J. Austral. Math. Soc. (Series A) 39 (1985), 194–215

REDUCIBLE $2-(11,5,4)$ AND $3-(12,6,4)$ DESIGNS

D. R. BREACH AND A. R. THOMPSON

(Received 15 October 1983)

Communicated by W. D. Wallis

Abstract

One way of constructing a $2-(11,5,4)$ design is to take together all the blocks of two $2-(11,5,2)$ designs having no blocks in common. We show that 58 non-isomorphic $2-(11,5,4)$ designs can be so made and that through extensions by complementation these can be packaged into just 12 non-isomorphic reducible $3-(12,6,4)$ designs.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 05 B 05.

1. Introduction

(i) The $t-(v,k,\lambda)$ designs

The *blocks* of $t-(v,k,\lambda)$ *design* are subsets of size k taken from a set of v *points* (or *varieties*, or *symbols*). The blocks between them contain each t-subset of the v points exactly λ times. *Repetition of blocks is not permitted.* A t-design is *trivial* it it has all the possible $\{\binom{v}{k}\}$ k-sets as blocks.

If s is a natural number less than t, then a t-design is also an s-design. In this paper only 2-designs and 3-designs are considered. Take an i-subset of the v points and let λ_i be the number of blocks of a design which contain this i-subset. Then $\lambda_0 = b$, the *number of blocks*; and $\lambda_1 = r$, the *replication number*, or number of times each point occurs in the design. For 2-designs and 3-designs the λ_i are given as follows,

$2-(v,k,\lambda)$	$3-(v,k,\lambda)$
$\lambda_0 = b = \dfrac{v(v-1)\lambda}{k(k-1)}$,	$\lambda_0 = b = \dfrac{v(v-1)(v-2)\lambda}{k(k-1)(k-2)}$,
$\lambda_1 = r = \dfrac{(v-1)\lambda}{(k-1)}$,	$\lambda_1 = r = \dfrac{(v-1)(v-2)\lambda}{(k-1)(k-2)}$,
$\lambda_2 = \lambda.$	$\lambda_2 = \dfrac{v-2}{k-2}\lambda$,
	$\lambda_3 = \lambda.$

These standard results are obtained by counting the number of incidences of i-tuples in two different ways (see for example [3], page 2).

A permutation of the point labels of a t-design D which maps blocks onto blocks is called an *automorphism* of D. The set of all automorphisms of D under successive applications forms the *automorphism group* of D, Aut D. If Aut D maps any ordered i-tuple of points onto any other i-tuple then Aut D and D are said to be *i-transitive*. 'One-transitive' is usually simplified to 'transitive'.

Given two t-designs with the same parameters there may exist a permutation of the point labels which maps all the blocks of one design onto all the blocks of one design onto all the blocks of the other. Then the two designs are *isomorphic* to each other. More precisely let $A = [a_{ij}]$ be the *incidence matrix* of a $t-(v,k,\lambda)$ design. Then $a_{ij} = 1$ if the ith points is on the jth block and $a_{ij} = 0$ otherwise. Two designs with incidence matrices A and B are isomorphic if there are permutation matrices P and Q such that $PAQ = B$. In the case where $A = B$ the effects of P and Q provide an automorphism of A and hence correspond to an automorphism of the design of which A is the incidence matrix.

(ii) Reducible designs

Suppose two $t-(v,k,\lambda)$ designs on the same set of points with the same parameters have no blocks in common. Then taken together the blocks of both designs form a $t-(v,k,2\lambda)$ design. It does not follows that any $t-(v,k,2\lambda)$ design can be decomposed into two $t-(v,k,\lambda)$ designs. If a $t-(v,k,\mu)$ design can be decomposed into $t-(v,k,\lambda)$ designs, with $\lambda < \mu$, then it is said to be *reducible*; otherwise it is *irreducible*. The terms *decompasable* and *non-decomposable* are also used.

Within the family of $2-(2n+1, n, n-1)$ designs those for which n is even are always irreducible. For n odd a $2-(2n+1, n, n-1)$ design is sometimes reducible to two $2-(2n+1, n, \frac{1}{2}n-\frac{1}{2})$ designs, i.e. to two *Hadamard designs*. A basic problem of design theory is that of determining all non-isomorphic designs for a given parameter set. For the $2-(2n+1, n, n-1)$ family it is known that

(i) there is a unique $2-(5,2,1)$ design (which is trivial),
(ii) there is a unique $2-(7,3,2)$ design and this is reducbile,
(iii) there are just eleven $2-(9,4,3)$ designs (Stanton, Mullin and Bate [6], and confirmed by others).

At the time of writing the number of irreducible non-isomorphic $2-(11,5,4)$ designs is not known although the authors have evidence that it exceeds 3000. For the reducible $2-(11,5,4)$ designs we assert that

there are just 58 *reducible* $2-(11,5,4)$ *designs.*

(iii) Extensions to 3-designs

It has been shown by Sprott [5] that any $2-(2n+1, n, \lambda)$ design can be *extended* to a $3-(2n+2, n+1, \lambda)$ design by *complementation.* In this process the same new point x is added to each block of the $2-(2n+1, n, \lambda)$ design and then more blocks (not containing x) are formed by taking the complements of the original blocks with respect to the point set. Thus in the 3-design the blocks are in complementary pairs and the whole design is said to be *self-complementary.* Conversely from a given $3-(2n+1, n+1, \lambda)$ design a $2-(2n+1, n, \lambda)$ design can be made by discarding all those blocks not containing a given point x and then deleting x from the remaining blocks. Relative to the 3-design the 2-design so formed is called a *restriction* on x. Thus *every* $2-(2n+1, n, \lambda)$ *design can be extended to a unique self-complementary* $3-(2n+2, n+1, \lambda)$ *design*. A 3-*design, however, may have several non-isomorphic restrictions.*

Dembowski [4] showed that if $\lambda = \frac{1}{2}(n-1)$, so the 2-design is a Hadamard design, then extension by complementation is the only way of extending to a 3-design. If relative to n, λ has larger values, then other methods of extension may be available. These need not lead to self-complementary 3-designs (Breach [1], [2]). By combining Dembowski's result with the notion of reducibility we have that

every reducible $2-(11,5,4)$ *design can be extended to a unique self-complementary reducible* $3-(12,6,4)$ *design.*

This provides a way of packaging $2-(11,5,4)$ designs up to twelve at a time by providing a minimal set of $3-(12,6,4)$ designs.

It will be shown that

there are just 12 *reducible* $3-(12,6,4)$ *designs.*

The non-isomorphic restrictions arising from these 12 3-designs are the 58 reducible $2-(11,5,4)$ designs.

https://doi.org/10.1017/S1446788700022473 Published online by Cambridge University Press

2. Block intersection numbers

(i) The block intersection equations

For a given block B of a $t-(v, k, \lambda)$ design, let n_i be the number of blocks having exactly i of the k points contained in B. Then $n_k = 1$ because there are no repeated blocks. Now count flags $(A : X)$ where A is a block and X is a j-set such that $X \subseteq A$ and $X \subseteq B$ (so X may be the null set). Then

$$\sum_{i=j}^{k} \binom{i}{j} n_i = \binom{k}{j} \lambda_j, \qquad 0 \leqslant j \leqslant t.$$

Thus we have a set of $(t+1)$ Diophantine equations for the block intersection numbers $n_i \geqslant 0$. Each solution set $(n_0, n_1, \ldots, n_k)$ corresponds to a *block type*. The blocks in a t-design need not all be of the same type and in fact a typical t-design involves blocks of several different types. A consideration of the number of blocks of each type present in each of two designs having the same parameters often provides a quick way of showing that the two designs are not isomorphic.

(ii) Block types for $2-(11,5,4)$ and $3-(12,6,4)$ designs

The existence of solution sets $(n_0, n_1, \ldots, n_k)$ for the block intersection numbers is merely a necessary condition for the existence of the corresponding design. Further considerations may show that blocks of a specified type cannot exist. For the designs of current interest the possible types are listed in Table I.

TABLE I

Design	Parameters				Intersection Numbers							Type
	λ_0	λ_1	λ_2	λ_3	n_0	n_1	n_2	n_3	n_4	n_5	n_6	
$2-(11,5,4)$	22	10	4	–	1	0	15	5	0	1	–	A
					0	3	12	6	0	1	–	B
					0	2	15	3	1	1	–	C
					0	1	18	0	2	1	–	D
$3-(12,6,4)$	44	22	10	4	1	1	5	30	5	1	1	AC
					1	0	9	24	9	0	1	B
					1	2	1	36	1	2	1	E
					0	4	3	28	8	0	1	F
$2-(11,5,2)$	11	5	2	–	0	0	10	0	0	1	–	–
$3-(12,6,2)$	22	11	5	2	1	0	0	20	0	0	1	–

3. Lemmas on the block types

(i) LEMMA. *A* $2-(11,5,4)$ *design cannot have blocks of type D.*

PROOF. In a $2-(11,5,4)$ design let N_4 be the number of occurrences of a given set of four points. Let N_i, $0 \leqslant i \leqslant 3$, be the number of occurrences of any i-subset of the set of four points. Let α be the number of blocks containing just three of the given four points. Then by the principle of inclusion and exclusion

$$\begin{aligned} N_0 &= b - N_1 + N_2 - N_3 + N_4 = \lambda_0 - 4\lambda_1 + 6\lambda_2 - N_3 + N_4 \\ &= 6 - (\alpha + 4N_4) + N_4 = 6 - \alpha - 3N_4. \end{aligned}$$

Since α, N_0, $N_4 \geqslant 0$ we have $N_4 \leqslant 2$. If $N_4 = 2$ then $\alpha = 0$. Now if a block (of five points) intersects two other blocks in four points then the three blocks must have a triple of points in common and we have a situation in which $N_4 \geqslant 2$ and $\alpha = 1$. Therefore D-type blocks cannot exist.

(ii) LEMMA. *If a block of a* $2-(2k+1, k, \lambda)$ *design is of type* $(n_0, n_1, \ldots, n_k)$ *then the corresponding block in the* $3-(2k+2, k+1, \lambda)$ *design obtained by complementation is of type* $(m_0, m_1, \ldots, m_{k+1})$ *where* $m_i = n_{k-i} + n_{i-1}$ *and* $n_{-1} = 0$. *That is, the sets of block intersection numbers for self-complementary* $3-(2k+2, k+1, \lambda)$ *designs are palindromic.*

PROOF. If in the 2-design two blocks intersect in i points, then in the 3-design obtained by complementation the corresponding blocks intersect in $(i+1)$ points. If block A of the 3-design intersects block B of the 3-design in j points then A intersects the complement of B in $k+1-j$ points. These two observations taken together produce the lemma.

(iii) LEMMA. *If a* $3-(12,6,4)$ *design is formed from a* $2-(11,5,4)$ *design by complementation then both A-type and C-type blocks from the* 2*-design correspond to AC type blocks of the* 3*-design; B-type blocks in the* 2*-design produce B-type blocks in the* 3*-design.*

PROOF. This follows from the previous lemma applied to the appropriate sets of intersection numbers.

(iv) LEMMA. *A self-complementary* $3-(12,6,4)$ *design cannot contain E-type blocks.*

https://doi.org/10.1017/S1446788700022473 Published online by Cambridge University Press

PROOF. Suppose the 3-design contains an E-type block. Then by the previous lemma a restriction on a point of this block canot lead to blocks of type A, B or C in the resulting 2-design. Therefore it must lead to D-type blocks. But a previous lemma denies the existence of such blocks.

4. All $3-(12,6,4)$ designs are self-complementary

(i) We establish that a $3-(12,6,4)$ design cannot have blocks for which $n_0 = 0$. This result is embodied in the

THEOREM. *No* $3-(12,6,4)$ *design can contain a block of type F.*

(ii) PROOF. Suppose a $3-(12,6,4)$ design does have a block $[abcdef]$ of type $F(0,4,3,28,8,0,1)$. Since $[abcdef]$ does not have 5 points in common with any block, a restriction on any of its points can only yield blocks of types $A(1,0,15,5,0,1)$ or $B(0,3,12,6,0,1)$ in the resulting $2-(11,5,4)$ design.

(iii) If a block intersects $[abcdef]$ in just one point x then a restriction on x leads to an A-type block, with $n_0 = 1$, in the $2-(11,5,4)$ design. Therefore the 4 blocks intersecting $[abcdef]$ in just one point each do so in a different point. These four blocks must have the structure

$$\begin{array}{ll} a\,.\,.\,.\,.\,. & \\ b\,.\,.\,.\,.\,. & n_1 = 4 \\ c\,.\,.\,.\,.\,. & \\ d\,.\,.\,.\,.\,. & \end{array}$$

where the dots represent numbers from $\{1,2,3,4,5,6\}$ (so the 12 points of the design fall into two classes of 6; numbers and letters).

(iv) Restrictions on any of a, b, c, d produce blocks of type A with $n_1 = 0$ in the $2-(11,5,4)$ design. Therefore none of a, b, c, d can occur on a block that intersects $[abcdef]$ in just two points. These three blocks have the structure

$$\begin{array}{ll} ef\,.\,.\,.\,. & \\ ef\,.\,.\,.\,. & n_2 = 3 \\ ef\,.\,.\,.\,. & \end{array}$$

(v) The pair ef must occur 10 times in the 3-design. Four of these occurrences are already accounted for. The remaining 6 are in blocks containing three or four letters. Triples efx where $x \in \{a,b,c\}$ occur 16 times in the 3-design. The block $[abcdef]$ accounts for 4 of these leaving 12 to lie in 6 blocks containing ef.

Therefore the blocks intersecting $[abcdef]$ in exactly 4 points have the structure

$$\begin{array}{ll} ef**\,.\,. & \\ ef**\,.\,. & \\ ef**\,.\,. & \\ ef**\,.\,. & n_4 = 8 \\ ef**\,.\,. & \\ abcd\,.\,. & \\ abcd\,.\,. & \end{array}$$

where the asterisks represent letters from $\{a, b, c, d\}$.

(vi) The 28 blocks containing just three letters are of three kinds

$$e**\ldots \quad (12), \qquad f**\ldots \quad (12) \qquad \text{and} \qquad ***\ldots \quad (4) \qquad n_3 = 28.$$

(vii) Having made a skeleton on the six letters let us now account for the 22 occurrences of a typical number, 1 say. We count all triples 1** and all pairs 1*. Further, in counting the triples we take particular note of the occurrences of 1ef. Let

u = # blocks with $1ef$ and no other letters,
v = # blocks with $1ef$ and two other letters,
x = # blocks $abcd$ and 1,
y = # blocks with three letters and 1,
z = # blocks with one letter and 1.

Then by counting in turn triples with 1 and two letters, pairs with 1 and a letter, triples $1ef$, and occurrences of 1, we have

$$\begin{aligned} u + 6v + 6x + 3y &= 60, \\ 2u + 4v + 4x + 3y + z &= 60, \\ u + v &= 4, \\ x + y + z &= 18. \end{aligned}$$

These equations imply $3x = 4u - 6$ from which $u > 2$ and u divides 3. But $u \leqslant 3$. Therefore $u = 3$. Thus the blocks of

$$\begin{array}{ll} ef\,.\,.\,.\,. & \\ ef\,.\,.\,.\,. & n_2 = 3 \\ ef\,.\,.\,.\,. & \end{array}$$

contain the number 1 thrice. Thus they contain *any* number thrice. But there are six numbers and these blocks do not provide enough spaces to contain all six thrice. Therefore a block of type F cannot exist in a $3 - (12, 6, 4)$ design. Consequently all $3 - (12, 6, 4)$ designs are self-complementary.

5. Constructing a reducible $2-(11,5,4)$ design: general method of attack

(i) Properties of the $2-(11,5,2)$ design

Up to isomorphisms there is only one $2-(11,5,2)$ design. This well-known design is symmetric, i.e. $b = v$, and is also a Hadamard design. It is usually presented in a cyclic form on the numbers $0, 1, \ldots, 10$ by giving a block [13459] containing the quadratic residues modulo 11. From this all blocks are generated through the point transformation $x \to x + 1 \pmod{11}$. We shall call this particular model of the $2-(11,5,2)$ design the design **D**. Any block of **D** intersects any other in exactly two points. The design **D** is 2-transitive with an automorphism group, Aut **D**, of order 660. This group contains elements such as (13459)(670108)(2) which fix a block [13459] of **D**. Another such element of Aut **D** is (1)(3)(459)(2710)(068) which not only fixes the block [13459] but also the block [101237]. Such group elements are useful in the construction of equivalence classes when **D** is embedded in a $2-(11,5,4)$ design.

Note that **D** can be extended by complementation to a unique $3-(12,6,2)$ design and that this is the only extension possible. Since **D** is unique all restrictions on the $3-(12,6,2)$ design are isomorphic and this 3-design must be 3-transitive.

(ii) The standard presentation of a reducible $2-(11,5,4)$ design

A reducible $2-(11,5,4)$ must contain **D** and another model of the $2-(11,5,2)$ design which will be called **D***. By convention the 11 blocks of **D** will always be presented on the left-hand side of the page while the 11 blocks of **D*** will be on the right. Thus the standard skeleton of a reducible skeleton of a reducible $2-(11,5,4)$ design is

1	3	4	5	9	.....
2	4	5	6	10	.....
3	5	6	7	0	.....
4	6	7	8	1	.....
5	7	8	9	2	.....
6	8	9	10	3	.....
7	9	10	0	4	.....
8	10	0	1	5	.....
9	0	1	2	6	.....
10	1	2	3	7	.....
0	2	3	4	8	.....

D $2-(11,5,2)$ **D*** $2-(11,5,2)$.

Now although Aut **D** and Aut **D*** as abstract groups are isomorphic, their representations as permutation groups may or may not have elements in common. Thus a permutation of $0, 1, \ldots, 10$ fixing **D** need not fix **D***. Moreover, it may fix

https://doi.org/10.1017/S1446788700022473 Published online by Cambridge University Press

patches or fragments of $\mathbf{D}^*$. In general we shall use the points from the first block of $\mathbf{D}$, [1 3 4 5 9] to fill in patches of $\mathbf{D}^*$; and will use all the elements of Aut $\mathbf{D}$ which fix its first block to keep the number of equivalence classes of partially completed skeletons for $\mathbf{D}^*$ as small as possible. After all the permutations fixing [1 3 4 5 9] and the patches of $\mathbf{D}^*$ are used up there remains the problem of determining the number of inequivalent ways that the symbols 0, 2, 6, 7, 8, 10 can be inserted into the skeleton.

(iii) The sieve of block types

A $2-(11,5,4)$ design can have blocks of any or all of types A, B, C. We start by determining all designs with at least one block of a specified type B say. Then having filtered these designs from the general pool we next determine all those designs with at least one C-type block but no B-type blocks. Thirdly we then determine all designs with no B-type or C-type blocks and therefore having all A-type blocks. The order in which the various types of block are sieved is somewhat arbitrary in principle but in practice is guided by preliminary investigations and the desirability of catching large numbers of designs in the first sieve.

(iv) Packaging into $3-(12,6,4)$ designs

To check that the census of reducible $2-(11,5,4)$ designs is complete each of them can be extended by complementation to a $3-(12,6,4)$ design. In general each such 3-design will have several non-isomorphic 2-designs as its restrictions. The totality of non-isomorphic reducible $3-(12,6,4)$ designs produced this way should exhaust the supply of reducible $2-(11,5,4)$ designs and furthermore when each such 3-design is analysed into its restrictions no new reducible $2-(11,5,4)$ designs should appear. Thus a necessary condition for the completeness of the list of reducible $2-(11,5,4)$ designs is that the packaging and unpackaging into and out of $3-(12,6,4)$ designs should be neat i.e. all the $2-(11,5,4)$ designs must be exactly accounted for.

(v) The number of A, B or C-type blocks is even or zero

For every block of type A in a $2-(11,5,4)$ design there is a unique block which does not intersect it and which must also be of type A. For each C-type block there is a unique block, also of C-type, intersecting it in four points. Thus A and C-type blocks each occur in pairs if at all. But the total number of blocks is 22. Therefore the number of B-type blocks must be even. However there does not seem to be any obvious way of pairing the B-type blocks.

6. The reducible $2-(11,5,4)$ designs with two or more B-type blocks

(i) Laying out the skeleton

Suppose that the first block, [1 3 4 5 9], of $\mathbf{D}$ in its standard form, is a B-type block in a $2-(11,5,4)$ design. Then this block intersects all other blocks of $\mathbf{D}$ in exactly two points. Consequently it must intersect three blocks of **D*** in one point, two blocks in two points and six blocks in three points. The blocks of **D*** must intersect amongst themselves in exactly two points. Thus three blocks of **D*** must have the shape

$$[1\ldots\ldots], [4\ldots\ldots], [5\ldots\ldots],$$

where the dots represent members of the set $\{0,2,6,7,8,10\}$. Aut **D** is 2-transitive so any pair of points can be deleted from [1 3 4 5 9] to yield a triple to place in these blocks. However having chosen 3,9 from here on we can use only the sub-group of Aut **D** which fixes $\{3,9\}$.

The symbol 5 must occur on four more blocks of **D***. Let x and y be the number of these blocks containing two and one respectively of the symbols 1, 3, 4, 9. Then by counting pairs containing 5 and occurrences of 5 we have $2x+y=8$ and $x+y=4$. Thus $x=4$ and $y=0$. The same results hold for the symbols 1 and 4. Therefore the two blocks of **D*** that intersect [1 3 4 5 9] in exactly two points both do so in the pair 39. The rest of the skeleton **D*** is then forced as far as 1, 3, 4, 5, 9 are concerned. This for our proposed $2-(11,5,4)$ design, gives the skeleton

D					D*					
1	3	4	5	9	1	.	.	.	.	(i)
2	4	5	6	10	4	.	.	.	.	(ii)
3	5	6	7	0	5	.	.	.	.	(iii)
4	6	7	8	1	3	9	.	.	.	(iv)
5	7	8	9	2	3	9	.	.	.	(v)
6	8	9	10	3	1	4	3	.	.	
7	9	10	0	4	4	5	3	.	.	
8	10	0	1	5	5	1	3	.	.	
9	0	1	2	6	1	4	9	.	.	
10	1	2	3	7	4	5	9	.	.	
0	2	3	4	8	5	1	9	.	.	

The elements of Aut **D** which leave this skeleton fixed are the identity and

$$\alpha = (1)\,(2)\,(8)\,(3\;9)\,(4\;5)\,(6\;10)\,(0\;7),$$
$$\beta = (0)\,(4)\,(6)\,(3\;8)\,(1\;5)\,(2\;7)\,(8\;10),$$
$$\gamma = (3)\,(9)\,(1\;5\;4)\,(6\;8\;10)\,(2\;7\;0),$$
$$\delta = (3)\,(9)\,(1\;4\;5)\,(6\;10\;8)\,(2\;0\;7),$$
$$\varepsilon = (7)\,(10)\,(5)\,(3\;9)\,(1\;4)\,(6\;8)\,(0\;2).$$

https://doi.org/10.1017/S1446788700022473 Published online by Cambridge University Press

(ii) Rectangular tops and triangular tops

The skeleton is to be completed by using the symbols 0, 2, 6, 7, 8, 10.The two blocks (iv) and (v) already intersect in two points so each of the remaining six points must occur exactly once in these two blocks. But **D** contains the block [3 9 6 8 10] so, to avoid repeated blocks, block (iv) must contain one symbol of the triple 6 8 10 and (v) must contain the other two. Then (v) must intersect [3 9 6 8 10] in four points and so must be a block of type C. Thus we have the lemma

> *every reducible* $2-(11,5,4)$ *design containing B-type blocks must also contain C-type blocks.*

The three blocks (i), (ii), (iii) mutually intersect in exactly two points so each of 0, 2, 6, 7, 8, 10 must occur exactly twice in these three blocks. Let a, b, $c \in \{0,2,7\}$ and d, e, $f \in \{6,8,10\}$ with all these letters distinct. Then in $\mathbf{D}^*$ there are two possible structures for the blocks (i), ..., (v); namely either

$$\begin{array}{cccccl} 1 & a & b & d & e & \text{(i)} \\ 4 & b & c & e & f & \text{(ii)} \\ 5 & c & a & f & d & \text{(iii)} \\ 3 & 9 & a & f & e & \text{(iv)} \\ 3 & 9 & b & c & d & \text{(v)} \end{array} \qquad \begin{array}{cccccl} 1 & b & d & e & f & \text{(i)} \\ 4 & a & c & d & f & \text{(ii)} \\ 5 & a & b & c & e & \text{(iii)} \\ 3 & 9 & a & f & e & \text{(iv)} \\ 3 & 9 & b & c & d & \text{(v)} \end{array}$$

Rectangular Top Triangular Top

(iii) Completing the rectangular top

For this case there are six ways of placing a, b, c into the blocks (i), (ii), (iii).

Then there are three ways of selecting a from $\{0,2,7\}$ to place in block (iv).

The elements α, β, γ, δ, ε of Aut **D** put these eighteen possibilities into four equivalence classes with typical members

1	0	2	.	.	1	0	7	.	.	1	0	7	.	.	1	0	2	.	.	(i)
4	2	7	.	.	4	7	2	.	.	4	2	0	.	.	4	7	0	.	.	(ii)
5	7	0	.	.	5	2	0	.	.	5	7	2	.	.	5	2	7	.	.	(iii)
3	9	0	.	.	3	9	0	.	.	3	9	0	.	.	3	9	0	.	.	(iv)
3	9	2	7	.	3	9	2	7	.	3	9	2	7	.	3	9	2	7	.	(v)

For each of these, once d has been chosen from $\{6,8,10\}$ to complete block (v) there are just two ways of completing the other four blocks. for the first two structures the element β reduces the resulting six possibilities to four. Hence the rectangular top can be completed in 20 non-equivalent ways.

For each completed rectangular top the remaining six blocks of $\mathbf{D}^*$ can be completed in just two ways according to the patterns

$$\begin{array}{ccccccc}
1 & 4 & 3 & & & f & d \\
4 & 5 & 3 & a & b & & \\
5 & 1 & 3 & c & & & e \\
1 & 4 & 9 & a & c & & \\
4 & 5 & 9 & & & d & e \\
5 & 1 & 9 & b & & & f
\end{array}
\quad \text{and} \quad
\begin{array}{ccccccc}
1 & 4 & 3 & a & c & & \\
4 & 5 & 3 & & & e & d \\
5 & 1 & 3 & b & & & f \\
1 & 4 & 9 & & & d & f \\
4 & 5 & 9 & a & b & & \\
5 & 1 & 9 & c & & & e.
\end{array}$$

Thus we obtain 40 reducible $2-(11,5,4)$ designs containing B-type blocks associated with rectangular tops. It is to be expected that isomorphic pairs will occur. In particular the possibility that there are point permutations mapping blocks of $\mathbf{D}$ onto $\mathbf{D}^*$ and vice versa has been ignored.

(iv) Completing the triangular top

Here the fragments abc, ac and b can be assigned to blocks (i), (ii) and (iii) in six ways. Then a, b, c can be assigned values from $\{0,2,7\}$ in six ways. Thus there are 36 possible patterns which however are assigned to six equivalence classes by the elements α, β, γ, δ, ε of $\operatorname{Aut}\mathbf{D}$. Representatives of these classes are

$$\begin{array}{ccccc}
1 & 2 & . & . & . \\
4 & 0 & 7 & . & . \\
5 & 0 & 2 & 7 & . \\
3 & 9 & 0 & . & . \\
3 & 9 & 2 & 7 & .
\end{array}
\qquad
\begin{array}{ccccc}
1 & 7 & . & . & . \\
4 & 0 & 2 & . & . \\
5 & 0 & 2 & 7 & . \\
3 & 9 & 0 & . & . \\
3 & 9 & 2 & 7 & .
\end{array}
\qquad
\begin{array}{ccccc}
1 & 0 & . & . & . \\
4 & 2 & 7 & . & . \\
5 & 0 & 2 & 7 & . \\
3 & 9 & 2 & . & . \\
3 & 9 & 0 & 7 & .
\end{array}
\qquad
\begin{array}{ccccc}
1 & 7 & . & . & . \\
4 & 0 & 2 & . & . \\
5 & 0 & 2 & 7 & . \\
3 & 9 & 2 & . & . \\
3 & 9 & 0 & 7 & .
\end{array}
\qquad
\begin{array}{l}
\text{(i)} \\ \text{(ii)} \\ \text{(iii)} \\ \text{(iv)} \\ \text{(v)}
\end{array}$$

$$\begin{array}{ccccc}
1 & 2 & . & . & . \\
4 & 0 & 7 & . & . \\
5 & 0 & 2 & 7 & . \\
3 & 9 & 7 & . & . \\
3 & 9 & 0 & 2 & .
\end{array}
\qquad
\begin{array}{ccccc}
1 & 0 & . & . & . \\
4 & 2 & 7 & . & . \\
5 & 0 & 2 & 7 & . \\
3 & 9 & 7 & . & . \\
3 & 9 & 0 & 2 & .
\end{array}
\qquad
\begin{array}{l}
\text{(i)} \\ \text{(ii)} \\ \text{(iii)} \\ \text{(iv)} \\ \text{(v).}
\end{array}$$

For each of these there are six ways of assigning values to the triple def from the set $\{0,2,7\}$ to complete the blocks (i), (ii), (iii), (iv), (v). There are then two ways of completing the remaining six blocks of $\mathbf{D}^*$ as in the rectangular top case. Thus we have 72 designs to examine for isomorphisms.

(v) The elimination of isomorphs

We note in passing that a reducible $2-(11,5,4)$ design with a B-type block having a rectangular top can be used to create a design with a B-type block having a triangular top. This is done by taking the rectangular top case and extending it with a new point x to a $3-(12,6,4)$ design. If a restriction on a is then made the result will be a $2-(11,5,4)$ design with a triangular type B-block. As a $2-(11,5,4)$ design can have both kinds of B-type block this extension-restriction process is not a good sorting mechanism. Nevertheless its existence suggests that there are many isomorphs among the 112 $2-(11,5,4)$ designs under current examination.

In practice the 112 designs can be put on paper very quickly with the help of a xerox machine and some coloured marker pens. The assignation of the blocks of each design to the types A, B, C can be done manually or by machine. The manual process is simplified and accelerated by the frequent re-occurrence of certain pairs of blocks throughout the collection of designs. A coarse sorting according to the number of blocks of each type puts the 112 designs in 16 equivalence classes. No two designs from different classes can be isomorphic. Within each class either a permutation mapping one design to another was found or it was shown that no such permutation exists. The 112 designs with B-type blocks reduce to 53 non-isomorphic designs.

The tabulation of these 53 designs will be postponed until all the reducible $2-(11,5,4)$ designs have been determined thus allowing a concise presentation in a single table.

7. Reducible $2-(11,5,4)$ designs with C-type blocks but with no B-type blocks

(i) The two possible skeletons

In the construction of the reducible $2-(11,5,4)$ designs in this section any partially completed design may be discarded if a B-type block appears. The designs with B-type blocks have already been accounted for in Section 6.

The first block [1 3 4 5 9] of $\mathbf{D}$, the standard $2-(11,5,2)$ design, is now required to be a block of type $C(0,2,15,3,1,1)$ of a $2-(11,5,4)$ design. Thus just one block of $\mathbf{D}^*$ must intersect it in four points. Since $\operatorname{Aut}\mathbf{D}$ is transitive we can take this block to be [1 3 4 5 .]. A pair count determines the unique placing of the symbol 9 in the blocks of $\mathbf{D}^*$. There are twelve ways to insert 1, 3, 4, 5.

Now the subgroup of $\operatorname{Aut}\mathbf{D}$ which fixes the block [1 3 4 5 9] and the set $\{1,3,4,5\}$ is of order 12 and is generated by

$$\rho=(1\,3)(4\,5)(2\,10)(0\,8),$$
$$\sigma=(4\,1)(3\,5)(2\,10)(6\,7),\qquad \tau=(3\,4\,5)(6\,0\,2)(10\,7\,8).$$

Under the action of this subgroup the twelve skeletons for $\mathbf{D}^*$ are put into two equivalence classes represented by

Case 1

1	3	4	5	a		(i)
1	3	9	d	f		
1	4	9	b	e		
3	5	9	b	c		
1	5	c	d	e		
3	4	c	e	f		
4	5	b	d	f		
4	9	a	c	d		(ii)
5	9	a	e	f		(iii)
1	a	b	c	f		(iv)
3	a	b	d	e		(v)

$\mathbf{D}^*$

and Case 2

1	3	4	5	a
1	3	9	b	e
1	4	9	d	f
4	5	9	b	c
1	5	c	d	e
3	4	c	e	f
3	5	b	d	f
3	9	a	c	d
5	9	a	e	f
1	a	b	c	f
4	a	b	d	e

$\mathbf{D}^*$

(ii) The completion of Case 1

Suppose block (i) is completed by the symbol $a \in \{0,2,6,7,8,10\}$. Then for a correct pairwise balance a must also occur in the last four blocks (ii), (iii), (iv), (v). Now element ρ of $\operatorname{Aut}\mathbf{D}$ fixes this skeleton so there are just four inequivalent choices for a. Then the completion of blocks (iv) and (v), avoiding a repeated triple, requires the repetition of a symbol B which can be chosen in five ways. For each of these there are 24 ways of completing blocks (ii), (iii), (iv), (v) with c, d, e, f. The rest of the design $\mathbf{D}^*$ is then forced. This gives $4 \times 5 \times 24 = 480$ cases. However to avoid repetition of blocks of $\mathbf{D}$ the conditions

$$\{c,d,e\} \neq \{8,10,0\}, \qquad \{c,e,f\} \neq \{0,2,8\}, \qquad \{b,d,f\} \neq \{2,6,10\},$$
$$\{a,c,d\} \neq \{7,10,0\}, \qquad \{a,e,f\} \neq \{2,7,8\}$$

must hold and these exclude many of the 480 cases. The requirement that B-type blocks do not occur will remove more. The systematic permutation of a, b, c, d, e, f under the set conditions can be done by hand or computer. The latter course was followed.

(iii) The completion of Case 2.

In this case the element σ fixes the skeleton without the letters. The procedure then follows that for Case 1. To avoid repeated blocks we must have

$$\{c,d,e\} \neq \{0,10,8\}, \qquad \{b,d,f\} \neq \{6,7,0\}, \qquad \{a,e,f\} \neq \{7,10,0\},$$
$$\{c,e,f\} \neq \{0,2,8\}, \qquad \{a,c,d\} \neq \{6,8,10\}.$$

(iv) The elimination of isomorphs

The systematic examination of Cases 1 and 2 yielded 28 acceptable designs which formed two equivalence classes according to the numbers of blocks of type A or C in each. Permutation techniques then gave a further reduction in numbers and only four new non-isomorphic designs were created.

8. Reducible $2-(11,5,4)$ designs with A-type blocks only

(i) A skeleton

If a reducible $2-(11,5,4)$ design is to contain A-type blocks only then the blocks must occur in disjoint pairs. Each such pair defines a unique symbol not in either member. Since there are eleven disjoint pairs of blocks and $r = 10$ it follows that each of the eleven symbols must be omitted just once from a disjoint pair of blocks.

As Aut **D** is transitive with the first block [1 3 4 5 9] of **D** we can pair [2 6 7 8 10] as the first block of **D***. Then recalling that **D*** is a $2-(11,5,2)$ design whose blocks mutually intersect in exactly two points we can construct a skeleton for the $2-(11,5,4)$ design by listing the blocks of **D** and **D*** in disjoint pairs, thus:

D					D*					
1	3	4	5	9	2	6	7	8	10	(i)
2	4	5	6	10	.	7	8	.	0	(ii)
3	5	6	7	0	.	.	.	.	.	(iii)
4	6	7	8	1	.	.	10	0	2	
5	7	8	9	2	6	10	0	.	.	
6	8	9	10	3	7	0	.	2	.	
7	9	10	0	4	.	.	.	.	.	
8	10	0	1	5	.	.	.	.	.	
9	0	1	2	6	.	.	.	.	.	
10	1	2	3	7	0	.	.	6	8	
0	2	3	4	8	.	.	.	.	.	

(ii) The completion

The non-trivial element of Aut **D** which fixes the skeleton is $\phi = (2\,6)(7\,8)(1\,9)(4\,5)$. Now block (ii) must be completed by a pair a, b from $\{1,3,9\}$. But under ϕ 1 and 9 are equivalent so either $\{a,b\} = \{3,9\}$ or $\{a,b\} = \{1,9\}$. If $\{a,b\} = \{1,9\}$ then block (iii) must be completed subject to the non-appearance

https://doi.org/10.1017/S1446788700022473 Published online by Cambridge University Press

of triples $\{1, 8, 9\}$ and $\{2, 8, 10\}$. The only possibility is [1 2 4 9 10] which cannot be allowed since the pair 2, 10 would appear three times in $\mathbf{D}^*$. Thus $\{a, b\} = \{3, 9\}$. But then the rest of $\mathbf{D}^*$ is forced. (Ask where the pairs 1, 0 must be.) The resultant design is generated cyclically under the action $x \to x + 1 \pmod{11}$ from the two starter blocks [1 3 4 5 9] and [2 6 7 8 10] containing the quadratic and non-quadratic residues, mod 11, respectively.

9. A catalogue of the reducible 2 − (11, 5, 4) designs

(i) The seven basic patterns

From Sections 6, 7 and 8 we have 53, 4 and 1 non-isomorphic reducible 2 − (11, 5, 4) designs respectively. We now present a systematic listing of these 58 designs, with a coded description of each that enables a specimen of it it be made if desired.

Each design contains the 11 blocks of the standard 2 − (11, 5, 2) design $\mathbf{D}$. In addition there are the 11 blocks of $\mathbf{D}^*$ formed on seven different patterns:

I					II					III					IV				
1	*a*	*b*	*d*	*e*	1	*a*	*b*	*d*	*e*	1	*b*	*d*	*e*	*f*	1	*b*	*d*	*e*	*f*
4	*b*	*c*	*e*	*f*	4	*b*	*c*	*e*	*f*	4	*a*	*c*	*f*	*d*	4	*a*	*c*	*f*	*d*
5	*c*	*a*	*f*	*d*	5	*c*	*a*	*f*	*d*	5	*a*	*b*	*c*	*e*	5	*a*	*b*	*c*	*e*
3	9	*a*	*f*	*e*	3	9	*a*	*f*	*e*	3	9	*a*	*e*	*f*	3	9	*a*	*e*	*f*
3	9	*b*	*c*	*d*	3	9	*b*	*c*	*d*	3	9	*b*	*c*	*d*	3	9	*b*	*c*	*d*
1	4	3	*f*	*d*	1	4	3	*a*	*c*	1	4	3	*a*	*b*	1	4	3	*c*	*e*
4	5	3	*a*	*d*	4	5	3	*e*	*d*	4	5	3	*d*	*e*	4	5	3	*b*	*f*
5	1	3	*c*	*e*	5	1	3	*b*	*f*	5	1	3	*c*	*f*	5	1	3	*a*	*f*
1	4	9	*a*	*c*	1	4	9	*d*	*f*	1	4	9	*c*	*e*	1	4	9	*a*	*b*
4	5	9	*d*	*e*	4	5	9	*a*	*b*	4	5	9	*b*	*f*	4	5	9	*d*	*e*
5	1	9	*b*	*f*	5	1	9	*c*	*e*	5	1	9	*a*	*d*	5	1	9	*c*	*f*

V					VI					VII				
1	3	4	5	*a*	1	3	4	5	*a*	2	6	7	8	10
1	3	9	*d*	*f*	1	3	9	*b*	*e*	3	7	8	9	0
1	4	9	*b*	*e*	1	4	9	*d*	*f*	4	8	9	10	1
3	5	9	*b*	*c*	4	5	9	*b*	*c*	5	9	10	0	2
1	5	*c*	*d*	*e*	1	5	*c*	*d*	*e*	6	10	0	1	3
3	4	*c*	*e*	*f*	3	4	*c*	*e*	*f*	7	0	1	2	4
4	5	*b*	*d*	*f*	3	5	*b*	*d*	*f*	8	1	2	3	5
4	9	*a*	*c*	*d*	3	9	*a*	*c*	*d*	9	2	3	4	6
5	9	*a*	*e*	*f*	5	9	*a*	*e*	*f*	10	3	4	5	7
1	*a*	*b*	*c*	*f*	1	*a*	*b*	*c*	*f*	0	4	5	6	8
3	*a*	*b*	*d*	*e*	9	*a*	*b*	*d*	*e*	1	5	6	7	9

For each of these, except pattern VII, values from $\{0, 2, 6, 7, 8, 10\}$ are to be assigned to the *ordered* set of points $\{a, b, c, d, e, f\}$. The resulting ordered set prefixed with the pattern number then specifies the reducible $2-(11, 5, 4)$ design (see Table II).

(ii) Comments on Table II

The designs are sorted into 19 classes according to the number of A, B and C-type blocks. For each class we give the number of designs found in that class by the methods of this paper. Then we give the number of non-isomorphic designs for each class followed by a pattern number and coded set from which a model of each distinct design can be made. In the final column a number has been assigned to each of the 58 non-isomorphic designs.

TABLE II. The 58 non-isomorphic reducible $2-(11, 5, 4)$ designs

Class	#Blocks $A\ B\ C$	#Designs found in class	#Distinct Designs	Representative Design Code	Design Number
1	12 2 8	1	1	III $\{7, 0, 2, 10, 8, 6\}$	1
2	10 2 10	2	2	I $\{0, 7, 2, 6, 8, 10\}$	2
				IV $\{2, 7, 0, 8, 6, 10\}$	3
3	6 2 14	8	4	I $\{0, 7, 2, 8, 6, 10\}$	4
				I $\{8, 10, 2, 7, 0, 6\}$	5
				IV $\{0, 2, 7, 6, 8, 10\}$	6
				IV $\{2, 0, 7, 6, 8, 10\}$	7
4	4 2 16	10	6	II $\{8, 6, 2, 7, 0, 10\}$	8
				II $\{10, 6, 2, 7, 0, 8\}$	9
				III $\{2, 0, 7, 10, 8, 6\}$	10
				III $\{7, 2, 0, 10, 6, 8\}$	11
				IV $\{0, 7, 2, 6, 10, 8\}$	12
				IV $\{2, 7, 0, 10, 6, 8\}$	13
5	2 2 18	12	7	I $\{0, 2, 7, 6, 10, 8\}$	14
				II $\{0, 7, 2, 8, 6, 10\}$	15
				II $\{6, 8, 2, 7, 0, 10\}$	16
				II $\{10, 8, 2, 7, 0, 6\}$	17
				III $\{0, 2, 7, 10, 8, 6\}$	18
				III $\{2, 7, 0, 8, 6, 10\}$	19
				IV $\{7, 0, 2, 10, 8, 6\}$	20
6	0 2 20	9	7	I $\{0, 2, 7, 8, 10, 6\}$	21
				I $\{6, 10, 7, 2, 0, 8\}$	22
				III $\{2, 7, 0, 6, 10, 8\}$	23
				III $\{2, 7, 0, 10, 6, 8\}$	24
				IV $\{0, 2, 7, 10, 6, 8\}$	25
				IV $\{0, 7, 2, 10, 8, 6\}$	26
				IV $\{2, 0, 7, 10, 6, 8\}$	27

TABLE II (continued)

Class	♯Blocks $A\ B\ C$	♯Designs found in class	♯Distinct Designs	Representative Design Code	Design Number
7	8 4 10	4	2	III {7, 0, 2, 6, 10, 8}	28
				III {7, 0, 2, 10, 6, 8}	29
8	6 4 12	8	2	II {0, 2, 7, 8, 10, 6}	30
				II {8, 6, 7, 2, 0, 10}	31
9	4 4 14	8	5	I {8, 6, 7, 2, 0, 10}	32
				I {8, 10, 7, 2, 0, 6}	33
				I {10, 8, 2, 7, 0, 6}	34
				III {2, 7, 0, 8, 10, 6}	35
				IV {0, 7, 2, 8, 6, 10}	36
10	2 4 16	14	4	I {0, 7, 2, 8, 10, 6}	37
				II {0, 7, 2, 8, 10, 6}	38
				II {10, 8, 7, 2, 0, 6}	39
				IV {0, 2, 7, 10, 8, 6}	40
11	0 4 18	12	6	I {6, 8, 2, 7, 0, 10}	41
				I {6, 8, 7, 2, 0, 10}	42
				II {6, 10, 2, 7, 0, 8}	43
				II {8, 10, 2, 7, 0, 6}	44
				III {0, 2, 7, 10, 6, 8}	45
				III {0, 7, 2, 8, 6, 10}	46
12	6 6 10	3	1	II {0, 2, 7, 6, 10, 8}	47
13	4 6 12	3	1	II {0, 2, 7, 8, 6, 10}	48
14	2 6 14	11	2	I {10, 8, 7, 2, 0, 6}	49
				II {0, 7, 2, 6, 8, 10}	50
15	0 6 16	3	1	II {8, 10, 7, 2, 0, 6}	51
16	2 10 10	4	2	II {0, 2, 7, 6, 8, 10}	52
				II {0, 7, 2, 6, 10, 8}	53
17	6 0 16	10	1	V {2, 10, 6, 7, 8, 0}	54
18	2 0 20	18	3	V {2, 6, 7, 8, 0, 10}	55
				V {2, 7, 6, 10, 0, 8}	56
				VI {0, 10, 2, 8, 7, 6}	57
19	22 0 0	1	1	VII	58

10. A catalogue of the reducible 3 − (12, 6, 4) designs

(i) Packaging the 2 − (11, 5, 4) designs

Each 2 − (11, 5, 4) design has a unqiue extension to a 3 − (12, 6, 4) design and each such 3-design may contain several non-isomorphic 2-designs. Thus the 58

https://doi.org/10.1017/S1446788700022473 Published online by Cambridge University Press

non-isomorphic reducible $2-(11,5,4)$ designs should pack neatly into $3-(12,6,4)$ designs with no 2-design unaccounted for. Going the other way, restrictions on the $3-(12,6,4)$ designs so formed should not produce any new reducible $2-(11,5,4)$ designs. This gives a check on the listing of the reducible $2-(11,5,4)$ designs.

The extension/restriction process is a mechanical one and, apart from the tedium involved, could be done by hand. However one of us (ART) designed a computer program that not only lists the restrictions on each 3-design but also assigns each block to its correct type. Meanwhile the other of us (DRB) working independently created all the reducible $3-(12,6,4)$ designs *ab initio*. This involved the investigation of about 100 cases more than half of which did not need to be completed leaving 43 designs which we then reduced to 12 non-isomorphic cases. It was found that these coincided with the 12 non-isomorphic reducible $3-(12,6,4)$ designs produced as extension from the reducible $2-(11,5,4)$ designs.

(ii) Listing the non-isomorphic reducible $3-(12,6,4)$ designs

Each of our models of the reducible $3-(12,6,4)$ designs contains the 22 *standard blocks* of the extension of **D**. These are generated from the starter block [1 3 4 5 9 11] by the permutation (0 1 2 3 4 5 6 7 8 9 10) (11) and complementation. The remaining 22 *non-standard blocks* of the 3-design are formed similarly by cycling and complementation from a *key block* of the extension of a **D*** design. Thus if the key block and an associated cycle are given then the whole of the reducible $3-(12,6,4)$ design can be constructed. In Table III we list key blocks and permutations to be used in making the 12 reducible non-isomorphic $3-(12,6,4)$ designs.

TABLE III. Key blocks and permutations for the twelve reducible $3-(12,6,4)$ designs.

Design Number	Key block	Key permutation	Block $\#AC$	types $\#B$
1	[0 1 2 6 8 11]	(0 1 2 3 6 5 7 8 10 9 4)	24	20
2	[0 1 2 8 10 11]	(0 1 2 3 4 7 10 6 5 8 9)	32	12
3	[0 1 6 8 10 11]	(1 2 3 0 7 10 8 6 4 5 9)	36	8
4	[0 1 6 8 10 11]	(1 2 3 0 7 6 8 10 4 5 9)	36	8
5	[0 1 7 8 10 11]	(1 2 3 9 8 10 0 4 7 6 5)	36	8
6	[1 6 7 8 10 11]	(1 2 3 6 0 7 8 10 4 5 9)	40	4
7	[0 1 2 3 4 11]	(1 2 3 6 0 9 8 4 7 10 5)	40	4
8	[0 1 6 8 10 11]	(1 2 3 6 7 10 0 8 5 4 9)	40	4
9	[0 1 6 7 8 11]	(1 2 3 9 6 8 0 5 7 10 4)	40	4
10	[0 1 6 7 8 11]	(1 2 3 6 10 0 8 7 4 5 9)	40	4
11	[1 2 3 4 5 11]	(1 2 3 7 9 6 5 8 0 4 10)	44	0
12	[0 1 3 4 5 11]	(0 1 2 5 10 8 4 6 7 9 3)	44	0

TABLE IV. The 2 – (11, 5, 4) designs associated with the twelve reducible 3 – (12, 6, 4) designs

	Transitivity sets	2 – (11, 5, 4) designs #A	#B	#C	2 – (11, 5, 4) design number
Design 1	{4, 11}	2	10	10	52
	{0, 1, 2, 3, 5, 6, 7, 8, 9, 10, 12}	2	10	10	53
Design 2	{11, 7}	0	6	16	51
	{6, 9}	6	6	10	47
	{0, 5}	4	6	12	48
	{1, 3}	2	6	14	50
	{2, 4, 8, 10}	2	6	14	49
Design 3	{2}	4	4	14	36
	{4}	0	4	18	41
	{0, 9}	0	4	18	43
	{1, 3}	4	4	14	32
	{6, 11}	8	4	10	28
	{8, 10, 7, 5}	2	4	16	38
Design 4	{5}	4	4	14	35
	{9}	0	4	18	45
	{3, 2}	4	4	14	33
	{1, 8}	0	4	18	42
	{6, 11}	8	4	10	29
	{7, 10, 4, 0}	2	4	16	37
Design 5	{1}	4	4	14	34
	{3}	0	4	18	46
	{2, 11}	0	4	18	44
	{8, 10}	2	4	16	40
	{4, 7}	2	4	16	39
	{6, 5}	6	4	12	30
	{0, 9}	6	4	12	31
Design 6	{1}	2	2	18	20
	{11}	10	2	10	3
	{8, 10}	6	2	14	7
	{2, 3}	4	2	16	9
	{5, 7}	0	2	20	27
	{4, 9}	2	2	18	15
	{6, 0}	2	2	18	16
Design 7	{7}	0	2	20	21
	{6}	0	2	20	26
	{3}	4	2	16	13
	{1}	12	2	8	1
	{8, 10}	6	2	14	6
	{1, 4}	4	2	16	8
	{0, 9}	2	2	18	18
	{2, 5}	0	2	20	22

https://doi.org/10.1017/S1446788700022473 Published online by Cambridge University Press

TABLE IV (continued)

	Transitivity sets	2 − (12, 5, 4) designs ♯A	♯B	♯C	2 − (11, 5, 4) design number
Design 8	{2, 4, 10, 3}	2	2	18	14
	{11, 0, 6, 8, 1, 5, 7, 9}	4	2	16	10
Design 9	{9}	0	2	20	23
	{10}	0	2	20	25
	{8}	4	2	16	12
	{5}	4	2	16	11
	{3, 7}	2	2	18	19
	{1, 2}	2	2	18	17
	{4, 6}	6	2	14	5
	{0, 11}	6	2	14	4
Design 10	{11, 6, 4, 0}	10	2	10	2
	{8, 2, 9, 3, 1, 5, 7, 10}	0	2	20	24
Design 11	{0, 7}	2	0	20	56
	{10, 6, 4, 3, 1}	2	0	20	55
	{11, 8, 2, 9, 5}	6	0	16	54
Design 12	{9}	22	0	0	58
	{0, 1, 2, 3, 4, 5, 6, 7, 8, 10, 11}	2	0	20	57

Note that AC type blocks occur in quartets. Within each quartet blocks intersect in 0, 1 or 5 points. The 1-point intersections define a unique pair of points for each quartet. A study of these pairs helps to distinguish non-isomorphic designs (see Table III).

For each $3-(12,6,4)$ design the transitivity sets (or points orbits) under the action of the automorphism group are given. Two points from the same transivity set give isomorphic restrictions. The $2-(11,5,4)$ design corresponding to each transitivity set is listed by the numbers of blocks of each type A, B, C and identified by the number assigned to it in Section 9 (see Table IV).

All 12 3-designs listed have non-trivial automorphims groups. The detailed description of these is not given. For an earlier version of this work see Thompson [7].

REFERENCES

[1] D. R. Breach, 'The $2-(9,4,3)$ and $3-(10,5,3)$ designs,' *J. Combin. Theory Ser. A* **27** (1979), 50–63.

[2] D. R. Breach, 'Some $2-(2n+1, n, n-1)$ designs with multiple extensions,' *Combinatorial Mathematics* VI (Proceedings, Armidale, Australia) (Springer-Verlag Lecture Notes in Mathematics 748, 1979, 32–40).

[3] P. J. Cameron and J. H. van Lint, 'Graphs, codes and designs' (*Lond. Math. Soc. Lecture Notes Series* 43, CUP, 1980).

[4] P. Dembowski, *Finite geometries* (Ergebnisse der Mathematik 44, Springer-Verlag, 1968).

[5] D. A. Sprott, 'Balanced incomplete block designs and tactical configurations', *Ann. of Math. Stat.* **26** (1955), 752–758.

[6] R. G. Stanton, R. C. Mullin and J. A. Bate,' Isomorphism classes of a set of prime BIBD parameters', *Ars Combinatoria* **2** (1976), 251–264.

[7] A. R. Thompson, *Decomposable* $2-(11,5,4)$ *and* $3-(12,6,4)$ *designs*, (M. Sc. thesis, University of Canterbury, 1982).

Department of Mathematics
University of Canterbury
Christchurch, New Zealand