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KLEIN'S OSCILLATION THEOREM FOR PERIODIC BOUNDARY CONDITIONS

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Multiparameter eigenvalue problems for systems of linear differential equations with homogeneous boundary conditions have been considered by Ince [**4**] and Richardson [**5, 6**], and more recently Faierman [**3**] has considered their completeness and expansion theorems. A survey of eigenvalue problems with several parameters, in mathematics, is given by Atkinson [**1**].

We consider the two differential equations:

$$(p_1u')' + (q_1 + \lambda A_1 + \mu B_1)u = 0, \tag{1a}$$

$$(p_2v')' + (q_2 + \lambda A_2 + \mu B_2)v = 0, \tag{1b}$$

where $p_1'(x), q_1(x), A_1(x), B_1(x)$ and $p_2'(y), q_2(y), A_2(y), B_2(y)$ are continuous for $x \in [a_1, b_1]$ and $y \in [a_2, b_2]$ respectively, and $p_1(x) > 0\,(x \in [a_1, b_1]), p_2(y) > 0$ $(y \in [a_2, b_2])$, $p_1(a_1) = p_1(b_1), p_2(a_2) = p_2(b_2)$. The differential equations (1) will be subjected to the periodic boundary conditions.

$$u(a_1) = u(b_1), \qquad u'(a_1) = u'(b_1) \tag{2a}$$

$$v(a_2) = v(b_2), \qquad v'(a_2) = v'(b_2). \tag{2b}$$

Let us consider a single differential equation

$$(pu')' + Qu = 0, \tag{3}$$

where $Q(x, \lambda, \mu) = q(x) + \lambda A(x) + \mu B(x)$, $(x \in [a, b])$, the $p'(x), q(x), A(x)$, $B(x)$ are continuous for $x \in [a, b]$, $p(x) > 0$ $(x \in [a, b])$ and $p(a) = p(b)$. We denote by $[\lambda, \mu_i(\lambda)]$, $i \geqq 0$, the eigenvalues of (3) for the periodic boundary conditions

$$u(a) = u(b), \qquad u'(a) = u'(b), \tag{4}$$

and by $[\lambda, \nu_i(\lambda)]$, $i \geqq 0$, the eigenvalues of (3) for the boundary conditions

$$u(a) = u(b) = 0. \tag{5}$$

From Coddington and Levinson [**2**, pp. 213–220] we have

THEOREM 1. *Assuming that $B > 0$, the curves $\mu_i(\lambda)$ and $\nu_i(\lambda)$ for each fixed i are continuous functions of λ, $-\infty < \lambda < \infty$, with continuously turning tangent and form sequences such that*

$$-\infty < \mu_0(\lambda) < \nu_0(\lambda) < \mu_1(\lambda) \leqq \nu_1(\lambda) \leqq \mu_2(\lambda) < \dots \tag{6}$$

$$\dots < \mu_{2j+1}(\lambda) \leqq \nu_{2j+1}(\lambda) \leqq \mu_{2j+2}(\lambda) < \nu_{2j+2}(\lambda) < \mu_{2j+3}(\lambda) \leqq \nu_{2j+3}(\lambda) \leqq \dots.$$

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The eigenfunction ψ_i *corresponding to* $[\lambda, \nu_i(\lambda)]$ *has* i *zeros on* (a, b) *and the eigenfunction* ϕ_i *corresponding to* $[\lambda, \mu_i(\lambda)]$ *has an even number of zeros on* $[a, b)$, *either* i *or* $i + 1$ *zeros. If* $\mu_{2i+1}(\lambda) < \mu_{2i+2}(\lambda)$ *for some* i *and* λ, *then there is a unique eigenfunction* ϕ_{2i+1} *at* $[\lambda, \mu_{2i+1}(\lambda)]$ *and a unique eigenfunction* ϕ_{2i+2} *at* $[\lambda, \mu_{2i+2}(\lambda)]$. *However, if* $\mu_{2i+1}(\lambda) = \mu_{2i+2}(\lambda)$, *then there are two linearly independent eigenfunctions* ϕ_{2i+1}, ϕ_{2i+2} *at* $[\lambda, \mu_{2i+1}(\lambda)] = [\lambda, \mu_{2i+2}(\lambda)]$.

It is evident that the above theorem can be restated for each of the cases $B < 0$, $A > 0$ and $A < 0$, where in the later two cases the roles of λ and μ are interchanged.

Let us now consider (3) at $[\lambda, \nu_i(\lambda)]$ with $B > 0$. By Sturm's Comparison Theorem, for a sufficiently large definite increment Δ of Q, the number of zeros in (a, b) of the eigenfunction ψ_i for (5) will vary. If $\Delta > 0$, then for $\Delta = \Delta_1$, say, the corresponding eigenfunction for (5) will have $i + 1$ zeros on (a, b) and for $\Delta = \Delta_2 > \Delta_1$ the corresponding eigenfunction for (5) will have $i + 2$ zeros on (a, b). Suppose that $i = 2m + 1$ and that $\Delta(x, \delta, \epsilon) = \delta A(x) + \epsilon B(x)$, $(x \in [a, b])$, where the δ and ϵ are such that $\Delta > 0$. Then corresponding to Δ_1, (5) has eigenvalue $[\lambda_1, \nu_{2m+2}(\lambda_1)]$ and corresponding to Δ_2, (5) has eigenvalue $[\lambda_2, \nu_{2m+3}(\lambda_2)]$ where λ_1, λ_2 are some particular values of λ. By (6) there are Δ_1' and Δ_2' such that $0 \leqq \Delta_1' < \Delta_1 < \Delta_2' \leqq \Delta_2$ for which (4) has eigenvalues $[\lambda_1', \mu_{2m+2}(\lambda_1')]$, $[\lambda_2', \mu_{2m+3}(\lambda_2')]$, respectively, and whose corresponding eigenfunctions have $2m + 2$ zeros and $2m + 4$ zeros, respectively, on $[a, b)$. Similarly for $i = 2m + 2$, sufficiently large definite increments in Q will produce a change in the number of zeros on $[a, b)$ of eigenfunctions for (4). Therefore, beginning with some fixed $[\lambda, \mu_i(\lambda)]$, variations in λ and μ, i.e., δ and ϵ, must be such that the increment Δ in Q should change signs continually, in order that the corresponding eigenfunction ϕ_i for (4) have its number of zeros on $[a, b)$ preserved. We have proved

LEMMA 1. *Assuming that* $B > 0$, *in order that the number of zeros on the interval* $[a, b)$ *for the eigenfunction* ϕ_i *be preserved in perturbations* $\Delta = \delta A + \epsilon B$ *of the function* Q, Δ *must continually change sign.*

We now consider system (1) with boundary conditions (2).

THEOREM 2. *A sufficient condition for the eigenvalues of* (1) *with* (2) *to be real is that*

$$(7) \qquad \det \begin{pmatrix} A_1(x) B_1(x) \\ A_2(y) B_2(y) \end{pmatrix} \neq 0 \qquad \begin{matrix} (x \in [a_1, b_1]), \\ (y \in [a_2, b_2]). \end{matrix}$$

Proof. From (1a) we have the equations:

$$(p_1 u')\bar{u} + (q_1 + \lambda A_1 + \mu B_1) u\bar{u} = 0,$$
$$(p_1 \bar{u}')u + (q_1 + \bar{\lambda} A_1 + \bar{\mu} B_1)\bar{u}u = 0,$$

which on subtraction give

$$(p_1 u')'\bar{u} - (p_1 \bar{u}')'u + [(\lambda - \bar{\lambda})A_1 + (\mu - \bar{\mu})B_1]u\bar{u} = 0,$$

and by the periodic boundary condition, integration of the above equation gives

$$\int_{a_1}^{b_1} [(\lambda - \bar{\lambda})A_1 + (\mu - \bar{\mu})B_1]u\bar{u}\,dx = 0.$$

Similarly, for (1b) we have

$$\int_{a_2}^{b_2} [(\lambda - \bar{\lambda})A_2 + (\mu - \bar{\mu})B_2]v\bar{v}\,dy = 0.$$

A sufficient condition for the only solution of these last two equations to be $\lambda - \bar{\lambda} = 0$ and $\mu - \bar{\mu} = 0$ is that

$$\det\begin{pmatrix} \int_{a_1}^{b_1} A_1 u\bar{u}\,dx & \int_{a_1}^{b_1} B_1 u\bar{u}\,dx \\ \int_{a_2}^{b_2} A_2 v\bar{v}\,dy & \int_{a_2}^{b_2} B_2 v\bar{v}\,dy \end{pmatrix} \neq 0,$$

i.e.,

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1} [A_1B_2 - B_1A_2]u\bar{u}v\bar{v}\,dx\,dy \neq 0.$$

Since the functions A_1, B_1, A_2 and B_2 are continuous, a sufficient condition for the above statement to hold is that $A_1(x)B_2(y) - B_1(x)A_2(y) \neq 0$ for $x \in [a_1, b_1]$ and $y \in [a_2, b_2]$.

In the case where A_j and B_j, $j = 1, 2$, take both positive and negative values on their respective intervals of definition we make use of the following theorem.

THEOREM 3. *Under condition* (7), *there exists a constant k such that* $\epsilon_j(A_j + kB_j) > 0$ *where* $\epsilon_j = \pm 1$ *and* $j = 1, 2$.

Proof. We consider the curves $C_1 \equiv [A_1(x), B_1(x)]$ $(x \in [a_1, b_1])$ and $C_2 \equiv [A_2(y), B_2(y)]$ $(y \in [a_2b_2])$ in the (ξ, η)-plane. By condition (7), there is no straight line through the origin which intersects both the curves C_1 and C_2. In fact, if the line $\xi + k\eta = 0$ were to intersect both curves C_1 and C_2, there would exist x_1 and y_1, $a_1 \leqq x_1 \leqq b_1$, $a_2 \leqq y_1 \leqq b_2$, such that

$$\begin{aligned} A_1(x_1) + kB_1(x_1) &= 0, \\ A_2(y_1) + kB_2(y_1) &= 0. \end{aligned} \tag{8}$$

By (7), the determinant of coefficients $[A_1(x_1)B_2(y_1) - A_2(y_1)B_1(x_1)] \neq 0$ and hence there can be no such k satisfying the relations (8). Therefore the curves C_1 and C_2 lie in nonoverlapping sectors and a line $\alpha\xi + \beta\eta = 0$ can be drawn such that the curves C_1 and C_2 are either to one side of the line or separated by the line. The distances of C_1 and C_2 from this line are given by $(\alpha A_1 + \beta B_1)/\pm\sqrt{\alpha^2 + \beta^2}$ and $(\alpha A_2 + \beta B_2)/\pm\sqrt{\alpha^2 + \beta^2}$ respectively. The signs of the distances are the same when C_1 and C_2 are on the same side of the line and we have opposing signs when C_1 and C_2 are separated by the line. In completing the proof we take $k = \beta/\alpha\sqrt{\alpha^2 + \beta^2}$.

In view of Theorem 3, Lemma 1 can be applied to each of the differential equations (1) under condition (7). In fact, suppose that A_1 and B_1 have both positive and negative values and that $A_1 + kB_1 > 0$ for some constant k which is evidently neither 0 or ∞. We make the transformation

$$\begin{pmatrix}\lambda\\ \mu\end{pmatrix} = \begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix},$$

where $\tan\alpha = k$; then

$$\begin{aligned}\lambda A_1 + \mu B_1 &= (\xi\cos\alpha - \eta\sin\alpha)A_1 + (\xi\sin\alpha + \eta\cos\alpha)B_1\\ &= \xi(A_1 + B_1\tan\alpha)\cos\alpha + \eta(-A_1\tan\alpha + B_1)\cos\alpha\\ &\equiv \xi\mathcal{A}_1 + \eta\mathcal{B}_1,\end{aligned}$$

where $\mathcal{A}_1$ is of definite sign.

On an eigenvalue curve for (1a) with (2a), a perturbation of the form $\Delta_{(1)} = A_1\epsilon\cos\theta + B_1\epsilon\sin\theta$ where ϵ is small and θ is the tangential angle to the curve, must by Lemma 1 continually change sign. Similarly for an eigenvalue curve for (1b) with (2b). Under condition (7) there is no θ for which the expressions $A_1\cos\theta + B_1\sin\theta$ and $A_2\cos\theta + B_2\sin\theta$ vanish simultaneously and so the tangential angles of eigenvalue curves for (1a) with (2a) and the tangential angles to the eigenvalue curves for (1b) with (2b) lie in distinctly disjoint angular sectors. A common boundary would imply an angle θ for which $\sin\theta = \cos\theta = 0$. Hence any eigenvalue curve for (1a) with (2a) will intersect any eigenvalue curve for (1b) with (2b). In fact, suppose that the eigenvalue curves are given by $\mu = f(\lambda)$ and $\mu = g(\lambda)$. On the interval $-\infty < \lambda < \infty$ the functions f and g are such that $df/d\lambda \not\equiv dg/d\lambda$. Let us assume the contrary, i.e., that $f(\lambda) \not\equiv g(\lambda)$ for every λ. If $df/d\lambda$ and $dg/d\lambda$ lie in angular sectors which do not contain $\pi/2$ (and therefore $-\pi/2$). The function $M = f - g$ has continuous derivative $dM/d\lambda = df/d\lambda - dg/d\lambda$ which is of the one sign. Thus M is strictly monotonic, contradicting $f \not\equiv g$. Suppose that the angular sector for $df/d\lambda$ contains $\pi/2$ (and $-\pi/2$); then since the eigenvalue curves have continuously turning tangent, two possibilities arise. Firstly, the derivative $df/d\lambda$ has singularities all of which are of the same sign. At these singularities the curve $\mu = f(\lambda)$ has slope $\pi/2$. Here $dM/d\lambda$ is of the one sign and so M is strictly monotonic, again contradicting $f \not\equiv g$. In the second case f is double valued in λ with f having slope $\pi/2$ at the one single value point. The function M is double valued in λ with the two branches of $dM/d\lambda$ having opposing signs and so the corresponding branches of M are strictly monotonic with opposing monotonicity again contradicting $f \not\equiv g$.

The intersection of two eigenvalue curves is unique since otherwise there would be points at which the slopes to the curves are the same.

By Theorem 1 and the above we have proved

THEOREM 4. *Given a pair of positive even integers* (m_1, m_2), *the system of differential equations* (1) *with condition* (7) *and periodic boundary conditions* (2) *has four unique real eigenvalue pairs* (λ_j, μ_j), $j = 1, 2, 3, 4$, *which, depending on coalescing eigenvalue curves, may be distinct, occur in coincident pairs or be coincident. At a distinct eigenvalue pair,* (1a) *has a unique eigenfunction with* m_1 *zeros on* $[a_1, b_1)$ *and* (1b) *has a unique eigenfunction with* m_2 *zeros on* $[a_2, b_2)$. *In the case of two coincident eigenvalue pairs, suppose that the eigenvalue curves of* (1a) *coalesce while those of* (1b) *are distinct; then* (1a) *has two linearly independent eigenfunctions with* m_1 *zeros on* $[a_1, b_1)$ *and* (1b) *has a unique eigenfunction with* m_2 *zeros on* $[a_2, b_2)$. *Similarly when the coincident eigenvalue pairs are due to coalescing eigenvalue curves of* (1b) *and a distinct eigenvalue curve of* (1a). *When the four eigenvalue pairs coincide,* (1a) *has two linearly independent eigenfunctions with* m_1 *zeros on* $[a_1, b_1)$ *and* (1b) *has two linearly independent eigenfunctions with* m_2 *zeros on* $[a_2, b_2)$.

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