THE ASYMPTOTIC EXPANSION OF THE NUMBER OF TREE-LIKE POLYHEXES†

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We refer to (**1**) for the definitions of U_n and H_n. Our object is to find asymptotic expansions for U_n and H_n for large n. This enables us to improve the approximations to U_n and H_n for large n found in the last two pages of (**1**).

We write

$$(1-x)^{k+\frac{1}{2}} = 1+(-1)^{k-1}\sum_{n=1}^{\infty} c_{k,n}x^n,$$

so that

$$c_{k,n} = \frac{(2k+1)!(2n-2k-3)!}{2^{2n-2}k!(n-k-2)!n!} \sim \frac{(k+\frac{1}{2})(k-\frac{1}{2})\dots\frac{1}{2}}{\pi^{\frac{1}{2}}n^{k+\frac{3}{2}}} \tag{1}$$

for fixed k and large n. Clearly

$$c_{k+1,n}/c_{k,n} = (2k+3)/(2n-2k-3).$$

Also

$$(1-5x)^{k+\frac{1}{2}} = 1+(-1)^{k-1}\sum_{n=1}^{\infty} c_{k,n}5^nx^n.$$

Near $x = \frac{1}{5}$, we have

$$5^{k+\frac{1}{2}}(1-x)^{k+\frac{1}{2}} = (4+1-5x)^{k+\frac{1}{2}} = 2^{2k+1}\sum_{t=0}^{\infty}\binom{k+\frac{1}{2}}{t}2^{-2t}(1-5x)^t$$

and so, if

$$\{(1-x)(1-5x)\}^{k+\frac{1}{2}} = 1+(-1)^{k-1}\sum_{n=1}^{\infty} d_{k,n}x^n,$$

we have, by Abel's result (see (**2**)),

$$d_{k,n} = 2^{2k+1}5^{n-k-\frac{1}{2}}\left\{\sum_{t=0}^{T-1}(-1)^t\binom{k+\frac{1}{2}}{t}2^{-2t}c_{k+t,n}+O(c_{k+T,n})\right\}$$

$$= 2^{2k+1}5^{n-k-\frac{1}{2}}c_{k,n}\left\{\sum_{t=0}^{T-1}(-1)^t\frac{(k+t+\frac{1}{2})\dots(k-t+\frac{3}{2})2^{-2t}}{t!(n-k-\frac{3}{2})\dots(n-k-t-\frac{1}{2})}+O(n^{-T})\right\}.$$

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It follows from (9) of (1) that

$$U_n = \tfrac{1}{2}d_{0,n+1} = 5^{n+\frac{1}{4}}c_{0,n+1}\left\{\sum_{t=0}^{T-1}(-1)^t\frac{(t+\frac{1}{2})\dots(-t+\frac{3}{2})2^{-2t}}{t!(n-\frac{1}{2})\dots(n-t+\frac{1}{2})}+O(n^{-T})\right\}$$

and from (17) of (1) that

$$H_n = \tfrac{1}{24}d_{1,n+2}+O(5^{\frac{1}{2}n})$$

$$= \tfrac{1}{3}5^{n+\frac{1}{4}}c_{1,n+2}\left\{\sum_{t=0}^{T-1}(-1)^t\frac{(t+\frac{3}{2})\dots(-t+\frac{5}{2})2^{-2t}}{t!(n-\frac{1}{2})\dots(n-t+\frac{1}{2})}+O(n^{-T})\right\}.$$

These are the asymptotic expansions of U_n and H_n for large n. Harary and Read (1) show that

$$U_n/(H_n(n+2))\to 2 \tag{2}$$

as $n\to\infty$. From the above, the left hand side of (2) is $2+O(1/n)$. Using the first two terms in our asymptotic expansions for U_n and H_n, we find that

$$U_n/\{H_n(n+2{\cdot}75)\} = 2+O(n^{-2}). \tag{3}$$

Taking Harary and Read's example (and using their numerical results) we have

$$U_{40}/(42{\cdot}75H_{40}) = 2{\cdot}00281,$$

which verifies (3).

REFERENCES

(1) F. Harary and R. C. Read, The enumeration of tree-like polyhexes, *Proc. Edinburgh Math. Soc.* (2) **17** (1970), 1-13.

(2) E. C. Titchmarsh, *Theory of functions* 2nd edition (Oxford, 1939), 224.

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