MEAN VALUE AND LIMIT THEOREMS FOR GENERALIZED MATRIX FUNCTIONS

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1. Introduction. Let $A = [a_{ij}]$ denote an n-square matrix with entries in the field $\mathscr{C}$ of complex numbers. Denote by H a subgroup of S_n, the symmetric group on the integers $1, \ldots, n$, and by $\chi: H \to \mathscr{C}$ a character of degree 1 on H. Then

$$d_\chi^H(A) \equiv \sum_{\sigma \in H} \chi(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$$

is the generalized matrix function of A associated with H and χ; e.g., if $H = S_n$ and $\chi \equiv 1$, then $\sum_{\sigma \in S_n} \prod_{i=1}^n a_{i\sigma(i)} = \operatorname{per} A$, the permanent function. If the sequences $\omega = (\omega_1, \ldots, \omega_m)$ and $\gamma = (\gamma_1, \ldots, \gamma_m)$ are m-selections, $m \leqq n$, of integers $1, \ldots, n$, then $A[\omega|\gamma]$ denotes the m-square generalized submatrix $[a_{\omega_i\gamma_j}]$, $i, j = 1, \ldots, m$, of the n-square matrix A. If ω is an increasing m-combination, then $A[\omega|\omega]$ is an m-square principal submatrix of A. H is viewed as a permutation group on the set $\Gamma_{m,n} = \Gamma$ of all m-selections, where H permutes the selection $\omega = (\omega_1, \ldots, \omega_m)$ by permuting the subscripts $1, \ldots, m$. From each orbit of Γ relative to H is selected a representative minimal in the lexicographical order, and this system of distinct representatives is denoted by Δ. If, for example, $H = S_m$, then $\Delta_{m,n}(S_m) = G_{m,n}$, the set of all $\binom{n+m-1}{m}$ non-decreasing m-selections. For each $\omega \in \Delta$, $\nu(\omega)$ is the order of the subgroup H_ω of H fixing ω, and $\bar{\Delta}_{m,n}(H) = \bar{\Delta}$ is the set of those selections $\omega \in \Delta$ for which χ restricted to H_ω is identically 1. If $H = S_m$ and $\chi = \epsilon$, the alternating character, then $\bar{\Delta}_{m,n}(S_m) = Q_{m,n}$, the set of all $\binom{n}{m}$ m-combinations, and $\nu(\omega) = \prod_{i=1}^n m_i(\omega)!$, where $m_i(\omega)$ is the multiplicity of i in ω. We are concerned with proofs and some consequences of the following principal results.

THEOREM 1. *Let $A = [a_{ij}]$ denote an n-square positive semi-definite hermitian matrix with eigenvalues $\lambda_1 \geqq \ldots \geqq \lambda_n$ and corresponding orthonormal eigenvectors $u_j = [u_{1j}, \ldots, u_{nj}]^{\mathrm{T}}$, $j = 1, \ldots, n$. Let $\omega = (\omega_1, \ldots, \omega_m) \in \bar{\Delta}_{m,n}(H)$. Then the function*

$$g_\omega(t) = \begin{cases} (d_\chi^H(A^t[\omega|\omega])/\nu(\omega))^{1/t}, & t \neq 0, \\ \prod_{j=1}^{n} \lambda_j^{\sum_{i=1}^m |u_{\omega_i j}|^2}, & t = 0, \end{cases}$$

is continuous, monotone non-decreasing for all real t and satisfies the inequalities

$$\text{(1)} \qquad \min_{\gamma \in \bar{\Delta}} \left(\prod_{i=1}^{m} \lambda_{\gamma_i} \right) \leqq g_\omega(t) \leqq \max_{\gamma \in \bar{\Delta}} \left(\prod_{i=1}^{m} \lambda_{\gamma_i} \right).$$

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If $m = n$ and $\omega = (1, \ldots, n)$, then $g_\omega(t)$ is strictly increasing for all t when A is invertible unless the subgroup $S_n(A)$ of the symmetric group S_n generated by the transpositions (ij) for which $a_{ij} \neq 0$ is a subgroup of H and χ restricted to $S_n(A)$ is the alternating character.

THEOREM 2. *Let A denote an n-square real non-negative irreducible matrix with unique real maximal eigenvalue λ_1. Let H be any subgroup of S_m and let $\chi \equiv 1$. Then A is primitive if and only if for all $\omega, \gamma \in \bar{\Delta}_{m,n}(H)$,*

$$\lim_{k \to \infty} (d_\chi^H(A^k[\omega|\gamma]))^{1/k} = \lambda_1^m. \tag{2}$$

From Theorem 1 and its proof we obtain the following result.

THEOREM 3. *Let A denote an n-square positive-definite hermitian matrix and let $\kappa = \kappa(A)$ denote the condition number relative to the spectral norm of A. Then*

$$d_\chi^H(A) \leqq (\kappa^{n/(\kappa^n - 1)} \det A)/(e \log \kappa^{n/(\kappa^n - 1)}) \tag{3}$$

and

$$d_\chi^H(A) \leqq \det A + \lambda_n^n\{[(\kappa^n - 1)/(n \log \kappa)] \log[(\kappa^n - 1)/(en \log \kappa)] + 1\}, \tag{4}$$

where λ_n is the minimal eigenvalue of A.

From Theorem 2 we prove an extension of a result of Brualdi (**2**).

THEOREM 4. *Let A denote an n-square non-negative irreducible matrix with unique real maximal eigenvalue λ_1. Let H denote any subgroup of S_m and let $\chi \equiv 1$. Then for any $\omega \in \bar{\Delta}_{m,n}(H)$ for which $d_\chi^H(A[\omega|\omega]) \neq 0$,*

$$\lim_{k \to \infty} (d_\chi^H(A^k[\omega|\omega]))^{1/k} = \lambda_1^m. \tag{5}$$

Again specifying $\chi \equiv 1$, we obtain from Theorem 1 and its proof a generalization of a result of Marcus and Newman (**13**, Theorem 11).

THEOREM 5. *Let $A = [a_{ij}]$ denote a positive semi-definite hermitian matrix and let $\chi \equiv 1$. If an eigenvector $u_1 = [u_{11}, \ldots, u_{n1}]^T$ corresponding to a maximal eigenvalue λ_1 of A satisfies $u_{\omega_i 1} \neq 0$ for $\omega = (\omega_1, \ldots, \omega_m) \in \bar{\Delta}_{m,n}$, then*

$$\lim_{t \to \infty} g_\omega(t) = \lambda_1^m. \tag{6}$$

If an eigenvector $u_n = [u_{1n}, \ldots, u_{nn}]^T$ corresponding to a minimal eigenvalue λ_n of A satisfies $u_{\omega_i n} \neq 0$, $i = 1, \ldots, m$, then

$$\lim_{t \to -\infty} g_\omega(t) = \lambda_n^m. \tag{7}$$

Let $m = n$ and $\omega = (1, \ldots, n)$. If A is invertible, then $g_\omega(t)$ is strictly increasing for all t unless A is diagonal. If A is singular, then $g_\omega(t)$ is strictly increasing for positive t unless A has a zero row.

The functions $d_\chi^H(A)$ were introduced by Schur (**14**) who proved that

$$\det A \leqq d_\chi^H(A) \tag{8}$$

when A is positive semi-definite hermitian. Extensions of Schur's inequality were obtained by Marcus (**6**) and by Marcus and Minc (**12**). Further studies of the functions $d_\chi^H(A)$ are summarized in the survey (**8**), and our proofs which follow in § 2 are based on the multilinear methods described there.

In § 3 we give some consequences of our results which furnish new inequalities for the permanent and determinant functions including two new refinements of Hadamard's inequality. It will follow, for example, that the function $f(t) = (\mathrm{per}(A^t))^{1/t}$, A positive-definite, is in general not convex-concave relative to any value of t. Some well-known inequalities for $\det A[\omega|\omega]$, a principal minor of a positive-definite matrix A, are contained in a general description of the behaviour of the function $(\det(A^t[\omega|\omega]))^{1/t}$.

2. Proofs. Let V denote a unitary vector space of dimension n with inner product (x, y). For $m \leqq n$, let $(V_\chi^m(H), *)$, or simply $V_\chi^m(H)$, denote an mth symmetric power of V associated with the subgroup H of S_m and the character $\chi: H \to \mathscr{C}$. This vector space may be viewed as a subspace of the mth tensorial power $(\otimes^m V, \otimes)$ generated (linearly) by the decomposable elements $x_1 \otimes \ldots \otimes x_m \in \otimes^m V$ which, for every $\sigma \in H$, satisfy

$$x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(m)} = \chi(\sigma) x_1 \otimes \ldots \otimes x_m \tag{9}$$

and for which we write $x_1 * \ldots * x_m$. $(\otimes^m V, \otimes)$ is itself an mth symmetric power associated with the identity group. The mth Grassmann power $(\bigwedge^m V, \wedge)$ and mth symmetric power $(\bigvee^m V, \vee)$ are symmetric powers associated with the group S_m and the characters $\chi = \epsilon$ and $\chi \equiv 1$, respectively (**4**). The inner product in V induces an inner product in $(\otimes^m V, \otimes)$ and, hence, in $(V_\chi^m(H), *)$ which for decomposable tensors satisfies

$$\begin{aligned}(x_1 * \ldots * x_m, y_1 * \ldots * y_m) &= (1/h) \sum_{\sigma \in H} \chi(\sigma) \prod_{i=1}^m (x_i, y_{\sigma(i)}) \\ &= (1/h) d_\chi^H([(x_i, y_j)]),\end{aligned} \tag{10}$$

where h is the order of H. If $e_1, \ldots, e_n$ is a (orthonormal) basis of V, then the decomposable tensors

$$\sqrt{(h/\nu(\omega))}e_{\omega_1} * \ldots * e_{\omega_m} = \sqrt{(h/\nu(\omega))}e_\omega^*, \qquad \omega = (\omega_1, \ldots, \omega_m) \in \bar{\Delta}_{m,n}(H),$$

form a (orthonormal) basis of $V_\chi^m(H)$ (**12**). Relative to this orthonormal basis, Parseval's formula in $V_\chi^m(H)$ is, from (10),

$$\begin{aligned} d_\chi^H([(x_{\alpha_i}, y_{\beta_j})]) &= h(x_\alpha^*, y_\beta^*) \\ &= h \sum_{\gamma \in \bar{\Delta}} (x_\alpha^*, \sqrt{(h/\nu(\gamma))}e_\gamma^*)(\sqrt{(h/\nu(\gamma))}e_\gamma^*, y_\beta^*) \\ &= \sum_{\gamma \in \bar{\Delta}} d_\chi^H([(x_{\alpha_i}, e_{\gamma_j})]) d_\chi^H([(e_{\gamma_i}, y_{\beta_j})])/\nu(\gamma)\end{aligned} \tag{11}$$

for the decomposable tensors $x_\alpha^* = x_{\alpha_1} * \ldots * x_{\alpha_m}$ and $y_\beta^* = y_{\beta_1} * \ldots * y_{\beta_m}$. Let $T: V \to V$ be linear and let $K(T): V_\chi^m(H) \to V_\chi^m(H)$ denote the mth symmetric power of the linear map T associated with H and χ. The operative equation for the linear map $K(T)$ is

$$K(T)x_1 * \ldots * x_m = Tx_1 * \ldots * Tx_m \tag{12}$$

from which follows the well-known relation

$$K(ST) = K(S)K(T), \tag{13}$$

S: $V \to V$ linear. We prove Theorem 1. The methods of (**7**, Theorem 3) apply.

Denote by $\mathscr{C}^n$ the unitary space of n-tuples of complex numbers with standard inner product and standard orthonormal basis

$$e_i = [\delta_{i1}, \ldots, \delta_{in}]^{\mathrm{T}}, \qquad i = 1, \ldots, n.$$

Define T: $\mathscr{C}^n \to \mathscr{C}^n$ by $Tx = A^{\mathrm{T}}x$. Then T^t is self-adjoint and positive with orthonormal basis of eigenvectors $v_j = [\bar{u}_{1j}, \ldots, \bar{u}_{nj}]^{\mathrm{T}}, j = 1, \ldots, n$, relative to the real non-negative eigenvalues $\lambda_1^t \geqq \ldots \geqq \lambda_n^t$ as is $K(T^t)$ with orthonormal eigenvectors $\sqrt{(h/\nu(\gamma))}v_\gamma^*$, $\gamma \in \bar{\Delta}$, corresponding to non-negative eigenvalues $\lambda_\gamma^t = \prod_{i=1}^n \lambda_i^{m_i(\gamma)t}$. By (10)–(12),

$$\begin{aligned} d_\chi^H(A^t[\omega|\omega]) &= d_\chi^H((T^t e_{\omega_i}, e_{\omega_i})) \\ &= \nu(\omega)(\sqrt{(h/\nu(\omega))}K(T^t)e_\omega^*, \sqrt{(h/\nu(\omega))}e_\omega^*) \\ &= \nu(\omega) \sum_{\gamma\in\bar{\Delta}} c_{\omega,\gamma}\lambda_\gamma^t \end{aligned} \tag{14}$$

where

$$\begin{aligned} c_{\omega,\gamma} &= |(\sqrt{(h/\nu(\omega))}e_\omega^*, \sqrt{(h/\nu(\gamma))}v_\gamma^*)|^2 \\ &= |d_\chi^H([(e_{\omega_i}, v_{\gamma_j})])|^2/\nu(\omega)\nu(\gamma) \\ &= |d_\chi^H([u_{\omega_i\gamma_j}])|^2/\nu(\omega)\nu(\gamma) \\ &= |d_\chi^H(U[\omega|\gamma])|^2/\nu(\omega)\nu(\gamma). \end{aligned}$$

$U = [u_{ij}]$ is an n-square unitary matrix diagonalizing A. Thus, $\sum_{\gamma\in\bar{\Delta}} c_{\omega,\gamma} = 1$ and $g_\omega(t)$ is is a tth power mean [1, 5] of the non-negative reals λ_γ, $\gamma \in \bar{\Delta}$, with non-negative weights $c_{\omega,\gamma}$. Hence, $g_\omega(t)$ is monotone non-decreasing and satisfies the inequalities (1). By an identity of Marcus (**7**, Theorem 1),

$$\sum_{\gamma\in\Delta} m_j(\gamma)c_{\omega,\gamma} = \sum_{i=1}^m |u_{\omega_i j}|^2. \tag{15}$$

Interchanging the order of taking products in the geometric mean $\prod_{\gamma\in\bar{\Delta}} \lambda_\gamma^{c_{\omega,\gamma}}$ and applying (15) yields the indicated value for $g_\omega(0)$. $g_\omega(t)$ is either everywhere strictly increasing or constant. If $m = n$ and $\omega = (1, \ldots, n)$, then $\nu(\omega) = 1$, and $g_\omega(0) \leqq g_\omega(1)$ is Schur's inequality (8). If A is invertible, then the conditions stated in Theorem 1 in order that $g_\omega(t)$ be everywhere constant coincide with those given by Schur (**14**) in order that equality hold in (8).

To prove Theorem 5, we note that when $\chi \equiv 1$, then $\bar{\Delta} = \Delta \supseteq G_{m,n}$, the non-decreasing sequences of length m. Hence, $\max_{\gamma\in\bar{\Delta}}(\lambda_\gamma) = \lambda_1^m$ and $\min_{\gamma\in\bar{\Delta}}(\lambda_\gamma) = \lambda_n^m$.

The weights belonging to these eigenvalues are $c_{\omega,(1,\ldots,1)}$ and $c_{\omega,(n,\ldots,n)}$, respectively. Since $\nu(1,\ldots,1) = h$, the order of H, we have by (10)

$$\begin{aligned}\nu(\omega)c_{\omega,(1,\ldots,1)} &= |(\sqrt{h}e_{\omega_1} * \ldots * e_{\omega_m}, v_1 * \ldots * v_1)|^2 \\ &= |d_\chi^H([(e_{\omega_i}, v_1)])|^2/h \\ &= h \prod_{i=1}^{m} |u_{\omega_i 1}|^2 \\ &\neq 0.\end{aligned}$$

Similarly, $c_{\omega,(n,\ldots,n)} \neq 0$ so that (6) and (7) follow. Let $m = n$ and $\omega = (1,\ldots,n)$. Clearly, if A is diagonal, then $g_\omega(0) = \det A = d_\chi^H(A) = g_\omega(1)$ so that $g_\omega(t)$ is everywhere constant. On the other hand, if A is invertible and $g_\omega(0) = g_\omega(1)$, then the subgroup $S_n(A)$ of S_n generated by the transpositions (ij) for which $a_{ij} \neq 0$ is a subgroup of H. However, $\chi \equiv 1 \neq \epsilon$ on $S_n(A)$ means that $S_n(A)$ reduces to the identity permutation, i.e., $a_{ij} = 0$ for $i \neq j$. Let A be singular. If A has a zero row then, clearly, $g_\omega(t) \equiv 0$. Finally, let $g_\omega(t)$ be constant for $t \geqq 0$. Then $g_\omega(t) = \det A = 0$. Let $x_1, \ldots, x_n$ be vectors in $\mathscr{C}^n$ for which the positive semi-definite hermitian matrix A is the Gram matrix $[(x_i, x_j)]$. Then

$$\begin{aligned}(x_1 * \ldots * x_n, x_1 * \ldots * x_n) &= (1/h)d_\chi^H([(x_i, x_j)]) \\ &= (1/h)d_\chi^H(A) \\ &= 0,\end{aligned}$$

whence $x_1 * \ldots * x_n = 0$. However, this is possible if and only if some $x_k = 0$, $1 \leqq k \leqq n$ (**12**, Lemma 2.4); i.e., row k of A is zero.

Set $m = n$, $H = S_n$, and $\omega = (1,\ldots,n)$ to obtain the result of Marcus and Newman. Note that our proof does not require that λ_1 and λ_n be simple in order that (6) and (7) hold.

Given an inequality for power means, we can write an inequality for matrix functions. For example, since $\log g_\omega{}^t(t)$ is convex in t, we have the following result.

COROLLARY 1. *Let $t = \sum_{i=1}^{k} p_i s_i$ be a convex combination of k real numbers s_i. Then if A is invertible or if all the s_i are non-negative,*

$$d_\chi^H(A^t[\omega|\omega]) \leqq \prod_{i=1}^{k} (d_\chi^H(A^{s_i}[\omega|\omega]))^{p_i}. \tag{16}$$

A pair of recent results, so-called reversals, giving bounds on ratios of means (**3**) and on differences of means (**15**), yields inequalities for $g_\omega(t)$ complementary to its monotone property. These may be paraphrased for matrix functions as follows.

THEOREM 6. *Let A denote an n-square positive-definite hermitian matrix not a scalar matrix. Let λ_1 and λ_n denote the maximal and minimal eigenvalues of A, respectively, and let $\kappa = \kappa(A) = \lambda_1/\lambda_n$, the condition number relative to the spectral* *norm of* A. *Set* $\gamma = \kappa^n$. *Then for* H *any subgroup of* S_n *and* χ *a character of degree* 1 *on* H,

$$(17) \qquad 1 \leqq (d_\chi^H(A^s))^{1/s}/(d_\chi^H(A^r))^{1/r} \leqq \Upsilon_1$$

for $r < s$, *and if* $s \geqq 1$,

$$(18) \qquad 0 \leqq (d_\chi^H(A^s))^{1/s} - (d_\chi^H(A^r))^{1/r} \leqq \lambda_n^n \Upsilon_2,$$

where

$$\Upsilon_1 = \begin{cases} \{[r(\gamma^s - \gamma^r)]/[(s-r)(\gamma^r - 1)]\}^{1/s}\{[s(\gamma^r - \gamma^s)]/[(r-s)(\gamma^s - 1)]\}^{-1/r}, & rs \neq 0, \\ \{\gamma^{s/(\gamma^s - 1)}/[e \log \gamma^{s/(\gamma^s-1)}]\}^{1/s}, \quad r = 0, \\ \{\gamma^{r/(\gamma^r - 1)}/[e \log \gamma^{r/(\gamma^r-1)}]\}^{1/r}, \quad s = 0, \end{cases}$$

and

$$\Upsilon_2 = \begin{cases} [\theta(\gamma^s - 1) + 1]^{1/s} - [\theta(\gamma^r - 1) + 1]^{1/r}, & r \neq 0, \\ [\theta(\gamma^s - 1) + 1]^{1/s} - \gamma^\theta, & r = 0, \end{cases}$$

θ *the unique solution of*

$$[y(\gamma^s - 1) + 1]^{1/s-1}(\gamma^s - 1)/s - [y(\gamma^r - 1) + 1]^{1/r-1}(\gamma^r - 1)/r = 0, \qquad 0 < y < 1,$$

if $r \neq 0$ *or*

$$[y(\gamma^s - 1) + 1]^{1/s-1}(\gamma^s - 1)/s - \gamma^y \log \gamma = 0, \qquad 0 < y < 1,$$

if $r = 0$.

The inequalities of Theorem 3 follow from the preceding result by setting $r = 0$ and $s = 1$.

Consider next the matrix representation of a mapping $K(T)$ in $V_\chi^m(H)$ induced by an arbitrary linear mapping T: $V \to V$. Let $e_1, \ldots, e_n$ be any basis of V. If $Te_j = \sum_{i=1}^n a_{ij}e_i$, $j = 1, \ldots, n$, then the n-square matrix $A = [a_{ij}]$ is the matrix representation of T relative to the ordered basis $e_1, \ldots, e_n$. Let $\sqrt{(h/\nu(\omega))}e_\omega^*$, $\omega \in \bar{\Delta}_{m,n}(H)$, be the corresponding induced basis of $V_\chi^m(H)$ ordered lexicographically in the sequences $\omega \in \bar{\Delta}$. Then it is easy to show that for all $\gamma \in \bar{\Delta}$,

$$\sqrt{(h/\nu(\gamma))}K(T)e_\gamma^* = \sum_{\omega \in \bar{\Delta}} g_{\omega\gamma}\sqrt{(h/\nu(\omega))}e_\omega^*,$$

where

$$g_{\omega\gamma} = d_\chi^H(A^{\mathrm{T}}[\gamma|\omega])/\sqrt{(\nu(\omega)\nu(\gamma))}.$$

The entry in row ω and column γ of the matrix representation of $K(T)$ is then, with the indicated factor, the generalized matrix function of the m-square matrix whose entry in row i and column j is $a_{\omega_j\gamma_i}$. It is natural to write $K(A)$ for the matrix $[g_{\omega\gamma}]$. Clearly, by (13), $K(AB) = K(A)K(B)$ for any n-square matrices A and B.

If the character χ is real, then $d_\chi^H(A^{\mathrm{T}}[\gamma|\omega]) = d_\chi^H(A[\omega|\gamma])$. In particular, let $\chi \equiv 1$ and let A denote an arbitrary n-square matrix with real non-negative entries. Let λ_1 denote a real maximal non-negative eigenvalue of A guaranteed by the Perron-Frobenius theory. Then the associated induced matrix $K(A)$ is real and non-negative, and the maximal real non-negative eigenvalue of $K(A)$ is λ_1^m. Since every diagonal element $g_{\omega\omega}$, $\omega \in \bar{\Delta}_{m,n}$, of $K(A)$ is trivially a principal submatrix, we immediately obtain (**11**, p. 126) the inequality

$$d_\chi^H(A[\omega|\omega]) \leqq \lambda_1^m \nu(\omega) \tag{19}$$

which is (**2**, Lemma 1) when $m = n$, $H = S_n$, and $\omega = (1, \ldots, n)$. Of course, if we know that A is irreducible we can say more.

A is primitive if and only if a positive integer p exists for which A^p is positive. However, $K(A^p) = (K(A))^p$. Thus, A is primitive if and only if $K(A)$ is primitive. Now by (**11**, p. 128), since $K(A^k) = (K(A))^k$ for all non-negative integers k, $K(A)$ is primitive if and only if for all ω, $\gamma \in \bar{\Delta}$, we have

$$\lim_{k\to\infty} (d_\chi^H(A^k[\omega|\gamma])/\sqrt{(\nu(\omega)\nu(\gamma))})^{1/k} = \lambda_1^m$$

which is equivalent to (2). Thus, Theorem 2 is proved.

Proof of Theorem 4. Let A be imprimitive with index of imprimitivity h. Then (**11**, p. 128) there is a permutation matrix P and primitive n_i-square matrices A_i, $i = 1, \ldots, h$, each with maximal eigenvalue λ_i^h satisfying $P^{\mathrm{T}}A^hP = \sum\nolimits_{i=1}^{\cdot h} A_i$, the direct sum of the h matrices A_i. Now

$$K(P^{\mathrm{T}}A^hP) = K^{\mathrm{T}}(P)K(A^h)K(P) = K\left(\sum_{i=1}^{\cdot h} A_i\right)$$

and P is a unitary matrix with real non-negative entries. Hence,

$$K(A^h) = K(P)K\left(\sum_{i=1}^{\cdot h} A_i\right)K^{\mathrm{T}}(P),$$

and $K(P)$ is unitary with real non-negative entries, i.e., a permutation matrix. Therefore, there is a sequence $\gamma = (\gamma_1, \ldots, \gamma_m) \in \bar{\Delta}$ for which

$$d_\chi^H(A^h[\omega|\omega])/\nu(\omega) = d_\chi^H\left(\left(\sum_{i=1}^{\cdot h} A_i\right)[\gamma|\gamma]\right)\Big/\nu(\gamma). \tag{20}$$

Denote by $A_i[\gamma^i|\gamma^i]$ the m_i-square submatrix consisting of the rows and columns of the m-square submatrix $(\sum\nolimits_{i=1}^{\cdot h} A_i)[\gamma|\gamma]$ included in the n_i-square direct summand A_i, $i = 1, \ldots, h$. Then $\sum_{i=1}^h m_i = m$. Let $d_\chi^I(A) = \prod_{i=1}^n a_{ii}$ denote the generalized matrix function of the matrix A corresponding to the identity group I. Using (19) and applying Theorem 2 in the case that $H = I$ to each of the primitive matrices A_i, we obtain, from (20),

$$\begin{aligned}\lambda_1^m &= (\lambda_1^{mhk})^{1/hk} \geqq (d_\chi^H(A^{hk}[\omega|\omega])/\nu(\omega))^{1/hk} \\ &= \left(d_\chi^H\left(\left(\sum_{i=1}^{\cdot h} A_i^k\right)[\gamma|\gamma]\right)\Big/\nu(\gamma)\right)^{1/hk} \geqq \left(d_\chi^I\left(\left(\sum_{i=1}^{\cdot h} A_i^k\right)[\gamma|\gamma]\right)\Big/\nu(\gamma)\right)^{1/hk} \\ &= ((1/\nu(\gamma)) \prod_{i=1}^h d_\chi^I(A_i^k[\gamma^i|\gamma^i]))^{1/hk} \xrightarrow[k\to\infty]{} \left(\prod_{i=1}^h \lambda_1^{m_ih}\right)^{1/h} = \lambda_1^m.\end{aligned}$$

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Thus, (5) holds when the limit is taken through integral multiples of h. The proof is now completed in a fashion similar to that used by Brualdi in the case that $m = n$, $H = S_n$, and $\omega = (1, \ldots, n)$.

3. Further inequalities and examples. In (**13**), Marcus and Newman raised the question whether or not the function $f(t) = (\text{per}\,(A^t))^{1/t}$ is convex-concave relative to $t = 0$; i.e., convex for $t \leqq 0$ and concave for $t \geqq 0$. It was shown by Shniad (**16**) that mean value functions are, in general, not convex-concave relative to any value of t, and this is the case for $f(t)$ also. The 2-square matrix $A = UDU^*$, where $D = \text{diag}(e^{3/2}, e^{1/2})$ and

$$U = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

is positive-definite, symmetric for all θ. With $m = n$, $H = S_n$, $\chi \equiv 1$, and $\omega = (1, \ldots, n)$, we obtain from the formula (14)

$$\text{per}\,A = 2e^3|\sin\theta\cos\theta|^2 + e^2|\cos^2\theta - \sin^2\theta|^2 + 2e|\sin\theta\cos\theta|^2.$$

If $\theta = (1/2)\arctan(1/2)$, then $(\text{per}(A^t))^{1/t} = (0.1e^t + 0.8e^{2t} + 0.1e^{3t})^{1/t}$, Shniad's counterexample for tth power means. Perhaps the most that can be said of $\text{per}(A^t)^{1/t}$ is that it is, with power means, *star-shaped* relative to the origin, every line through the origin intersects the graph of $(\text{per}(A^t))^{1/t}$ at most once.

Marcus and Newman pointed out that the function $f(t)$ has no analogue for the determinant function, and this is certainly the case when $m = n$. However, suppose that $m < n$ and let $(V_\chi^m(H), *)$ be $(\bigwedge^m V, \wedge)$, the mth Grassmann power of V. In this case, $d_\chi^H(A[\omega|\omega]) = \det A[\omega|\omega]$, a principal minor of the n-square matrix A, and $K(A^T)$ is $C_m(A^T)$, the mth compound of A^T. Let $\lambda_1 \geqq \ldots \geqq \lambda_n$. Then λ_γ, $\gamma \in Q_{m,n}$, are the eigenvalues of $K(A^T)$ with maximal eigenvalue $\prod_{i=1}^m \lambda_i$ and minimal eigenvalue $\prod_{i=1}^m \lambda_{n-i+1}$. Thus,

$$g_\omega(t) = (\det A^t[\omega|\omega])^{1/t}$$

is a continuous, monotone non-decreasing function which satisfies for all t the inequalities

$$\prod_{i=1}^m \lambda_{n-i+1} \leqq (\det A^t[\omega|\omega])^{1/t} \leqq \prod_{i=1}^m \lambda_i. \tag{21}$$

When $t = 1$, (21) is an inequality of Marcus and McGregor (**10**). The weights $c_{\omega,\gamma}$ belonging to the maximal and minimal eigenvalues of $C_m(A)$ and are, respectively, $|\det U[\omega|(1, \ldots, m)]|^2$ and $|\det U[\omega|(n - m + 1, \ldots, n)]|^2$. Hence, equality is attained in (21) in the limiting cases if the m-square submatrices $U[\omega|(1, \ldots, m)]$ and $U[\omega|(n - m + 1, \ldots, n)]$ of the n-square unitary matrix U are invertible. The inequalities of (21) can be reversed in the manner of Theorem 6. Let $\gamma = \prod_{i=1}^{m} \lambda_i/\lambda_{n-i+1}$ and let $\omega \in Q_{m,n}$. Then, if the eigenvalues of A satisfy $\lambda_1 \geqq \ldots \geqq \lambda_n$ and $\lambda_1 > \lambda_n$, we have, for $r < s$,

$$\det{}^{1/s}(A^s[\omega|\omega])/\det{}^{1/r}(A^r[\omega|\omega]) \leqq \Upsilon_1, \tag{22}$$

and if $s \geqq 1$,

$$\det{}^{1/s}(A^s[\omega|\omega]) - \det{}^{1/r}(A^r[\omega|\omega]) \leqq \Upsilon_2 \prod_{i=1}^{m} \lambda_{n-i+1}. \tag{23}$$

If in (22) we set $r = -1$ and $s = 1$, we have

$$1 \leqq \det(A[\omega|\omega])\det(A^{-1}[\omega|\omega]) \leqq (\gamma^{1/2} + \gamma^{-1/2})^2/4,$$

an inequality of Marcus and Khan (**9**).

Finally, the inequalities (3) and (4) of Theorem 3 are complements of Schur's inequality (8). If H is the identity permutation, then $(V_\chi{}^m(H), *)$ is $(\otimes^m V, \otimes)$, the mth tensorial power of V. With $m = n$, Schur's inequality is

$$\det A \leqq \prod_{i=1}^{n} a_{ii},$$

Hadamard's inequality, of which Theorem 3 provides the following refinements:

$$1 \leqq \left(\prod_{i=1}^{n} a_{ii}\right)\Big/\det A \leqq \kappa^{n/(\kappa^n-1)}/(e \log \kappa^{n/(\kappa^n-1)})$$

and

$$\begin{aligned} 0 &\leqq \prod_{i=1}^{n} a_{ii} - \det A \\ &\leqq \lambda_n{}^n\{[(\kappa^n - 1)/(n \log \kappa)] \log[(\kappa^n - 1)/(en \log \kappa)] + 1\}. \end{aligned}$$

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