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THE FARKAS LEMMA OF SHIMIZU, AIYOSHI AND KATAYAMA

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Shimizu, Aiyoshi and Katayama have recently given a finite dimensional generalization of the classical Farkas Lemma. In this note we show that a result of Pshenichnyi on convex programming can be used to give a generalization of the result of Shimizu, Aiyoshi and Katayama to infinite dimensional spaces. A generalized Farkas Lemma of Glover is also obtained.

In [7], Theorem 2.1, Shimizu, Aiyoshi and Katayama have given a finite dimensional generalization of the classical lemma of Farkas ([2]). In this note we show that an infinite dimensional generalization of the result of Shimizu, Aiyoshi and Katayama can be obtained from a global characterization of a minimum in a convex programming problem due to Pshenichnyi ([6], II.2.1). We also show that a generalized Farkas Lemma due to Glover can be obtained from Pschenichnyi's result ([3], Theorem 1).

We begin by describing the result of Shimizu, Aiyoshi and Katayama. Throughout this paper, let X be a (real) Hausdorff locally convex topological vector space with dual X' and duality form $\langle \ , \ \rangle$. We assume throughout that X' is equipped with the weak* topology from X. Let A and B be compact subsets of X such that 0 is not in the convex hull of A $(0 \notin \text{co } A)$. The result of Shimizu, Aiyoshi and

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Katayama ([7], 2.1) then states that in a finite dimensional space the following conditions are equivalent:

(1) $\max\{\langle x', b\rangle : b \in B\} \le 0$ implies $\max\{\langle x', a\rangle : a \in A\} \ge 0$;

(2) there exist $x_0 \in \text{co } A$, $x_1 \in \text{co } B$ and $t \le 0$ such that $x_0 = tx_1$.

If E is a subset of X , let $S(\cdot|E)$ be the support functional of E , $S(x'|E) = \sup\{\langle x', x\rangle : x \in E\}$ for $x' \in X'$ ([1], 2.1.3). Note that $S(\cdot|E) = S(\cdot|\overline{\text{co}}\, E)$, where $\overline{\text{co}}\, E$ is the closed convex hull of E . Using the notation of support functional, (1) becomes:

$$S(x'|B) \le 0 \text{ implies } S(x'|A) \ge 0 .$$

Now the support functional is always convex so this suggests replacing condition (1) by the condition:

(1') $h(x) \le 0$ implies $f(x) \ge 0$,

where h and f are convex functions defined on X . We then need to obtain an appropriate generalization of (2) for this situation. This is given in Theorem 2 (also Corollary 3).

We now fix the notation and terminology which will be used. Let h and f be real valued convex functions defined on X . The subgradient of f at x_0 is $\partial f(x_0) = \{x' \in X' : f(x)-f(x_0) \ge \langle x', x-x_0\rangle \text{ for } x \in X\}$ ([6], I.1.1; [4], 6D). If f is continuous at x_0 , then $\partial f(x_0)$ is a non-void, weak* compact subset of X' ([4], 14B). A similar statement holds if f is a lower semicontinuous sublinear function (f is sublinear if $f(x+y) \le f(x) + f(y)$ and $f(tx) = tf(x)$ for $x, y \in X$ and $t \ge 0$); that is, $\partial f(x_0)$ is a non-void weak* compact subset of X' ([10], Proposition 1 and [7], .4).

If Ω is a subset of X and $x_0 \in \Omega$, then the cone of feasible directions to Ω at x_0 is

$$F(\Omega, x_0) = \{e : \text{there exists } \alpha > 0 \text{ such that } x_0+te \in \Omega \text{ for } 0 \le t \le \alpha\}$$

([6], II; [4], 14E). For a convex set,

$$F(\Omega, x_0) = \{t(x-x_0) : t \geq 0, x \in \Omega\} .$$

If $C \subseteq X$ then the dual cone of C is given by

$$C^* = \{x' \in X' : \langle x', x \rangle \geq 0 \text{ for all } x \in C\}$$

([6], I, p. 30).

Pshenichnyi gives a necessary and sufficient condition for a convex programming problem to have a solution in terms of subgradients and the feasible direction cone ([6], II.2.1; [4], 14E). If f is a convex function with $\partial f(x_0)$ a non-void, weak* compact subset, then f attains its minimum at $x_0 \in \Omega$ if and only if $F(\Omega, x_0)^* \cap \partial f(x_0) \neq 0$. (Pshenichnyi proves this result for f continuous but the continuity is only used to guarantee that $\partial f(x_0)$ is weak* compact and non-empty; actually Pshenichnyi's proof applies to the class of quasi-differentiable functions with $\partial f(x_0)$ weak* compact ([6], 3.1).)

Set $\Omega = \{x : h(x) \leq 0\}$ and let $x_0 \in \Omega$ be such that $h(x_0) = 0$. Concerning the cone $F(\Omega, x_0)$, we have that the inclusion $F(\Omega, x_0)^* \supseteq \mathbb{R}_- \partial h(x_0)$ holds always, where $\mathbb{R}_-$ is the set of non-positive real numbers ([6], 2.2; [4], 14E). Concerning the equality, we make the following constraint qualification.

DEFINITION 1. The function h is *regular* at x_0 if and only if $F(\Omega, x_0)^* = \mathbb{R}_- \partial h(x_0)$.

A sufficient condition for h to be regular at x_0 is that h satisfy Slater's condition; that is, there exists $x_1 \in \Omega$ such that $h(x_1) < 0$ ([6], II.2.2; [4], 14E).

From Pshenichnyi's condition given above, we have

THEOREM 2. *Let h be regular at x_0 and let f be such that $\partial f(x_0)$ is weak* compact and non-void. The following conditions are equivalent:*

(3) $h(x) \leq 0$ *implies* $f(x) \geq f(x_0)$;

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(4) *there exist* $x_0' \in \partial f(x_0)$, $x_1' \in \partial h(x_0)$ *and* $t \le 0$ *such that* $x_0' = tx_1'$.

Proof. If (3) holds, x_0 solves the convex programming problem $\min\{f(x) : x \in \Omega\}$. By Pshenichnyi's condition and the regularity assumption, (4) follows immediately.

If (4) holds and $x \in \Omega$, then

$$f(x) - f(x_0) \ge \langle x_0', x-x_0 \rangle = t\langle x_1', x-x_0 \rangle .$$

But $0 \ge h(x) - h(x_0) \ge \langle x_1', x-x_0 \rangle$ so $f(x) - f(x_0) \ge 0$, and (3) holds. (This implication does not require regularity.)

This result is given in [4], 14E, under the assumption that f is continuous and Slater's condition is satisfied.

We now indicate haw an infinite dimensional version of the result of Shimizu, Aiyoshi and Katayama can be obtained from Theorem 2. Let A and B be weak* compact subsets of X' with A and B such that $\overline{\text{co}}\, A$ and $\overline{\text{co}}\, B$ are also weak* compact (this condition is automatically satisfied if X is barrelled ([5], 3.6.2)). Set $h = S(\cdot|B) = S(\cdot|\overline{\text{co}}\, B)$ and $f = S(\cdot|A) = S(\cdot|\overline{\text{co}}\, A)$, where we compute these support functionals in the duality between X and X' . Note that both f and h are lower semi-continuous and sublinear with $\partial f(0) = \overline{\text{co}}\, A$. From Theorem 2 we obtain the following generalization of the result of Shimizu, Aiyoshi and Katayama ([7], 2.1).

COROLLARY 3. *Suppose* $0 \notin \overline{\text{co}}\, B$. *The following conditions are equivalent:*

(5) $h(x) \le 0$ *implies* $f(x) \ge f(0) = 0$;

(6) *there exist* $x_0' \in \overline{\text{co}}\, A$, $x_1' \in \overline{\text{co}}\, B$ *and* $t \le 0$ *such that* $x_0' = tx_1'$.

Proof. Since $0 \notin \overline{\text{co}}\, B$, by the Hahn-Banach Theorem there is an $x_0 \in X$ such that $h(x_0) = \max\{\langle x', x_0 \rangle : x' \in B\} < 0$; that is, h satisfies Slater's condition. Since $\partial h(0) = \overline{\text{co}}\, B$ and $\partial f(0) = \overline{\text{co}}\, A$, Theorem 2 gives the result.

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Shimizu, Aiyoshi and Katayama treat the case when $X = \mathbb{R}^n$ is finite dimensional. In this case the weak* topology on $X' = \mathbb{R}^n$ coincides with the norm topology and if $A, B \subseteq \mathbb{R}^n$ are compact, then co A and co B are also compact. Thus, in this case, condition (6) can be stated in stronger form:

(6') there exist $x'_0 \in \text{co } A$, $x'_1 \in \text{co } B$ and $t \leq 0$ such that $x'_0 = tx'_1$.

Condition (6') is clearly equivalent to condition (3) of Shimizu, Aiyoshi and Katayama ([7]).

The classical finite dimensional version of the Farkas Lemma corresponds to the case when B is finite and A is a singleton.

The Farkas result of Shimizu, Aiyoshi and Katayama for locally convex spaces has also been obtained by somewhat different methods in [8].

We now also show that a generalized Farkas Lemma of Glover can be obtained from Pshenichnyi's result. Let $f : X \to \mathbb{R}$ be a lower semi-continuous sublinear function. Let Y be a locally convex space, $S \subseteq Y$ a closed convex cone and $g : X \to Y$ S-sublinear (that is, $g(tx) = tg(x)$ for $t \geq 0$ and $-g\big(tx+(1-t)y\big) + tg(x) + (1-t)g(y) \in S$ for $0 \leq t \leq 1$, $x, y \in X$). We assume that g is such that $y'g$ is lower semicontinuous on X for each $y' \in S^*$ (weakly S^*-lower semicontinuous in Glover's terminology). Glover's result ([3], Theorem 1) is given by

THEOREM 4. *Let* $x' \in X'$. *The following conditions are equivalent:*

(7) $-g(x) \in S$ *implies that* $f(x) \geq \langle x', x\rangle$;

(8) $x' \in \partial f(0) + \overline{\bigcup_{s' \in S^*} \partial(s'g)(0)}$.

Proof. If $\Omega = g^{-1}(-S)$, then (7) is equivalent to the fact that 0 is a solution of the convex program: $\min\{(f-x')(x) : x \in \Omega\}$. By Pshenichnyi's condition, since $\partial(f-x')(0) = \partial f(0) - x'$, (7) is equivalent to $\big(\partial f(0)-x'\big) \cap F(\Omega, 0)^* \neq 0$. But Ω is a convex cone so $F(\Omega, 0) = \Omega$, and by Lemma 1 of [3], $F(\Omega, 0)^* = -\overline{\bigcup_{s' \in S^*} \partial(s'g)(0)}$. Thus, by Pshenichnyi's condition, (7) and (8) are equivalent.

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