
AMS Subject Classification: **17B65, 17B68, 17B70.**
Keywords: Heisenberg–Virasoro superalgebra, Harish-Chandra module, Verma module, tensor product, intermediate series.

researches for the $\mathbb{Z}$-graded superconformal current algebra were completed in [11] by the A-cover theory, which cannot be refer to the $\frac{1}{2}\mathbb{Z}$ case.

To study tensor products of well-known modules is a basic task in representation theory of Lie (super)algebras. Recently, the tensor products of highest weight modules and Harish-Chandra modules of the intermediate series, which are not Harish-Chandra modules, over the Virasoro algebra, the twisted Heisenberg–Virasoro algebra, and the super Virasoro algebra, the Schrödinger-Virasoro algebra were studied in [9, 16, 18, 25, 26], etc. In this paper, we solve this problem for the $N = 1$ Heisenberg–Virasoro superalgebra with singular vectors constructed in [1, 2]. The necessary and sufficient conditions for these tensor products to be simple are studied and an isomorphism theorem for such tensor products is given.

The paper is organized as follows: Section 2 presents basic results for our study, Sections 3 and 4 classify all simple Harish-Chandra modules (Theorems 3.2, 4.3). The necessary and sufficient conditions for the tensor product of the highest weight modules, and the modules of the intermediate series over $\mathfrak{g}$ to be simple are studied (Theorem 6.6) and an isomorphism theorem (Theorem 6.8) for such tensor products is given in Sections 5 and 6.

Throughout this paper, we will use the following notations: $\mathbb{C}$, $\mathbb{N}$, $\mathbb{Z}_+$, and $\mathbb{Z}$ refer to the sets of complex numbers, nonnegative integers, positive integers, and integers, respectively.

2 Preliminaries

In this section, we recall some definitions and results for later use.

2.1 Basic definitions

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be any $\mathbb{Z}_2$-graded vector space. Then any element $u \in V_{\bar{0}}$ (resp. $u \in V_{\bar{1}}$) is said to be even (resp. odd). We define $\bar{u} = 0$ if u is even and $\bar{u} = 1$ if u is odd. Elements in $V_{\bar{0}}$ or $V_{\bar{1}}$ are called homogeneous. Whenever $\bar{u}$ is written, it is understood that u is homogeneous.

Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a Lie superalgebra, an L-module is a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ together with a bilinear map, $L \times V \to V$, denoted $(x, v) \mapsto xv$ such that

$$x(yv) - (-1)^{\bar{x}\bar{y}} y(xv) = [x, y]v,$$

for all $x, y \in L, v \in V$, and $L_i V_j \subseteq V_{i+j}$ for all $i, j \in \mathbb{Z}_2$. It is clear that there is a parity change functor Π on the category of L-modules, which interchanges the $\mathbb{Z}_2$-grading of a module. We use $U(L)$ to denote the universal enveloping algebra of the Lie superalgebra L.

2.2 The $N = 1$ Heisenberg–Virasoro superalgebra

Definition 2.1 [12] The $N = 1$ Heisenberg–Virasoro superalgebra $\mathfrak{g} := \mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$, where $\mathfrak{g}_{\bar{0}} := \mathbb{C}\{L_m, H_m, C_1, C_2, C_3 \mid m \in \mathbb{Z}\}$ and $\mathfrak{g}_{\bar{1}} = \mathbb{C}\{G_p, F_p \mid p \in \mathbb{Z} + \frac{1}{2}\}$, with the

https://doi.org/10.4153/S0008439524000559 Published online by Cambridge University Press

following relations:

$$
\begin{aligned}
&[L_m, L_n] = (n-m)L_{n+m} + \delta_{m+n,0}\frac{1}{12}(m^3-m)C_1,\\
&[L_m, H_n] = nH_{n+m} - \delta_{m+n,0}(m^2+m)C_2,\\
&[H_m, H_n] = m\delta_{m+n,0}C_3,\\
&[L_m, G_p] = (p-\frac{m}{2})G_{p+m}, [L_m, F_p] = (p+\frac{m}{2})F_{m+p},\\
&[G_p, G_q] = -2L_{p+q} + \frac{1}{3}(p^2-\frac{1}{4})\delta_{p+q,0}C_1,\\
&[G_p, F_q] = H_{p+q} + (2p+1)\delta_{p+q,0}C_2,\\
&[H_m, G_p] = mF_{m+p},\ [F_m, F_n] = \delta_{m+n,0}C_3,
\end{aligned}
$$

for $m, n \in \mathbb{Z},\ p, q \in \mathbb{Z} + \frac{1}{2}$.

Note that $\mathfrak{g}$ is equipped with a triangular decomposition and $\frac{1}{2}\mathbb{Z}$-graded structure: $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}^-$, where

$$
\begin{aligned}
\mathfrak{g}^{\pm} &= \bigoplus_{n\in\mathbb{Z}_+, r\in\mathbb{N}+\frac{1}{2}} (\mathbb{C}L_{\pm n} \oplus \mathbb{C}H_{\pm n} \oplus \mathbb{C}G_{\pm r} \oplus \mathbb{C}F_{\pm r}),\\
\mathfrak{g}_0 &= \mathbb{C}H_0 \oplus \mathbb{C}L_0 \oplus \oplus_{i=1}^{3}\mathbb{C}C_i.
\end{aligned}
$$

Moreover,

(2.1) $$\mathfrak{g} = \bigoplus_{k\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}_k,$$

is $\frac{1}{2}\mathbb{Z}$-graded with $\mathfrak{g}_i = \mathbb{C}L_i \oplus \mathbb{C}H_i$, $\mathfrak{g}_r = \mathbb{C}G_r \oplus \mathbb{C}F_r$ for $i \in \mathbb{Z}^*, r \in \mathbb{Z} + \frac{1}{2}$.

The $N = 1$ Heisenberg–Virasoro superalgebra $\mathfrak{g}$ has a 5-dimensional canonical Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_0$ and C_i, $i = 1, 2, 3$ with H_0 span the center of $\mathfrak{g}$.

For convenience, we set

$$\mathfrak{b}^+ = \mathfrak{g}^+ \oplus \mathfrak{h},\ \ \mathfrak{b}^- = \mathfrak{g}^- \oplus \mathfrak{h}.$$

Now we consider the following subalgebras of $\mathfrak{g}$:

$$
\begin{aligned}
\mathfrak{v} &:= \mathrm{span}_{\mathbb{C}}\{L_m, C_1 |\ m \in \mathbb{Z}\},\\
\mathfrak{ns} &:= \mathrm{span}_{\mathbb{C}}\{L_m, G_r, C_1 |\ m \in \mathbb{Z}, r \in \mathbb{Z} + \frac{1}{2}\},\\
\mathfrak{hc} &:= \mathrm{span}_{\mathbb{C}}\{H_m, F_p, C_3 \mid m \in \mathbb{Z}, p \in \mathbb{Z} + \frac{1}{2}\},\\
\mathfrak{q} &:= \mathrm{span}_{\mathbb{C}}\{L_m, F_p, C_1, C_3 \mid m \in \mathbb{Z}, p \in \mathbb{Z} + \frac{1}{2}\},\\
\mathfrak{p} &:= \mathrm{span}_{\mathbb{C}}\{L_m, H_m, F_p, C_1, C_2, C_3 \mid m \in \mathbb{Z}, p \in \mathbb{Z} + \frac{1}{2}\}.
\end{aligned}
$$

Clearly, $\mathfrak{v}$ is isomorphic to the Virasoro algebra. $\mathfrak{g}_{\bar{0}}$, denoted by $\mathfrak{t}$ for the following use, is isomorphic to the twisted Heisenberg–Virasoro algebra [3]. $\mathfrak{ns}$ is isomorphic to the super Virasoro algebra (also called the $N = 1$ Neveu-Schwarz algebra, see [4]). $\mathfrak{q}$ is the

https://doi.org/10.4153/S0008439524000559 Published online by Cambridge University Press

Ferminon-Virasoro superalgebra defined in [24] (also see [10]). Moreover, $U(\mathfrak{g})$ has a natural $\frac{1}{2}\mathbb{Z}$-gradation and an induced $\mathbb{Z}_2$-gradation. For homogeneous $u \in U(\mathfrak{g}_-)$, we denote by $|u|$ and $\bar{u}$ the degree of u according as $\frac{1}{2}\mathbb{Z}$-gradation and $\mathbb{Z}_2$-gradation, respectively.

2.3 Harish-Chandra modules

For any $\mathfrak{g}$-module V and $\lambda \in \mathbb{C}$, set $V_\lambda := \left\{v \in V \,\middle|\, L_0 v = \lambda v\right\}$, which is generally called the weight space of V corresponding to the weight λ. a $\mathfrak{g}$-module V is called a weight module if V is the sum of all its weight spaces.

For a weight module $V = V_{\bar{0}} + V_{\bar{1}}$, we define

$$\mathrm{Supp}(V) := \left\{\lambda \in \mathbb{C} \,\middle|\, V_\lambda \neq 0\right\}. \tag{2.2}$$

Obviously, if V is a simple weight $\mathfrak{g}$-module, then there exists $\lambda \in \mathbb{C}$ such that $\mathrm{Supp}(V) \subset \lambda + \frac{1}{2}\mathbb{Z}$. So $V = \sum_{i \in \frac{1}{2}\mathbb{Z}} V_i$ is $\frac{1}{2}\mathbb{Z}$-graded, where $V_i = V_{\lambda+i}$. An simple weight $\mathfrak{g}$-module $V = \sum V_i$ is called *Harish-Chandra module* if all V_i are finite-dimensional. If, in addition, there exists a positive integer N such that

$$\dim(V_i)_\tau \leq N, \ \forall i \in \frac{1}{2}\mathbb{Z}, \ \forall \tau \in \mathbb{Z}_2, \tag{2.3}$$

the module V is called *cuspidal*. If $N \leq 1$, the cuspidal module V is called *intermediate series*.

A $\mathfrak{g}$-module V is called a highest (resp. lowest) weight module, if there exists a nonzero $v \in V_\lambda$ such that

1) V is generated by v as $\mathfrak{g}$-module with $L_0 w = hw$ and $Cw = cw$ for some $h, c \in \mathbb{C}$;
2) $\mathfrak{g}_+ v = 0$ (resp. $\mathfrak{g}_- v = 0$), where $\mathfrak{g}_+ = \sum_{i>0} \mathfrak{g}_i$, $\mathfrak{g}_- = \sum_{i<0} \mathfrak{g}_i$.

Next we define the Verma module, which is a highest weight module. For any $c_1, c_2, c_3, \lambda, h \in \mathbb{C}$, let $\mathbb{C}\mathbf{1}$ be the one-dimensional module over the subalgebra $\mathfrak{g}_+ \oplus \mathfrak{g}_0$ defined by

$$\mathfrak{g}_+ \mathbf{1} = 0, C_i \mathbf{1} = c_i \mathbf{1}, \ H_0 \mathbf{1} = h\mathbf{1}, \ \text{and } L_0 \mathbf{1} = \lambda \mathbf{1}, i = 1, 2, 3.$$

Then we get the induced $\mathfrak{g}$-module, called Verma module:

$$M(\lambda, h, c_1, c_2, c_3) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_+ + \mathfrak{g}_0)} \mathbb{C}\mathbf{1}.$$

It is well known that the Verma module $M(\lambda, h, c_1, c_2, c_3)$ has a unique maximal submodule $J(\lambda, h, c_1, c_2, c_3)$, and the corresponding simple quotient module is denoted by $L(\lambda, h, c_1, c_2, c_3)$. A nonzero weight vector $u' \in M(\lambda, h, c_1, c_2, c_3)$ is called a singular vector if $\mathfrak{g}_+ u' = 0$. It is clear that $J(\lambda, h, c_1, c_2, c_3)$ is generated by all homogenous singular vectors in $M(\lambda, h, c_1, c_2, c_3)$ not in $\mathbb{C}\mathbf{1}$, and that $M(\lambda, h, c_1, c_2, c_3) = L(\lambda, h, c_1, c_2, c_3)$ if and only if $M(\lambda, h, c_1, c_2, c_3)$ does not contain any other singular vectors besides those in $\mathbb{C}\mathbf{1}$.

For the super Virasoro algebra $\mathfrak{ns}$, we can also define the Verma module over $\mathfrak{ns}$ by $M(\lambda, c_1) := U(\mathfrak{ns}) \otimes_{U(\mathfrak{ns}_+ \oplus \mathfrak{ns}_0)} \mathbb{C}\mathbf{1}_{\mathfrak{ns}}$, where $\mathfrak{ns}_+ = \mathfrak{ns} \cap \mathfrak{g}_+$ and $\mathfrak{ns}_0 = \mathfrak{ns} \cap \mathfrak{g}_0$. Certainly, $M(\lambda, c_1)$ can be regarded as a $\mathfrak{g}$-module by trivial action of $\mathfrak{hc}$, denoted it by $M(\lambda, c_1)^{\mathfrak{g}}$.

https://doi.org/10.4153/S0008439524000559 Published online by Cambridge University Press

From [4] we know that there exist homogeneous elements

$$P_1, P_2 \in U(\mathfrak{ns}_-),$$

such that the unique maximal submodule $J(\lambda, c_1)$ of Verma module $M(\lambda, c_1)$ can be generated by singular vectors $P_1\mathbf{1}_{\mathfrak{ns}}$ and $P_2\mathbf{1}_{\mathfrak{ns}}$:

$$J(\lambda, c_1) = U(\mathfrak{ns}_-)P_1\mathbf{1}_{\mathfrak{ns}} + U(\mathfrak{ns}_-)P_2\mathbf{1}_{\mathfrak{ns}}, \tag{2.4}$$

where P_1, P_2 are unique up to nonzero scalars; moreover, $P_1 = P_2$ if $J(\lambda, c_1)$ can be generated by a singular vector and $P_1 = P_2 = 0$ if $M(\lambda, c_1)$ itself is simple.

Meanwhile, $M(c_3) := U(\mathfrak{hc}) \otimes_{U(\mathfrak{hc}_+ + \mathfrak{hc}_0)} \mathbb{C}\mathbf{1}_{\mathfrak{hc}}$ is the Verma module over the Fermion–Clifford superalgebra $\mathfrak{hc}$, where $\mathfrak{hc}_+ = \mathfrak{hc} \cap \mathfrak{g}_+$ and $\mathfrak{hc}_0 = \mathfrak{hc} \cap \mathfrak{g}_0$. By constructions in [12], $M(c_3)$ can be lifted as a $\mathfrak{g}$-module, denoted it by $M(c_3)^{\mathfrak{g}}$ (also see [15]).

Theorem 2.2 [1, 2] (1) *For $c_3 \neq 0$, $M(\lambda, h, c_1, c_2, c_3) \cong M(\lambda - \frac{1}{2c_3}h^2 + \frac{c_2}{c_3}h, c_1 - \frac{3}{2} + \frac{12c_2}{c_3})^{\mathfrak{g}} \otimes M(c_3)^{\mathfrak{g}}$, and then $M(\lambda, h, c_1, c_2, c_3)$ is simple if and only if $M(\lambda - \frac{1}{2c_3}h^2 + \frac{c_2}{c_3}h, c_1 - \frac{3}{2} + \frac{12c_2}{c_3})$ is a simple $\mathfrak{ns}$-module. If $M(\lambda, h, c_1, c_2, c_3)$ is not simple, the maximal submodule $J = U(\mathfrak{g}_-)P_1\mathbf{1}_{\mathfrak{ns}} \otimes \mathbf{1}_{\mathfrak{hc}} + U(\mathfrak{g}_-)P_2\mathbf{1}_{\mathfrak{ns}} \otimes \mathbf{1}_{\mathfrak{hc}}$, where $U(\mathfrak{ns}_-)P_1\mathbf{1}_{\mathfrak{ns}} + U(\mathfrak{ns}_-)P_2\mathbf{1}_{\mathfrak{ns}}$ is the maximal submodule of $M(\lambda - \frac{1}{2c_3}h^2 + \frac{c_2}{c_3}h, c_1 - \frac{3}{2} + \frac{12c_2}{c_3})$ given by (2.4).*

(2) *$M(\lambda, h, c_1, c_2, 0)$ is simple if and only if $h + (n+1)c_2 \neq 0$ for any $n \in \mathbb{Z}^*$. Moreover, if $h + (n+1)c_2 = 0$ for some $n \in \mathbb{Z}^*$, then the maximal submodule $J = U(\mathfrak{g}_-)P_1\mathbf{1} + U(\mathfrak{g}_-)P_2\mathbf{1}$, where P_1, P_2 are given in [2] (in many cases, $P_1 = P_2$).*

Clearly, simple highest or lowest weight modules are Harish-Chandra modules.

For the Virasoro algebra $\mathfrak{v}$, the intermediate series module $\mathcal{A}_{a,\,b}$ for some $a, b \in \mathbb{C}$ is given by as follows (see [13]):

$$\mathcal{A}_{a,\,b} = \sum_{i \in \mathbb{Z}} \mathbb{C}v_i : \ L_m v_i = (a + i + bm)v_{m+i}, \ \forall i, m \in \mathbb{Z}. \tag{2.5}$$

It is well known that $\mathcal{A}_{a,\,b} \cong \mathcal{A}_{a+1,\,b}$, $\forall a, b \in \mathbb{C}$, then we can always suppose that $a \notin \mathbb{Z}$ or $a = 0$ in $\mathcal{A}_{a,\,b}$. Moreover, the module $\mathcal{A}_{a,\,b}$ is simple if $a \notin \mathbb{Z}$ or $b \neq 0, 1$. In the opposite case, the module contains two simple subquotients namely the trivial module and $[t, t^{-1}]/\mathbb{C}$. It is also clear that $\mathcal{A}_{0,0}$ has $\mathbb{C}v_0$ as a submodule, and its corresponding quotient is denoted by $\mathcal{A}'_{0,0}$. Dually, $\mathcal{A}_{0,1}$ has $\mathbb{C}v_0$ as a quotient module, and its corresponding submodule is isomorphic to $\mathcal{A}'_{0,0}$. For convenience, we simply write $\mathcal{A}'_{a,b} = \mathcal{A}_{a,b}$ when $\mathcal{A}_{a,b}$ is simple.

All simple Harish-Chandra modules over the Virasoro algebra $\mathfrak{v}$ were mainly classified in [19].

Theorem 2.3 [19] *Let V be a simple Harish-Chandra module over the Virasoro algebra $\mathfrak{v}$. Then V is a highest weight module, lowest weight module, or a module of the intermediate series.*

Based on this classification, all simple Harish-Chandra modules over the twisted Heisenberg–Virasoro algebra $\mathfrak{t}$ were also classified.

https://doi.org/10.4153/S0008439524000559 Published online by Cambridge University Press

Theorem 2.4 [17] *Let V be a simple Harish-Chandra module over the twisted Heisenberg–Virasoro algebra $\mathfrak{t}$. Then V is a highest weight module, a lowest weight module, or a module of the intermediate series.*

Remark 2.5 The $\mathfrak{t}$-module of the intermediate series, denoted by $\mathcal{A}_{a,\,b,\,c}$ for some $a, b, c \in \mathbb{C}$, was given in [17] as follows:

$$\mathcal{A}_{a,\,b,\,c} = \sum_{i\in\mathbb{Z}} \mathbb{C}v_i : \; L_m v_i = (a + i + bm)v_{m+i}, H_m v_i = cv_{m+i}, \;\forall\, i, m \in \mathbb{Z}. \tag{2.6}$$

Moreover, the module $\mathcal{A}_{a,\,b,\,c}$ is simple if $a \notin \mathbb{Z}$ or $b \neq 0, 1$ or $c \neq 0$. For convenience, we also use $\mathcal{A}'_{a,b,c}$ to denote by the simple subquotient of $\mathcal{A}_{a,b,c}$.

For the super Virasoro algebra $\mathfrak{ns}$, its simple Harish-Chandra modules were classified in [7, 8, 22].

Theorem 2.6 [7, 8, 22] *Let V be a simple Harish-Chandra module over the super Virasoro algebra $\mathfrak{ns}$. Then V is a highest weight module, a lowest weight module, or a module of the intermediate series.*

The module of the intermediate series over the super Virasoro algebra $\mathfrak{ns}$ was determined by [22] as follows (up to parity-change): $\mathcal{S}_{a,b} := \sum_{i\in\mathbb{Z}} \mathbb{C}x_i + \sum_{k\in\mathbb{Z}} \mathbb{C}y_k$ with

$$L_n x_i = (a + bn + i)x_{i+n},\; L_n y_k = (a + (b - \frac{1}{2})n + k)y_{k+n},$$
$$G_r x_i = y_{r+i},\; G_r y_k = (a + k + 2r(b - \frac{1}{2}))x_{r+k},$$

for all $n, i \in \mathbb{Z}, r, k \in \mathbb{Z} + \frac{1}{2}$, where $a, b \in \mathbb{C}$.

Moreover, $\mathcal{S}_{a,b}$ is not simple if and only if $a = 0, b = 1$ or $a = b = \frac{1}{2}$. We also use $\mathcal{S}'_{a,b}$ to denote by the simple subquotient of $\mathcal{S}_{a,b}$.

The following result plays a key role in classification of Harish-Chandra modules for many Lie superalgebras.

Theorem 2.7 [10] *Let V be a simple Harish-Chandra module over the Lie superalgebra $\mathfrak{q}$. Then V is a highest weight module, a lowest weight module, or a module of the intermediate series $\mathcal{A}'_{a,b}$ with the trivial action of F_r for any $r \in \mathbb{Z} + \frac{1}{2}$.*

3 Simple cuspidal modules

In order to achieve our main result, we first do such researches for the subalgebra $\mathfrak{p}$. Clearly, $\mathfrak{p}_{\bar{0}}$ is isomorphic to the twisted Heisenberg–Virasoro algebra $\mathfrak{t}$ and $\mathfrak{p}_{\bar{1}} = \operatorname{span}_{\mathbb{C}}\{F_r \mid r \in \mathbb{Z} + \frac{1}{2}\}$.

Proposition 3.1 *Let V be a simple cuspidal $\mathfrak{p}$-module. Then V is a Harish-Chandra module of the intermediate series and $V = \sum v_i \cong \mathcal{A}'_{a,b,c}$ for some $a, b, c \in \mathbb{C}$ with $H_m v_i = cv_{m+i}, F_{m+\frac{1}{2}} V = 0$ for all $m \in \mathbb{Z}$.*

https://doi.org/10.4153/S0008439524000559 Published online by Cambridge University Press

Proof Clearly, the subalgebra $\mathrm{span}\{L_m, F_r, C_1, C_3 \mid m \in \mathbb{Z}, r \in \mathbb{Z} + \frac{1}{2}\}$ is isomorphic to $\mathfrak{q}$. By Theorem 2.7, we can choose a simple $\mathfrak{q}$-module V' with $F_r V' = 0$ for all $r \in \mathbb{Z} + \frac{1}{2}$. In this case, we have $V = \mathrm{Ind}_{\mathfrak{q}}^{\mathfrak{p}} V'$. Moreover, we have $F_r V = 0$ for all $r \in \mathbb{Z} + \frac{1}{2}$ by the definition of $\mathfrak{p}$. Then V is a simple $\mathfrak{p}$-module if and only if V is a simple $\mathfrak{k}$-module. So the proposition follows from Theorem 2.4. ∎

Theorem 3.2 *Let V be a simple cuspidal $\mathfrak{g}$-module. Then V is a module of the intermediate series.*

Proof Clearly, C_1, C_2, C_3 act on V as zero's [17]. Now we consider the subalgebra $\mathfrak{p}$ of $\mathfrak{g}$.

By Proposition 3.1, we can choose a simple $\mathfrak{p}$-module $U = \sum \mathbb{C} u_i$ of V such that $H_m u_i = c u_{m+i}$ for all $m, i \in \mathbb{Z}$, and $F_r U = 0$ for all $r \in \mathbb{Z} + \frac{1}{2}$. In this case, $V = \sum_{i \geq 0} G^i U$, where $G = \{G_r \mid r \in \mathbb{Z} + \frac{1}{2}\}$, the subspace of $\mathfrak{g}$.

Case 1. $c = 0$. In this case, $H_m U = 0$ for all $m \in \mathbb{Z}$ and then $H_m V = F_r V = 0$ for all $m \in \mathbb{Z}, r \in \mathbb{Z} + \frac{1}{2}$. Then V becomes a simple cuspidal $\mathfrak{ns}$-module. So it follows by [8, 22] directly.

Case 2. $c \neq 0$.

Now we can suppose that $GU \neq 0$ (otherwise V is a trivial $\mathfrak{g}$-module). Set $G^0 U = U$ and $G^{i+1} U = G G^i U$ for all $i \geq 0$. Then

$$V = \sum_{i \geq 0} G^i U. \tag{3.1}$$

Moreover,

$$G^i U \subset G^{i+2} U. \tag{3.2}$$

Since V is cuspidal, there exists $p \in \mathbb{N}$ such that

$$G^p U = G^{p+2} U. \tag{3.3}$$

By $QU = 0$ and $[F_r, G_s] u_i = H_{r+s} u_i = c u_{i+r+s} \neq 0$, where $Q = \{F_r \mid r \in \mathbb{Z} + \frac{1}{2}\}$, the subspace of $\mathfrak{g}$, we get $HGU = GU$ and then $HG^2 U = U + G^2 U = G^2 U$. By induction, we can get $HG^n U = G^n U$ for any $n \in \mathbb{N}$.

Similarly, by $QU = 0$ and $[F_r, G_s] u_i = H_{r+s} u_i = c u_{i+r+s} \neq 0$, we get $QGU = U$ and then $GQGU = GU$. So $QG^2 U = HGU + GQGU = GU$. By induction, we can get

$$QG^n U = G^{n-1} U, \tag{3.4}$$

for any $n \geq 1$.

If $p = 0$ in (3.3), then $V = U + GU$ and then $\dim(V_i)_\tau \leq 1$ for any $i \in \mathbb{Z}$ and $\tau \in \mathbb{Z}_2$.

If $p > 0$ in (3.3), then we can get $G^{p-1} U = G^{p+1} U$ by (3.4). So we can also get $V = U + GU$. Then the proposition is obtained. ∎

By direct calculation, we can get the precise module structure on $V = U + GU$ as follows (up to parity-change): $V = \mathcal{S}_{a,b,c} := \sum_{i \in \mathbb{Z}} \mathbb{C} x_i + \sum_{k \in \mathbb{Z} + \frac{1}{2}} \mathbb{C} y_k$ with

$$L_n x_i = (a + bn + i) x_{i+n}, \; L_n y_k = (a + (b - \frac{1}{2}) n + k) y_{k+n},$$

$$G_r x_i = y_{r+i}, \; G_r y_k = (a + k + 2r(b - \frac{1}{2})) x_{r+k},$$

https://doi.org/10.4153/S0008439524000559 Published online by Cambridge University Press

$$H_n x_i = c x_{n+i},\ H_n y_k = c y_{n+k},$$
$$F_r x_i = 0,\ F_r y_k = c x_{r+k},$$

for all $n, i \in \mathbb{Z}, r, k \in \mathbb{Z} + \frac{1}{2}$, where $a, b, c \in \mathbb{C}$.

Note that $\mathcal{S}_{a,b,c}$ is not simple if and only if $a \in \mathbb{Z}, b = 1, c = 0$ or $a \in \mathbb{Z} + \frac{1}{2}, b = \frac{1}{2}$, $c = 0$. If $a \in \mathbb{Z}$, then $\mathcal{S}_{a,1,c} \cong \mathcal{S}_{0,1,c}$ and $\mathcal{S}_{0,1,0}$ has a unique simple submodule $\mathcal{S}'_{0,1,0}$ spanned by $\{x_j \mid j \in \mathbb{Z}^*\} \cup \{y_k \mid k \in \mathbb{Z} + \frac{1}{2}\}$.

Moreover, by direct calculation, we can get that $\mathcal{S}_{a,b,c} \cong \mathcal{S}_{a',b',c'}$ if and only if one of the following holds:

(1) $a - a' \in \mathbb{Z}, b = b', c = c'$;
(2) $a \notin \mathbb{Z}, a - a' \in \frac{1}{2} + \mathbb{Z}, b = 1, b' = \frac{1}{2}$;
(3) $a \notin \frac{1}{2} + \mathbb{Z}, a - a' \in \frac{1}{2} + \mathbb{Z}, b = \frac{1}{2}, b' = 1$.

Especially, if $a \in \mathbb{Z} + \frac{1}{2}$, then $\mathcal{S}_{a,\frac{1}{2},c} \cong \mathcal{S}_{\frac{1}{2},\frac{1}{2},c}$ and $\mathcal{S}_{\frac{1}{2},\frac{1}{2},0}$ has a unique simple quotient module $\mathcal{S}'_{\frac{1}{2},\frac{1}{2},0} := \mathcal{S}_{\frac{1}{2},\frac{1}{2},0}/\mathbb{C}y_{-\frac{1}{2}}$.

Let $\mathcal{S}'_{a,b,c} = \mathcal{S}_{a,b,c}$ if $\mathcal{S}_{a,b,c}$ is simple and $\mathcal{S}'_{0,1,0}, \mathcal{S}'_{\frac{1}{2},\frac{1}{2},0}$ be defined as above.

4 Simple Harish-Chandra modules

In this section, we shall classify all simple Harish-Chandra modules over the $N = 1$ Heisenberg-Virasoro super algebra. The following result is well known.

Lemma 4.1 *Let M be a Harish-Chandra module over the Virasoro algebra with* $\mathrm{supp}(M) \subseteq \lambda + \mathbb{Z}$. *If for any $v \in M$, there exists $N(v) \in \mathbb{N}$ such that $L_i v = 0, \forall i \geq N(v)$, then* $\mathrm{supp}(M)$ *is upper bounded.*

With the previous result, we can easily get the following result.

Theorem 4.2 *Let V be a Harish-Chandra module over $\mathfrak{g}$. If V is not a highest and lowest module, then V is uniformly bounded.*

Proof It is essentially the same as that of Lemma 4.2 in [7].

Fix $\lambda \in \mathrm{supp}(M)$. Since M is not cuspidal, there exists $k \in \frac{1}{2}\mathbb{Z}$ such that $\dim M_{-k+\lambda} > 2(\dim M_\lambda + M_{\lambda+\frac{1}{2}} + \dim M_{\lambda+1} + M_{\lambda+\frac{3}{2}} + \dim M_{\lambda+2})$. Without loss of generality, we may assume that $k \in \mathbb{N}$. Then there exists a nonzero element $w \in M_{-k+\lambda}$ such that $L_k w = L_{k+1} w = H_{k+2} w = G_{k+\frac{1}{2}} w = F_{k+\frac{3}{2}} w = 0$. Therefore, $L_i w = H_i w = G_{i-\frac{1}{2}} w = F_{i-\frac{1}{2}} w = 0$ for all $i \geq k^2$, since $[\mathfrak{g}_i, \mathfrak{g}_j] = \mathfrak{g}_{i+j}$.

It is easy to see that $M' = \{v \in M \,|\, \dim \mathfrak{g}_+ v < \infty\}$ is a nonzero submodule of M, where $\mathfrak{g}_+ = \sum_{n \in \mathbb{Z}_+} (\mathbb{C}L_n + \mathbb{C}H_n + \mathbb{C}G_{n-\frac{1}{2}} + \mathbb{C}F_{n-\frac{1}{2}})$. Hence, $M = M'$. So, Lemma 4.1 tells us that $\mathrm{supp}(M)$ is upper bounded, that is, M is a highest weight module. ∎

Combining with Theorems 3.2 and 4.2 and we get the main result of this paper.

Theorem 4.3 *Let V be a simple weight $\mathfrak{g}$-module with finite dimensional weight spaces. Then V is a highest weight module, a lowest weight module, or a module of the intermediate series.*

https://doi.org/10.4153/S0008439524000559 Published online by Cambridge University Press

5 Tensor product of weight modules

In this section, we study the tensor product of highest weight modules with intermediate series modules over the $N=1$ Heisenberg–Virasoro superalgebra.

Let $M=M(\lambda,h,c_1,c_2,c_3)$ be the Verma module with highest weight vector $\mathbf{1}$, and $\mathcal{S}_{a,b,c}=\sum_{i\in\mathbb{Z}}\mathbb{C}x_i+\sum_{k\in\mathbb{Z}+\frac{1}{2}}\mathbb{C}y_k$ be the module of the intermediate series. Without loss of generality, we may assume that $\mathbf{1}\in M_{\bar{0}}$ in the following. We will consider the tensor product modules $M\otimes\mathcal{S}'_{a,b,c}$, and $L(\lambda,h,c_1,c_2,c_3)\otimes\mathcal{S}'_{a,b,c}$.

Since M and $\mathcal{S}'_{a,b,c}$ are L_0-diagonalizable, so is $M\otimes\mathcal{S}'_{a,b,c}$:

$$M\otimes\mathcal{S}'_{a,b,c}=\bigoplus_{m\in\frac{1}{2}\mathbb{Z}}(M\otimes\mathcal{S}'_{a,b,c})_{m+h+a},$$

where

$$(M\otimes\mathcal{S}'_{a,b,c})_{m+h+a}=\bigoplus_{n\in\frac{1}{2}\mathbb{Z}}M_{h+n}\otimes\mathbb{C}v_{m-n},$$

and

$$v_{m-n}=\begin{cases}x_{m-n}, & if \quad m-n\in\mathbb{Z},\\ y_{m-n}, & if \quad m-n\in\mathbb{Z}+\frac{1}{2}.\end{cases}$$

Remark 5.1 If M is nontrivial, then $(M\otimes\mathcal{S}'_{a,b,c})_{m+h+a}$ is infinite dimensional for all $m\in\frac{1}{2}\mathbb{Z}$.

Lemma 5.2 *The module $M\otimes\mathcal{S}'_{a,b,c}$ is generated by $\{\mathbf{1}\otimes v_k\mid k\in\frac{1}{2}\mathbb{Z}\}$.*

Proof Note that $M\otimes\mathcal{S}'_{a,b,c}$ is spanned by $\{u\mathbf{1}\otimes v_k\mid k\in\frac{1}{2}\mathbb{Z},u\in U(\mathfrak{g}_-)\}$, so the lemma holds. ∎

Lemma 5.3 *$M\otimes\mathcal{S}'_{a,b,c}$ is reducible for all $a,b,c,\lambda,h,c_1,c_2,c_3\in\mathbb{C}$.*

Proof It sufficient to prove that every $\mathbf{1}\otimes v_k$ generates a proper submodule of $M\otimes\mathcal{S}'_{a,b,c}$, where $\mathbf{1}$ is the highest weight vector of M. Assume that $M\otimes\mathcal{S}'_{a,b,c}$ is cyclic on $\mathbf{1}\otimes v_k$, i.e.,

$$M\otimes\mathcal{S}'_{a,b,c}=U(\mathfrak{g})(\mathbf{1}\otimes v_k)=U(\mathfrak{g}_-)U(\mathfrak{g}_+)(\mathbf{1}\otimes v_k).$$

Then there must exists $w\in U(\mathfrak{g}_-)U(\mathfrak{g}_+)$ such that

$$\mathbf{1}\otimes v_{k-\frac{1}{2}}=w(\mathbf{1}\otimes v_k).$$

Let

$$w=\sum_{\mathbf{i},\mathbf{j}\in\mathbb{M},\mathbf{k},\mathbf{l}\in\mathbb{M}_1}L^{\mathbf{i}}H^{\mathbf{j}}G^{\mathbf{k}}F^{\mathbf{l}}u_{\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}},$$

where $u_{\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}}\in U(\mathfrak{g}_+)$ is a homogeneous element. Since

$$u_{\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}}\mathbf{1}=0,\quad \forall u_{\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}}\in U(\mathfrak{g}_+)/\mathbb{C},$$

https://doi.org/10.4153/S0008439524000559 Published online by Cambridge University Press

we can assume there exists some $u_{\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}} v_k \neq 0$. Let $L^{\mathbf{i}} H^{\mathbf{j}} G^{\mathbf{k}} F^{\mathbf{l}}$ be a term in the expression of w such that $\mathrm{w}(\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l})$ is maximal. By comparing two sides of $\mathbf{1} \otimes v_{k-\frac{1}{2}} = w(\mathbf{1} \otimes v_k)$, we have

$$L^{\mathbf{i}} H^{\mathbf{j}} G^{\mathbf{k}} F^{\mathbf{l}} u_{\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}} v_k = 0.$$

Since M is a free $U(\mathfrak{g}_-)$-module, it follows that $u_{\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}} v_k = 0$, which is a contradiction. This completes the proof. ∎

Theorem 5.4 *$L(\lambda, h, c_1, c_2, c_3) \otimes \mathcal{S}'_{a,b,c}$ is simple if and only if it is cyclic on every vector $\mathbf{1} \otimes v_k$.*

Proof The only if part is trivial.

Assume that $L(\lambda, h, c_1, c_2, c_3) \otimes \mathcal{S}'_{a,b,c}$ is cyclic on every $\mathbf{1} \otimes v_k$. Let U be a submodule and $0 \neq x \in U$ homogenous vector. Then

$$x = x_0 \otimes v_k + x_{-\frac{1}{2}} \otimes v_{k+\frac{1}{2}} + \cdots + x_{-s} \otimes v_{k+s},$$

for some $x_j \in L(\lambda, h, c_1, c_2, c_3)_j$, $j = 0, \frac{1}{2}, \cdots, s$. We use induction on n to show that there is $\mathbf{1} \otimes v_k \in U$ for some $k \in \frac{1}{2}\mathbb{Z}$.

Case 1. $c \neq 0$.

Replacing x with ux for some $u \in U(\mathfrak{g}_+)$ if necessary, we may assume that $x_0 = \mathbf{1}$. Choose n such that $L_j x_{-i} = G_{j+\frac{1}{2}} x_{-i} = H_j x_{-i} = F_{j+\frac{1}{2}} x_{-i} = 0, \forall j \geq n, i = 0, \frac{1}{2}, \ldots, s$. Note that $\mathcal{S}_{a,b,c}$ is simple as $(\mathfrak{g}^{(n)} + \mathfrak{h}\mathfrak{c} + \mathbb{C}C_2)$-module. Therefore from Density Lemma [17], we may choose some $u \in U(\mathfrak{g}^{(n)} + \mathfrak{h}\mathfrak{c} + \mathbb{C}C_2)$ with $uv_{k+i} = \delta_{0,i} v_0$ for all $i = 0, \frac{1}{2}, \ldots, s$. Rewrite $u = \sum_i u_i u'_i$ with $u_i \in U(\mathfrak{h}\mathfrak{c})$ and $u'_i \in U(\mathfrak{g}^{(n)})$. Note that

$$H_j H_i X = c H_{i+j} X, \ \forall i, j \in \mathbb{Z}, X \in \mathcal{S}_{a,b,c}.$$

For sufficient large l, replacing $H_j H_i$ with cH_{i+j} in $H_l u$, we obtain $u' \in U(\mathfrak{g}^{(n)})$ with $u' v_{k+i} = H_l u v_{k+i} = c\delta_{0,i} v_l, \forall i = 0, \frac{1}{2}, \ldots, s$. Now $0 \neq u'\beta = c\mathbf{1} \otimes v_l \in M$.

Case 2. $c = 0$.

If $s = 0$, then $x = x_0 \otimes v_n \in U$. Assume $s > 0$. Recall that $F_{j-\frac{1}{2}} v_i = 0$ for any $j \in \mathbb{Z}$, $i \in \frac{1}{2}\mathbb{Z}$.

If $F_{l-\frac{1}{2}} x \neq 0$ for some $l \in \mathbb{Z}_+$, we have

$$F_{l-\frac{1}{2}} x = y_{-\frac{1}{2}} \otimes v_{k+\frac{1}{2}} + \cdots + y_{-s} \otimes v_{k+s},$$

where $y_i = F_{l-\frac{1}{2}} x_{-i} \in L(\lambda, h, c_1, c_2, c_3)_{-i+l-\frac{1}{2}}$. By inductive hypothesis, now there must be some $\mathbf{1} \otimes v_k \in U$.

So $F_{i-\frac{1}{2}} x = 0$ for any $i \in \mathbb{Z}_+$. Since $G_{\frac{1}{2}}, G_{\frac{3}{2}}, F_{\frac{1}{2}}$ generate $\mathfrak{g}_+$, vectors $G_{\frac{1}{2}} x$ and $G_{\frac{3}{2}} x$ cannot both equal zero, for otherwise x would be a singular vector in $L(\lambda, h, c_1, c_2, c_3)$ other than $\mathbf{1}$. But now we can follow the proof of [26, Lemma 3.4] (also see [21, Theorem 28]). This completes the proof. ∎

https://doi.org/10.4153/S0008439524000559 Published online by Cambridge University Press

6 Simplicity of tensor product modules

In this section, we shall consider the simplicity of tensor product modules defined in Section 5.

Let us first introduce an auxiliary module, using the called "shifting technique" in [9].

Lemma 6.1 *The vector space $\mathcal{V} = L(\lambda, h, c_1, c_2, c_3) \otimes \mathbb{C}[t^{\pm\frac{1}{2}}]$ can be endowed with a $\mathfrak{g}$-module structure via*

$$L_k(u\mathbf{1} \otimes t^s) = \begin{cases} (L_k + a + s + kb - |u|)u\mathbf{1} \otimes t^{s+k}, & if \quad s + |u| \in \mathbb{Z}, \\ \left(L_k + a + s + k(b - \frac{1}{2}) - |u|\right) u\mathbf{1} \otimes t^{s+k}, & if \quad s + |u| \in \mathbb{Z} + \frac{1}{2}. \end{cases}$$

$$H_k(u\mathbf{1} \otimes t^s) = (H_k + c)u\mathbf{1} \otimes t^{s+k}, \quad s + |u| \in \frac{1}{2}\mathbb{Z}.$$

$$G_{k+\frac{1}{2}}(u\mathbf{1} \otimes t^s) = \begin{cases} (G_{k+\frac{1}{2}} + (-1)^{\bar{u}})u\mathbf{1} \otimes t^{s+k+\frac{1}{2}}, & if \quad s + |u| \in \mathbb{Z}, \\ \left(G_{k+\frac{1}{2}} + (-1)^{\bar{u}}(a + s + (2k+1)(b - \frac{1}{2}) - |u|)\right) u\mathbf{1} \otimes t^{s+k+\frac{1}{2}}, & if \quad s + |u| \in \mathbb{Z} + \frac{1}{2}. \end{cases}$$

$$F_{k+\frac{1}{2}}(u\mathbf{1} \otimes t^s) = \begin{cases} F_{k+\frac{1}{2}} u\mathbf{1} \otimes t^{s+k+\frac{1}{2}}, & if \quad s + |u| \in \mathbb{Z}, \\ \left(F_{k+\frac{1}{2}} + (-1)^{\bar{u}} c\right) u\mathbf{1} \otimes t^{s+k+\frac{1}{2}}, & if \quad s + |u| \in \mathbb{Z} + \frac{1}{2}. \end{cases}$$

Proof It can be checked by straightforward but tedious calculations. ∎

Lemma 6.2 *The $\mathfrak{g}$-module $L(\lambda, h, c_1, c_2, c_3) \otimes \mathcal{S}_{a,b,c}$ is isomorphic to $\mathcal{V} = L(\lambda, h, c_1, c_2, c_3) \otimes \mathbb{C}[t^{\pm\frac{1}{2}}]$ via the following map: for any $m \in \mathbb{Z}$,*

$$f : M \otimes \mathcal{S}_{a,b,c} \to M \otimes \mathbb{C}[t^{\pm\frac{1}{2}}]$$
$$u\mathbf{1} \otimes x_m \mapsto u\mathbf{1} \otimes t^{m+|u|},$$
$$u\mathbf{1} \otimes y_{m+\frac{1}{2}} \mapsto u\mathbf{1} \otimes t^{m+\frac{1}{2}+|u|},$$

for all $m \in \mathbb{Z}, u \in U(\mathfrak{g}_-)_{-m}$.

Proof It can be checked directly. ∎

We identify $\mathcal{V}$ (resp., $\mathcal{V}'$) with $L(\lambda, h, c_1, c_2, c_3) \otimes \mathcal{S}_{a,b,c}$ (resp., $L(\lambda, h, c_1, c_2, c_3) \otimes \mathcal{S}'_{a,b,c}$) in this section.

Clearly, $\mathcal{V} = L(\lambda, h, c_1, c_2, c_3) \otimes \mathbb{C}[t^{\pm\frac{1}{2}}]$ is the weight space decomposition, that is

$$L(\lambda, h, c_1, c_2, c_3) \otimes t^s = \{v \in \mathcal{V} \mid L_0 v = (a + h + s)v, s \in \frac{1}{2}\mathbb{Z}\}.$$

Moreover, we see that $\mathcal{V}'$ is generated by $\{\mathbf{1} \otimes t^s \mid s \in \frac{1}{2}\mathbb{Z}\}$.

For $k \in \frac{1}{2}\mathbb{Z}$, we define

$$W^{(k)} = \sum_{i \in \frac{1}{2}\mathbb{N}} U(\mathfrak{g})(w \otimes v_{k+i}) \subset L(\lambda, h, c_1, c_2, c_3) \otimes \mathcal{S}_{a,b,c},$$
$$W_s^{(k)} = W^{(k)} \cap (L(\lambda, h, c_1, c_2, c_3) \otimes t^s), \forall s \in \frac{1}{2}\mathbb{Z}.$$

The proof of Theorem 5.4 actually shows the following result.

https://doi.org/10.4153/S0008439524000559 Published online by Cambridge University Press

Corollary 6.3 *Let W be a nontrivial submodule in $L(\lambda, h, c_1, c_2, c_3) \otimes \mathcal{S}'_{a,b,c}$, then W contains W_k for some $k \in \frac{1}{2}\mathbb{Z}$.*

Lemma 6.4

(1) $W^{(k)} = \sum_{i \in \frac{1}{2}\mathbb{N}} U(\mathfrak{g}_-)(w \otimes t^{k+i})$.

(2) $W^{(k)} \supset \oplus_{i \geq k} L(\lambda, h, c_1, c_2, c_3) \otimes t^i$.

(3) $L(\lambda, h, c_1, c_2, c_3) \otimes t^{k-\frac{1}{2}} = W^{(k)}_{k-\frac{1}{2}} \oplus \mathbb{C}(w \otimes t^{k-\frac{1}{2}})$.

(4) *Suppose that P is a weight vector in $U(\mathfrak{g}_-)$ such that $Pw \otimes t^{k-\frac{1}{2}} \in W^{(k)}_{k-\frac{1}{2}} \subset L(\lambda, h, c_1, c_2, c_3) \otimes \mathcal{S}_{a,b,c}$, then $(U(\mathfrak{g}_-)Pw) \otimes t^{k-\frac{1}{2}} \subset W^{(k)}_{k-\frac{1}{2}}$.*

Proof (1) It follows from $U(\mathfrak{g})(w \otimes t^i) = U(\mathfrak{g}_-)U(\mathfrak{g}_+ + \mathfrak{g}_0)(w \otimes t^i) \subset \sum_{j \in \mathbb{Z}_+} U(\mathfrak{g}_-)(w \otimes t^{i+j})$.

(2) Using (1) and Lemma 6.1, by induction on $s + m$ it is straight forward to prove that $H_{-j_1} \ldots H_{-j_s} L_{-l_1} \cdots L_{-l_m} w \otimes t^i \in W^{(k)}$ for all $i \geq k$ and $j_1, \ldots, j_s, l_1, \ldots, l_m \in \mathbb{Z}_+$.

(3) This follows from (2) and the proof of Lemma 5.3.

(4) Suppose that $P \in U(\mathfrak{g}_-)_{-p}$, $p \in \frac{1}{2}\mathbb{Z}_+$. From (2) and Lemma 6.1, we have

$$(L_{-i}Pw) \otimes t^{k-1} = L_{-i}(Pw \otimes t^{k+i-1})) - (a - m + k - 1 - ib)(Pw) \otimes t^{k-1} \in W^{(k)}_{k-1},$$
$$(H_{-i}Pw) \otimes t^{k-1} = H_{-i}(Pw \otimes t^{k+i-1})) - cPw \otimes t^{k-1} \in W^{(k)}_{k-1}, \forall i \in \mathbb{Z}_+.$$

Therefore, we may prove (4) by induction on p. ■

For any $s \in \frac{1}{2}\mathbb{Z}$, from Lemma 6.2, similar to φ_s in [9], we may define the linear map $\varphi_s : U(\mathfrak{g}_-) \to \mathbb{C}$ by

$$\varphi_s(\mathbf{1}) = 1,$$
$$\varphi_s(H_{-i}u\mathbf{1}) = -c\varphi(u),$$
$$\varphi_s(L_{-i}u\mathbf{1}) = \begin{cases} -(a + s - ib - |u| + i)\varphi_s(u), & if \quad s + |u| \in \mathbb{Z}, \\ -(a + s - i(b - \frac{1}{2}) - |u| + i)\varphi_s(u), & if \quad s + |u| \in \mathbb{Z} + \frac{1}{2}, \end{cases}$$

$$\varphi_s(G_{-i-\frac{1}{2}}u\mathbf{1}) = \begin{cases} -(-1)^{\bar{u}}\varphi_s(u), & if \quad s + |u| \in \mathbb{Z}, \\ -(-1)^{\bar{u}}(a + s - (2i + 1)(b - \frac{1}{2}) - |u| + i)\varphi_s(u), & if \quad s + |u| \in \mathbb{Z} + \frac{1}{2}, \end{cases}$$
$$\varphi_s(F_{-i-\frac{1}{2}}u\mathbf{1}) = \begin{cases} 0, & if \quad s + |u| \in \mathbb{Z}, \\ -(-1)^{\bar{u}}c\varphi_s(u), & if \quad s + |u| \in \mathbb{Z} + \frac{1}{2}. \end{cases}$$

It is clear that φ_s depends only on a, b, c, s.

Lemma 6.5 *Let $P \in U(\mathfrak{g}_-)$. Then*

(1) $Pw \otimes t^n \equiv \varphi_n(P)w \otimes t^n \pmod{W^{(n+\frac{1}{2})}}$;

(2) $Pw \otimes t^n \in W^{(n+\frac{1}{2})}$ *if and only if* $\varphi_n(P) = 0$.

https://doi.org/10.4153/S0008439524000559 Published online by Cambridge University Press

Proof The proof for (1) is similar to that of [9, Lemma 8]. Part (2) follows from (1). ∎

For $a, b, c, \lambda, h, c_1, c_2, c_3 \in \mathbb{C}$, by Lemma 5.3, $M(\lambda, h, c_1, c_2, c_3) \otimes \mathcal{S}'_{a,b,c}$ is always reducible, even if $M(\lambda, h, c_1, c_2, c_3)$ is simple. We will give necessary and sufficient conditions for the simplicity of $L(\lambda, h, c_1, c_2, c_3) \otimes \mathcal{S}'_{a,b,c}$.

Theorem 6.6 *(1) If $(a, b, c) \neq (\frac{1}{2}, \frac{1}{2}, 0)$, then $L(\lambda, h, c_1, c_2, c_3) \otimes \mathcal{S}'_{a,b,c}$ is simple as a $\mathfrak{g}$-module if and only if $(\varphi_s(P_1), \varphi_s(P_2)) \neq (0, 0)$ for all $s \in \frac{1}{2}\mathbb{Z}$, where P_1, P_2 are given in Theorem 2.2.*

(2) $L(\lambda, h, c_1, c_2, c_3) \otimes \mathcal{S}'_{\frac{1}{2},\frac{1}{2},0}$ is simple as a $\mathfrak{g}$-module if and only if $(\varphi_s(P_1), \varphi_s(P_2)) \neq (0, 0)$ for all $s \in \frac{1}{2}\mathbb{Z} \setminus \{-\frac{1}{2}\}$.

Proof (1) By Theorem 5.4 and Lemmas 6.4, 6.5, it is clear that $L(\lambda, h, c_1, c_2, c_3) \otimes \mathcal{S}'_{a,b,c}$ is simple if and only if $J \otimes t^s + W_s^{(s+\frac{1}{2})} = M(\lambda, h, c_1, c_2, c_3) \otimes t^s$ for all $s \in \frac{1}{2}\mathbb{Z}$, where $J = U(\mathfrak{g}_-)P_1 w + U(\mathfrak{g}_-)P_2 w$ is the maximal submodule of $M(\lambda, h, c_1, c_2, c_3)$ given in Theorem 2.2. It is equivalent to that $(U(\mathfrak{g}_-)P_1 w + U(\mathfrak{g}_-)P_2 w) \otimes t^s \not\subseteq W_s^{(s+\frac{1}{2})}$ for all $s \in \frac{1}{2}\mathbb{Z}$, and is equivalent to that $(\varphi_s(P_1), \varphi_s(P_2)) \neq (0, 0)$ for all $s \in \frac{1}{2}\mathbb{Z}$. So the statement (1) follows.

(2) It is similar to (1), the only difference is that $\varphi_{-\frac{1}{2}} = 0$. ∎

Example 6.7 (1) If $\lambda = h = 0$ and $c_3 \neq 0$, then J is generated by $P_1 = P_2 = G_{-\frac{1}{2}}$. In this case, $M(0, 0, c_1, c_2, c_3) \otimes \mathcal{S}'_{a,b,c}$ is simple if and only if $\phi_s(G_{-\frac{1}{2}}) \neq 0$ if and only if $a - b \notin \mathbb{Z}$.

(2) If $c_3 = 0, h = -2c_2 \neq 0$, then J is generated by $P_1 = H_{-1}\mathbf{1}$ and $P_2 = F_{-\frac{1}{2}}\mathbf{1}$. In this case, $M(\lambda, h, c_1, c_2, c_3) \otimes \mathcal{S}'_{a,b,c}$ is not simple since

$$(\varphi_s(H_{-1}), \varphi_s(F_{-\frac{1}{2}})) = (0, 0),$$

for all $s \in \frac{1}{2}\mathbb{Z}$.

Theorem 6.8 *Let $V(\lambda, h, c_1, c_2, c_3)$ and $V(\lambda', h', c_1', c_2', c_3')$ be the highest weight $\mathfrak{g}$-modules (not-necessarily simple) with highest weight $(\lambda, h, c_1, c_2, c_3)$ and $(\lambda', h', c_1', c_2', c_3')$, where $0 \leq \mathfrak{Rea}, \mathfrak{Rea}' < 1, b, b' \neq 1$. Then*

$$V(\lambda, h, c_1, c_2, c_3) \otimes \mathcal{S}'_{a,b,c} \cong V(\lambda', h', c_1', c_2', c_3') \otimes \mathcal{S}'_{a',b',c'}$$

if and only if

$$\lambda = \lambda', \ h = h', \ c_i = c_i', \ a = a', \ b = b', \ c = c', i = 1, 2, 3.$$

Proof The 'if' part is trivial. We only need to prove the 'only if' part.

Assume that

$$\sigma : V(\lambda, h, c_1, c_2, c_3) \otimes \mathcal{S}'_{a,b,c} \to V(\lambda', h', c_1', c_2', c_3') \otimes \mathcal{S}'_{a',b',c'}.$$

https://doi.org/10.4153/S0008439524000559 Published online by Cambridge University Press

Fix any $k \in \frac{1}{2}\mathbb{Z}$ such that $k \neq -\frac{1}{2}$ when $(a, b, c) = (\frac{1}{2}, \frac{1}{2}, 0)$. Since $\sigma(\mathbf{1} \otimes t^k)$ and $\mathbf{1} \otimes t^k$ are of the same weight, we can assume that

$$\sigma(\mathbf{1} \otimes t^k) = \sum_{i=1}^{r} p_{i,k}\mathbf{1}' \otimes t^l,$$

where $p_{i,k}$ are homogeneous elements of $U(\mathfrak{g}_-)$ and

$$a + h + k = a' + h' + l, c_i = c_i', i = 1, 2, 3. \tag{6.1}$$

Claim 1 $c = c'$.

For $k \in \frac{1}{2}\mathbb{Z}$, we have

$$\begin{aligned} \sigma(H_0(\mathbf{1} \otimes t^k)) = c\sigma(\mathbf{1} \otimes t^k) &= H_0\sigma(\mathbf{1} \otimes t^k) \\ &= H_0(\sum_{i=1}^{r} p_{i,k}\mathbf{1}' \otimes t^l) \\ &= c' \sum_{i=1}^{r} p_{i,k}\mathbf{1}' \otimes t^l. \end{aligned}$$

Then we get $c = c'$.

Claim 2 $\lambda = \lambda', h = h', a = a'$ *and* $b = b'$.

For $m, n \in \mathbb{Z}$, $k \in \mathbb{Z}$, we have

$$\begin{aligned} &\sigma(L_{m+n+1}(\mathbf{1} \otimes t^k)) \\ &= (a + k + (m + n + 1)b)\sigma(\mathbf{1} \otimes t^{m+n+1+k}), \\ &= L_{m+n+1}\sigma(\mathbf{1} \otimes t^k) \\ &= L_{m+n+1}(\sum_{i=1}^{r} p_{i,k}\mathbf{1}' \otimes t^l) \\ &= \sum_{i=1}^{r}(a' + l + (m + n + 1)b' - |u|_{i,k})p_{i,k}\mathbf{1}' \otimes t^{m+n+1+l}. \end{aligned} \tag{6.2}$$

As the proof of [9, Theorem 2], we get $\sigma(\mathbf{1} \otimes t^k) = \mathbf{1}' \otimes t^k$. So σ is an isomorphism from $V(\lambda, h, c_1, c_2, c_3)$ to $V(\lambda', h', c_1', c_2', c_3')$. Thus, $\lambda = \lambda', h = h'$. Then by (6.1), we get $a = a'$. By (6.2), we get $b = b'$. ■

References

[1] D. Adamović, B. Jandrić, and G. Radobolja, *On the N = 1 super Heisenberg-Virasoro vertex algebra, Lie Groups, Number Theory, and Vertex Aglebras*. In: Proceedings of the Conference "Representation Theory XVI," June 24–29, IUC-Dubrovnik, Contemporary Mathematics, Croatia, 2019.

[2] D. Adamovic, B. Jandric, and G. Radobolja, *The N = 1 super Heisenberg-Virasoro vertex algebra at level zero*. J. Algebra Appl. **21**(2022), no. 12, 2350003.

[3] E. Arbarello, C. De Concini, V. G. Kac, and C. Procesi, *Moduli spaces of curves and representation theory*. Commun. Math. Phys. **117**(1988), 1–36.

[4] A. Astashkevich, *On the structure of Verma modules over Virasoro and Neveu–Schwarz algebras*. Commun. Math. Phys. **186**(1997), no. 3, 531–562.

https://doi.org/10.4153/S0008439524000559 Published online by Cambridge University Press

[5] Y. Billig and V. Futorny, *Classification of simple representations of Lie algebra of vector fields on a torus*. J. Reine Angew. Math. **720**(2016), 199–216.

[6] Y. Billig, V. Futorny, K. Iohara, and I. Kashuba, *Classification of simple strong Harish-Chandra W(m,n)-modules*.

[7] Y. Cai, D. Liu, and R. Lü, *Classification of simple Harish-Chandra modules over the N = 1 Ramond algebra*. J. Algebra **567**(2021), 114–127.

[8] Y. Cai and R. Lü, *Classification of simple Harish-Chandra modules over the super Virasoro algebra and its contact subalgebra*. J. Pure Appl. Algebra **226**(2022), 106866.

[9] H. Chen, X. Guo, and K. Zhao, *Tensor product weight modules over the Virasoro algebra*. J. Lond. Math. Soc. (2). **88**(2013), 829–834.

[10] M. Dilxat, L. Chen, and D. Liu, *Classification of simple Harish-Chandra modules over the Ovsienko–Roger superalgebra*. Proc. R. Soc. Edinb. A: Math. **154**(2024), no. 2, 483–493.

[11] Y. He, D. Liu, and Y. Wang, *Simple Harish-Chandra modules over the superconformal current algebras*. J. Algebra **658**(2024), no. 15, 227–246.

[12] V. G. Kac and I. T. Todorov, *Superconformal current algebras and their unitary representations*. Commun. Math. Phys. **102**(1985), 337–347.

[13] I. Kaplansky and L. J. Santharoubane, *Harish-Chandra modules over the Virasoro algebra*. In: Infinite-dimensional groups with applications (Berkeley, CA, 1984), Mathematical Sciences Research Institute Publications, 4, Springer, New York, 1985, pp. 217–231.

[14] D. Liu, Y. Pei, and L. Xia, *Classification of simple modules with finite-dimensional weight spaces for the N = 2 Ramond algebra (in Chinese)*. Sci. Sin. Math. **53**(2023), 1181–1194.

[15] D. Liu, Y. Pei, L. Xia, and K. Zhao, *Simple smooth modules over the superconformal current algebra*.

[16] D. Liu and X. Zhang, *Tensor product weight modules of Schrödinger–Virasoro algebras*. Front. Math. China **14**(2019), no. 2, 381–393.

[17] R. Lü and K. Zhao, *Classification of irreducible weight modules over the twisted Heisenberg–Virasoro algebra*. Commun. Contemp. Math. **12**(2010), 183–205.

[18] R. Lü and K. Zhao, *Generalized oscillator representations of the twisted Heisenberg–Virasoro algebra*. Algebra Represent. Theory **23**(2020), 1417–1442.

[19] O. Mathieu, *Classification of Harish-Chandra modules over the Virasoro Lie algebra*. Invent. Math. **107**(1992), 225–234.

[20] Y. Pei and C. Bai, *Balinsky-Novikov superalgebras and infinite-dimensional Lie superalgebras*. J. Algebra Appl. **11**(2012), no. 6, 1250119.

[21] G. Radobolja, *Subsingular vectors in Verma modules, and tensor product modules over the twisted Heisenberg–Virasoro algebra and W(2,2) algebra*. J. Math. Phys. **54**(2013), 071701.

[22] Y. Su, *Classification of Harish-Chandra modules over the super Virasoro algebras*. Commun. Algebra **23**(1995), 3653–3675.

[23] Y. Xue and R. Lü, *Simple weight modules with finite dimensional weight spaces over Witt superalgebras*. J. Algebra **574**(2021), 92–116.

[24] Y. Xue and K. Zhao, *Simple representations of the Fermion-Virasoro algebras*.

[25] H. Zhang, *A class of representations over the Virasoro algebra*. J Algebra **190**(1997), no. 1, 1–10.

[26] X. Zhang, *Tensor product weight representations of the Neveu–Schwarz algebra*. Commun. Algebra **43**(2015), 3754–3775.

College of Mathematics and System Sciences, Xinjiang University, Urumqi 830046, Xinjiang, China
e-mail: lanchao1027@163.com

Department of Mathematics, Huzhou University, Zhejiang Huzhou, 313000, China
e-mail: liudong@zjhu.edu.cn