COMPOSITIO MATHEMATICA

Sharp bounds on the height of K-semistable Fano varieties I, the toric case

Rolf Andreasson and Robert J. Berman

Compositio Math. **160** (2024), 2366–2406.

doi:10.1112/S0010437X2400736X

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY
EST. 1865

https://doi.org/10.1112/S0010437X2400736X Published online by Cambridge University Press

Compositio Math. 160 (2024) 2366–2406
doi:10.1112/S0010437X2400736X

Sharp bounds on the height of K-semistable Fano varieties I, the toric case

Rolf Andreasson and Robert J. Berman

Abstract

Inspired by K. Fujita's algebro-geometric result that complex projective space has maximal degree among all K-semistable complex Fano varieties, we conjecture that the height of a K-semistable metrized arithmetic Fano variety $\mathcal{X}$ of relative dimension n is maximal when $\mathcal{X}$ is the projective space over the integers, endowed with the Fubini–Study metric. Our main result establishes the conjecture for the canonical integral model of a toric Fano variety when $n \leq 6$ (the extension to higher dimensions is conditioned on a conjectural 'gap hypothesis' for the degree). Translated into toric Kähler geometry, this result yields a sharp lower bound on a toric invariant introduced by Donaldson, defined as the minimum of the toric Mabuchi functional. Furthermore, we reformulate our conjecture as an optimal lower bound on Odaka's modular height. In any dimension n it is shown how to control the height of the canonical toric model $\mathcal{X}$, with respect to the Kähler–Einstein metric, by the degree of $\mathcal{X}$. In a sequel to this paper our height conjecture is established for any projective diagonal Fano hypersurface, by exploiting a more general logarithmic setup.

1. Introduction

1.1 The height of K-semistable Fano varieties

Let $(\mathcal{X}, \mathcal{L})$ be a projective flat scheme $\mathcal{X}$ over $\mathbb{Z}$ of relative dimension n, endowed with a relatively ample line bundle $\mathcal{L}$. The complexification of $(\mathcal{X}, \mathcal{L})$ will be denoted by (X, L). In other words, X is the complex projective variety consisting of the complex points of $\mathcal{X}$ and L is the corresponding ample line bundle over X.

A central role in arithmetic and Diophantine geometry is played by the *height* of $(\mathcal{X}, \mathcal{L})$, which is defined with respect to a continuous metric $\|\cdot\|$ on L. This is an arithmetic analog of the algebro-geometric degree of (X, L), i.e. of the top intersection number L^n on X. The height of $(\mathcal{X}, \mathcal{L}, \|\cdot\|)$ – also known as the Faltings height – is defined as the $(n+1)$-fold arithmetic intersection number of the metrized line bundle $(\mathcal{L}, \|\cdot\|)$ on $\mathcal{X}$, introduced by Gillet and Soulé in the context of Arakelov geometry [Fal91, BGS94] (see § 1.1). We recall that in Arakelov geometry the metric $\|\cdot\|$ on L plays the role of a 'compactification' of $\mathcal{X}$. Accordingly, a metrized line bundle $(\mathcal{L}, \|\cdot\|)$ is usually denoted by $\overline{\mathcal{L}}$. The definition of height naturally extends to any $\mathbb{Q}$-line bundle $\mathcal{L}$, using homogeneity.

Received 30 May 2023, accepted in final form 14 March 2024, published online 10 October 2024.
2020 Mathematics Subject Classification 14G40 (primary), 32Q20, 53C25, 11G50, 14J45 (secondary).
Keywords: Arakelov geometry, Faltings heights, Kähler–Einstein metrics, Fano varieties, K-stability.

In contrast to the algebro-geometric degree of L, the height of $\overline{\mathcal{L}}$ can rarely be computed explicitly and all one can hope for, in general, is explicit bounds on the height. When $\mathcal{L}$ is the relative canonical line bundle, which we shall denote by $\mathcal{K}_{\mathcal{X}}$ and $n = 1$, such conjectural upper bounds are motivated by the Bogolomov–Miyaoka–Yau inequality on X and imply, in particular, the effective Mordell conjecture, concerning explicit upper bounds on the number of rational points on $X_{\mathbb{Q}}$, and the abc conjecture [Par88, Voj88, Sou94]. Here we shall be concerned with the opposite situation where $\mathcal{X}$ is an *arithmetic Fano variety*, in the sense that the relative anti-canonical line bundle is defined as a relative ample $\mathbb{Q}$-line bundle that we denote by $-\mathcal{K}_{\mathcal{X}}$, using additive notation for tensor products (see § 2.2.1). In particular, X is a complex *Fano variety*, a variety whose canonical line bundle $-K_X$ defines an ample $\mathbb{Q}$-line bundle. We will also, for simplicity, assume that X is normal. As shown in [BB17] in the toric case and then in [Fuj18] in general, for any complex Fano variety X,

$$(-K_X)^n \leq (-K_{\mathbb{P}^n_{\mathbb{C}}})^n \tag{1.1}$$

under the assumption that X is *K-semistable.* Moreover, equality holds if and only if $X = \mathbb{P}^n_{\mathbb{C}}$ [Liu18]. In contrast, when X is not K-semistable the degree $(-K_X)^n$ can be arbitrarily large in any given dimension n, for singular X (see [Deb03, Ex 4.2] for simple two-dimensional toric examples). The notion of K-stability first arose in the context of the Yau–Tian–Donaldson conjecture for Fano manifolds, saying that a Fano manifold admits a Kähler–Einstein metric if and only if it is K-polystable [Tia97, Don02]. The conjecture was settled in [CDS15] and very recently also established for singular Fano varieties [Li22, LXZ22]. From a purely algebro-geometric perspective K-stability can be viewed as a limiting form of Chow and Hilbert–Mumford stability [RT07], which enables a good theory of moduli spaces (see the survey [Xu21]).

Is there an arithmetic analog of inequality (1.1)? More precisely, it seems natural to ask if, under appropriate assumptions, the height $(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1}$ is bounded from above by the height $(\overline{-\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}})^{n+1}$ of the relative anti-canonical line bundle on the projective space $\mathbb{P}^n_{\mathbb{Z}}$ over the integers, endowed with its standard Kähler–Einstein metric (the Fubini–Study metric). This would yield an explicit bound on the height $(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1}$, since the height of Fubini–Study metric on projective space was explicitly calculated in [GS90, § 5.4], giving, after volume normalization,

$$(\overline{-\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}})^{n+1} = \frac{1}{2}(n+1)^{n+1}\left((n+1)\sum_{k=1}^{n} k^{-1} - n + \log\left(\frac{\pi^n}{n!}\right)\right). \tag{1.2}$$

However, if such a universal bound is to hold, one needs to impose a normalization condition on the metric on $-K_X$. Indeed, $\overline{\mathcal{L}}^{n+1}$ is additively equivariant with respect to scalings of the metric. Accordingly, the metric $\|\cdot\|$ on $-K_X$ will henceforth be assumed to be *volume-normalized* in the sense that the corresponding volume form on X has total unit volume. As it turns out, the supremum of the height $\overline{-\mathcal{K}_{\mathcal{X}}}^{n+1}$ over all volume-normalized metrics on $-K_X$ with positive curvature current is finite if and only if X is K-semistable (Theorem 2.5). It thus seems natural to make the following conjecture.

CONJECTURE 1.1. Let $\mathcal{X}$ be an arithmetic Fano variety of relative dimension n over $\mathbb{Z}$. If the complexification X of $\mathcal{X}$ is K-semistable, then the following height inequality holds for any volume-normalized continuous metric on $-K_X$ with positive curvature current:

$$(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1} \leq (\overline{-\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}})^{n+1},$$

https://doi.org/10.1112/S0010437X2400736X Published online by Cambridge University Press

where $-K_{\mathbb{P}^n_{\mathbb{C}}}$ is endowed with the volume-normalized Fubini–Study metric. Moreover, if $\mathcal{X}$ is normal, equality holds if and only if $\mathcal{X} = \mathbb{P}^n_{\mathbb{Z}}$ and the metric is Kähler–Einstein, i.e. coincides with the Fubini–Study metric, modulo the action of an automorphism.

More generally, when $\mathbb{Z}$ is replaced by the ring of integers of a number field F, i.e. a finite field extension F of $\mathbb{Q}$, the height $(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1}$ should be divided by the degree $[F:\mathbb{Q}]$. But, for simplicity, we will focus on the case when $F = \mathbb{Q}$ (see § 6.2 for a generalization of the previous conjecture). The converse 'only if' statement to the previous conjecture does hold (as a consequence of Theorem 2.5). Moreover, the conjecture is compatible with taking products (Proposition 2.10). The inequality in the previous conjecture is equivalent to the following inequality for any continuous metric on $-K_X$ with positive curvature current, as follows from a simple scaling argument:

$$\frac{(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1}}{(n+1)} + \frac{(-K_X)^n}{2}\log\mu(X) \le c_n, \tag{1.3}$$

where $\mu(X)$ denotes the volume of X with respect to the measure μ on X corresponding to the metric $\|\cdot\|$ on $-K_X$ and c_n denotes the constant in the right-hand side of formula (1.2). Some intriguing relations between the conjectural bound (1.3) and the Manin–Peyre conjecture, concerning the density of rational points on Fano varieties, are discussed in [Ber23].

Our main result concerns the case when X is toric and $\mathcal{X}$ is its canonical toric integral model (see [Mai00, § 2] and [BPS14, Def 3.5.6]).

Theorem 1.2. *Let X be an n-dimensional K-semistable toric Fano variety and denote by $\mathcal{X}$ its canonical model over $\mathbb{Z}$. Then the previous conjecture holds under any one of the following conditions:*

- *$n \le 6$ and X is $\mathbb{Q}$-factorial (equivalently, X is non-singular or has abelian quotient singularities);*
- *X is not Gorenstein or has some abelian quotient singularity.*

Note that when $n = 2$ any toric variety is, in fact, $\mathbb{Q}$-factorial. More generally, we will show that the curvature assumption may be dispensed with if the height $(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1}$ is replaced by the χ-arithmetic volume $\widehat{\mathrm{vol}}_\chi(\overline{-\mathcal{K}_{\mathcal{X}}})$ of $\overline{-\mathcal{K}_{\mathcal{X}}}$ (whose definition is recalled in § 2.2.2). We expect that the maximum of $\widehat{\mathrm{vol}}_\chi(\overline{-\mathcal{K}_{\mathcal{X}}})$ over all integral models $(\mathcal{X}, -\mathcal{K}_{\mathcal{X}})$ of a given toric Fano variety $(X, -K_X)$ is attained at the canonical integral model $\mathcal{X}$ featuring in the previous theorem. This expectation is inspired by a conjecture of Odaka discussed in § 1.4 below.

The key ingredient in the proof of Theorem 1.2 is the following bound estimating the arithmetic volume $\widehat{\mathrm{vol}}_\chi(\overline{-\mathcal{K}_{\mathcal{X}}})$ of any volume-normalized metric on $-K_X$ in terms of the algebro-geometric volume $\mathrm{vol}(X)$ (Proposition 3.7):

$$\widehat{\mathrm{vol}}_\chi(\overline{-\mathcal{K}_{\mathcal{X}}}) \le -\frac{1}{2}\mathrm{vol}(X)\log\left(\frac{\mathrm{vol}(X)}{(2\pi^2)^n}\right)\mathrm{vol}(X) := (-K_X)^n/n!. \tag{1.4}$$

Since $\mathrm{vol}(X)$ is maximal for $X = \mathbb{P}^n$ the right-hand side above is bounded by a constant C_n only depending on the dimension n. Under the 'gap hypothesis' that $\mathbb{P}^{n-1}\times\mathbb{P}^1$ has the second largest volume among all n-dimensional K-semistable X, we show that the bound (1.4) implies Conjecture 1.1 for the canonical integral model $\mathcal{X}$ of a toric Fano variety X. The proof of Theorem 1.2 is concluded by verifying the gap hypothesis under the conditions in Theorem 1.2. But we do expect that the gap hypothesis above holds for any toric Fano variety (see § 3.2.1).

In a sequel [AB23] to the present paper Conjecture 1.1 is established for any diagonal Fano hypersurface $\mathcal{X}$ in $\mathbb{P}^{n+1}_{\mathbb{Z}}$ (i.e. $\mathcal{X}$ is the subscheme cut out by a homogeneous polynomial of the

https://doi.org/10.1112/S0010437X2400736X Published online by Cambridge University Press

form $a_0x_0^d + \cdots + a_{n+1}x_{n+1}^d$ for any given integers a_i, with no common divisors, and $d \leq n+1$). Although $\mathcal{X}$ is not toric the proof, somewhat surprisingly, is reduced to a simple toric logarithmic case.

1.2 The height of toric Kähler–Einstein metrics

In the toric case, X is K-semistable if and only if it is K-polystable and thus admits a toric Kähler–Einstein metric [WZ04, BB13], i.e. a toric continuous metric on $-K_X$ whose curvature form defines a Kähler metric with constant positive Ricci curvature on the regular locus of X. Moreover, in general, any volume-normalized Kähler–Einstein metric maximizes $(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1}$. This means that the inequality in the previous theorem is equivalent to the corresponding inequality for the volume-normalized toric Kähler–Einstein metric on $-K_X$. The special role of the Kähler–Einstein condition in arithmetic (Arakelov) geometry – as an analog of minimality of $\mathcal{X}$ over Spec$\mathbb{Z}$ – was emphasized in the early days of Arakelov geometry by Manin [Man85]. It is, however, rare that the Kähler–Einstein metric and the corresponding height, can be explicitly computed. In fact, in the Fano case this seems to only have been achieved when X is homogeneous [Mai95, CM00, KK02, Tam99a, Tam99b, Tam00]. The following result, complementing the general upper bound (1.4), yields a rather precise control on its height $(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1}$ in the toric case.

Theorem 1.3. *Let X be an n-dimensional toric Fano variety and denote by $\mathcal{X}$ its canonical model over $\mathbb{Z}$. Then the height $(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1}$ of any volume-normalized Kähler–Einstein metric satisfies*

$$\frac{(n+1)!}{2}\mathrm{vol}(X)\log\left(\frac{n!m_n\pi^n}{\mathrm{vol}(X)}\right) \leq (\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1} \leq \frac{(n+1)!}{2}\mathrm{vol}(X)\log\left(\frac{(2\pi)^n\pi^n}{\mathrm{vol}(X)}\right),$$

where m_n denotes the largest lower bound on the Mahler volume of a convex body. In particular, $(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1} > 0$.

We also provide an infinite family of toric varieties X for which the height of the corresponding Kähler–Einstein can be explicitly computed as a function $f(v)$ of vol(X) of the same form as in the previous theorem: $f(v) = v\log(av^{-1})$ for some constant a. The constant m_n in the previous theorem is the largest constant satisfying

$$m_n \leq \mathrm{vol}(P)\mathrm{vol}(P^*),$$

where P^* denotes the polar dual of any given convex body P containing the origin in its interior (the role of P in the present setting is played by the moment polytope of X). According to Mahler's conjecture, the constant m_n is equal to $(n+1)^{n+1}/(n!)^2$ (which is realized for a simplex P). The case $n=2$ was settled in [Mah39], but for our purposes the following general bound from [Kup08] will be enough:

$$m_n \geq \left(\frac{\pi}{2e}\right)^{n-1}(n+1)^{n+1}/(n!)^2,$$

which implies the strict positivity of $(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1}$. Combining the previous theorem with the upper bound (1.1) thus yields the following universal bounds.

Corollary 1.4. *Let X be an n-dimensional toric Fano variety and denote by $\mathcal{X}$ its canonical model over $\mathbb{Z}$. Then the height $(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1}$ of any volume-normalized Kähler–Einstein metric* *satisfies the following universal bounds:*

$$0 < \big(\overline{-\mathcal{K}_{\mathcal{X}}}\big)^{n+1} \leq \frac{n(n+1)^{n+1}}{2}\log\left(\frac{2\pi^2 n!}{n+1}\right).$$

Incidentally, the upper bound above is related to a question posed in [NT96], asking whether $(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1}$ is bounded from above by a universal constant C_n, under the assumption that X be non-singular and $\overline{-\mathcal{K}_{\mathcal{X}}}$ be relatively ample. This is a stronger condition than having positive curvature, as we assume. We also allow singularities, but our results concern only the toric case. Under the conditions in Theorem 1.2 our upper bound may be improved to the sharp bound $(\overline{-\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}})^{n+1}$ (given by formula (1.2)). As for the lower bound, it is sharp in any dimension n. Indeed, there are n-dimensional K-semistable ($\mathbb{Q}$-factorial) Fano varieties X such that $\mathrm{vol}(X)$ and thus (by Theorem 1.3) $(\overline{-\mathcal{K}_{\mathcal{X}}})^{n+1}$ is arbitrarily close to 0; see Example 3.1.

1.3 Donaldson's toric invariant

Let (X, L) now be a polarized complex projective manifold. A prominent role in Kähler geometry is played by Mabuchi's K-energy functional $\mathcal{M}$ [Mab86], defined on the space $\mathcal{H}(X, L)$ of all smooth metrics $\|\cdot\|$ on L with positive curvature. Its critical points are the metrics whose curvature form ω defines a Kähler metric on X with constant scalar curvature. The precise definition of $\mathcal{M}$ is recalled in § 4.1. Since the definition of $\mathcal{M}$ only involves its differential, the functional $\mathcal{M}$ is only defined up to addition by a real constant. However, when (X, L) is toric, Donaldson [Don02] exploited the toric structure to define the Mabuchi functional $\mathcal{M}$ as a canonical functional on toric metrics:

$$\mathcal{M}_L := \int_{\partial P} u\,d\sigma - a\int_P u\,dx - \int_P \log\det(\nabla^2 u)\,dx, \quad a := \int_{\partial P} d\sigma \Big/ \int_P dx, \tag{1.5}$$

where P is the moment polytope in $\mathbb{R}^n$ corresponding to the polarized toric manifold (X, L), whose boundary ∂P comes with a measure $d\sigma$ induced by Lebesgue measure dx on $\mathbb{R}^n$ and the lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$ and u is the smooth bounded convex function on P corresponding to a toric metric on L under Legendre transformation (see § 3.1.2). In particular, in the last section of [Don02] Donaldson introduced an invariant of a polarized toric manifold (X, L), defined as the infimum of the toric Mabuchi functional $\mathcal{M}_L$ defined by formula (1.5). Here we show that Theorem 1.2 implies that when X is a Fano variety and $L = -K_X$, a slight perturbation of Donaldson's invariant is minimal when X is complex projective space, under the conditions on X appearing in Theorem 1.2.

Theorem 1.5. *Let X be a K-semistable toric Fano variety of dimension n, satisfying the conditions in Theorem 1.2. Then the invariant*

$$X \mapsto \inf_{\mathcal{H}(X,-K_X)} \mathcal{M}_{-K_X} - \frac{(-K_X)^n}{n!}\log\left(\frac{(-K_X)^n}{n!}\right)$$

is minimal for $X = \mathbb{P}^n$ (and only then), where the infimum is attained at the metric on $-K_{\mathbb{P}^n}$ induced by the Fubini–Study metric.

In the previous theorem the Fano variety X is allowed to be singular. The Mabuchi functional for singular general Fano varieties was introduced in [DT92, BBEGZ19], and Donaldson's formula (1.5) was extended to singular toric Fano varieties in [BB13]. In general, for Fano varieties the Mabuchi functional $\mathcal{M}$ is bounded from below if and only if X is K-semistable [Li17] (see the discussion following Theorem 2.5).

https://doi.org/10.1112/S0010437X2400736X Published online by Cambridge University Press

1.4 The arithmetic Mabuchi functional and Odaka's modular height

For a general polarized manifold (X, L) the infimum of the Mabuchi functional $\mathcal{M}$ is not canonically defined (since $\mathcal{M}$ is only defined up to addition by a constant). But to any given integral model $(\mathcal{X}, \mathcal{L})$ of a polarized complex variety (X, L) one may, as shown by Odaka [Oda18], attach a particular Mabuchi functional $\mathcal{M}_{(\mathcal{X},\mathcal{L})}$ which (up to a multiplicative normalization) is given as the following sum of arithmetic intersection numbers:

$$\mathcal{M}_{(\mathcal{X},\mathcal{L})}(\overline{\mathcal{L}}) := \frac{a}{(n+1)!}\overline{\mathcal{L}}^{n+1} - \frac{1}{n!}(-\overline{\mathcal{K}}_{\mathcal{X}}) \cdot \overline{\mathcal{L}}^{n}, \quad a = -n(K_X \cdot L^{n-1})/L^n, \tag{1.6}$$

where, as in the previous section, $\overline{\mathcal{L}}$ denotes the metrized line bundle $(\mathcal{L}, \|\cdot\|)$. In the definition of the second arithmetic intersection number above one also needs to endow $-K_X$ with a metric and one is confronted with two different natural choices: either the metric induced by the volume form $\omega^n/n!$ of the Kähler metric ω defined by the curvature form of $(\mathcal{L}, \|\cdot\|)$ or the *normalized* volume form ω^n/L^n (which has unit total volume). The first choice is the one adopted in [Oda18], and we show that when X is a toric Fano variety and $(\mathcal{X}, \mathcal{L})$ is the canonical integral model of (X, L) this choice coincides with Donaldson's one (formula (1.5)). However, for our purposes the second volume-normalized choice turns out to be the appropriate one. It yields, in particular, the shift by the logarithm of $(-K_X)^n$ appearing in Theorem 1.5:

$$2\mathcal{M}_{(\mathcal{X},-\mathcal{K}_X)} = \mathcal{M}_{-K_X} - \frac{(-K_X)^n}{n!}\log\left(\frac{(-K_X)^n}{n!}\right)$$

(Proposition 5.2). The point is that with this choice the following formula holds in the arithmetic setting:

$$\sup \frac{\left(\overline{-\mathcal{K}_{\mathcal{X}}}\right)^{n+1}}{(n+1)!} = -\inf_{\mathcal{H}(X,-K_X)} \mathcal{M}_{(\mathcal{X},-\mathcal{K}_{\mathcal{X}})}, \tag{1.7}$$

where the supremum ranges over all volume-normalized metrics in $\mathcal{H}(X, -K_X)$ (see Proposition 5.3). As a consequence, Conjecture 1.1 is equivalent to the inequality

$$\inf_{\mathcal{H}(X,-K_X)} \mathcal{M}_{(\mathcal{X},-K_{\mathcal{X}})} \geq \inf_{\mathcal{H}(\mathbb{P}^n,-K_{\mathbb{P}^n})} \mathcal{M}_{(\mathbb{P}^n_{\mathbb{Z}},\dots)}. \tag{1.8}$$

Theorem 1.5 thus follows from Theorem 1.2.

1.4.1 *Odaka's modular height.* Let (X_F, L_F) be an n-dimensional polarized variety defined over a number field F. In [Oda18] Odaka introduced the following invariant of (X_F, L_F), dubbed the *intrinsic K-modular height of* (X_F, L_F)*:*

$$h(X_F, L_F) = \inf_{(\mathcal{X},\mathcal{L})} \inf_{\mathcal{H}(X,L)} \mathcal{M}_{(\mathcal{X},\mathcal{L})}, \tag{1.9}$$

where $(\mathcal{X}, \mathcal{L})$ is a model of (X_F, L_F) over the rings of integers $\mathcal{O}_{F'}$ of a finite field extension F' of F and $\mathcal{M}_{(\mathcal{X},\mathcal{L})}$ now denotes the arithmetic K-energy (1.6), divided by the degree $[F' : \mathbb{Q}]$. In contrast to [Oda18], we will employ the volume-normalized metric on $-K_X$ in the definition of $\mathcal{M}_{(\mathcal{X},\mathcal{L})}$, discussed in the previous section. As shown in (1.6), for a polarized abelian variety (X_F, L_F), Odaka's modular height $h(X_F, L_F)$ essentially coincides with Faltings's stable modular height of (X_K, L_K) [Fal83a] (see § 6.4). Furthermore, as explained in [Oda18], $h(X_F, L_F)$ can be viewed as a 'large rank limit' of Bost and Zhang's intrinsic heights appearing in [Bos94, Bos96, Zha96], where the role of K-semistability is played by Chow semistability (see formula (6.8)). We propose the following conjecture.

https://doi.org/10.1112/S0010437X2400736X Published online by Cambridge University Press

Conjecture 1.6. Let $X_{\mathbb{Q}}$ be a Fano variety defined over $\mathbb{Q}$. Then Odaka's modular invariant $h(X_{\mathbb{Q}}, -K_{X_{\mathbb{Q}}})$, normalized as above, is minimal when $X_{\mathbb{Q}} = \mathbb{P}^n_{\mathbb{Q}}$.

According to a conjecture of Odaka [Oda20], any globally K-semistable integral model $(\mathcal{X}, -\mathcal{K}_{\mathcal{X}})$ of $(X, -K_X)$ minimizes $\mathcal{M}_{(\mathcal{X},\mathcal{L})}$ over all models $(\mathcal{X}, \mathcal{L})$ (the function field analog of this minimization property is established in [BX19]; see also [Xu21, Remark 7.9]). Global K-semistability means that all the fibers of $\mathcal{X} \to \mathrm{Spec}\mathcal{O}_F$ are K-semistable. In other words, in addition to the K-semistability of the generic fiber X_F, this means that the variety $X_{\mathbb{F}_p}$ over the finite field $\mathbb{F}_p$, corresponding to the integral model $\mathcal{X}$, is K-semistable for any prime ideal p. For example, as pointed out to us by Odaka, the canonical model $\mathcal{X}$ of a K-semistable toric Fano variety $X_{\mathbb{Q}}$ appearing in Theorem 1.2 is globally K-semistable. Thus if Odaka's minimization conjecture holds, then Theorem 1.2 implies Conjecture 1.6 for any toric Fano variety $X_{\mathbb{Q}}$ satisfying the conditions in Theorem 1.2.[1] In any case, the positivity statement in Theorem 1.3 implies that the modular invariant $h(X_{\mathbb{Q}}, -K_{X_{\mathbb{Q}}})$ is negative for any K-semistable toric Fano variety $X_{\mathbb{Q}}$.

1.5 Organization

In § 2 we start by recalling the complex geometric and arithmetic setup before proving Theorem 2.5, relating upper bounds on the height of Fano varieties to K-semistability. The proof leverages an arithmetic analog of the Ding functional. In § 3 we specialize to the toric situation and prove the sharp height inequality in Theorem 1.2 and the height bounds for Kähler–Einstein metrics in Theorem 1.3. We also show that Conjecture 1.1 is compatible with taking products. We then go on, in § 4, to deduce Theorem 1.5 concerning the sharp lower bound on Donaldson's toric Mabuchi functional. In § 5 Donaldson's functional is related to Odaka's arithmetic Mabuchi functional, which in turn is related to the arithmetic Ding functional. In the last section we make a comparison with the function field case, formulate a generalized version of Conjecture 1.1 and compare with previous work of Bost and Zhang, Odaka and Faltings.

We have made an effort to make the paper readable for the reader with a background in arithmetic geometry, as well as for complex geometers, by including most of the background material needed for the proofs of the main results.

2. Heights, arithmetic volumes and K-stability of Fano varieties

In this section we show, in particular, that the height of a polarized integral model $(\mathcal{X}, \mathcal{L})$ of a Fano manifold $(X, -K_X)$ is bounded from above – as the metric on $\mathcal{L}$ ranges over all volume-normalized metrics with positive curvature current – if and only if $(X, -K_X)$ is K-semistable (Theorem 2.5). See also [Oda18] for further connections between K-stability of polarized varieties (X, L) and arithmetic geometry. The main new feature here, compared to [Oda18], is that we leverage an arithmetic version of the Ding functional in Kähler geometry, while [Oda18] considers an arithmetic version of the Mabuchi functional (the two functionals are compared in § 5).

2.1 Complex geometric setup

Throughout the paper X will denote a compact connected complex normal variety, assumed to be $\mathbb{Q}$-Gorenstein. This means that the canonical divisor K_X on X is defined as a $\mathbb{Q}$-line bundle: there exist some positive integer m and a line bundle on X whose restriction to the regular locus X_{reg} of X coincides with the mth tensor power of $K_{X_{\mathrm{reg}}}$, i.e. the top exterior

[1] During the revision of the first preprint version of the present paper, Odaka's minimization conjecture was settled in [HO22] under slightly stronger assumptions than global K-semistability.

power of the cotangent bundle of X_{reg}. We will use additive notation for tensor powers of line bundles.

2.1.1 *Metrics on line bundles.* Let (X, L) be a polarized complex projective variety, i.e. a complex normal variety X endowed with an ample line bundle L. We will use additive notation for metrics on L. This means that we identify a continuous Hermitian metric $\|\cdot\|$ on L with a collection of continuous local functions ϕ_U associated to a given covering of X by open subsets U and trivializing holomorphic sections e_U of $L \to U$:

$$\phi_U := -\log(\|e_U\|^2), \tag{2.1}$$

which defines a function on U. Of course, the functions ϕ_U on U do not glue to define a global function on X, but the current

$$dd^c\phi_U := \frac{i}{2\pi}\partial\bar{\partial}\phi_U$$

is globally well defined and coincides with the normalized curvature current of $\|\cdot\|$ (the normalization ensures that the corresponding cohomology class represents the first Chern class $c_1(L)$ of L in the integral lattice of $H^2(X, \mathbb{R})$). Accordingly, as is customary, we will symbolically denote by ϕ a given continuous Hermitian metric on L and by $dd^c\phi$ its curvature current. The space of all continuous metrics ϕ on L will be denoted by $\mathcal{C}^0(L)$. We will denote by $\mathcal{C}^0(L) \cap \mathrm{PSH}(L)$ the space of all continuous metrics on L whose curvature current is positive, $dd^c\phi \geq 0$ (which means that ϕ_U is plurisubharmonic (psh)). Then the exterior powers of $dd^c\phi$ are defined using the local pluripotential theory of Bedford and Taylor [BB10]. The *volume* of an ample line bundle L may be defined by

$$\mathrm{vol}(L) := \lim_{k\to\infty} k^{-n} \dim H^0(X, L^{\otimes k}) = \frac{1}{n!}L^n = \frac{1}{n!}\int_X (dd^c\phi)^n \tag{2.2}$$

using in the second equality the Hilbert–Samuel theorem and where ϕ denotes any element in $\mathcal{C}^0(L) \cap \mathrm{PSH}(L)$.

More generally, metrics ϕ are defined for a $\mathbb{Q}$-line bundle L: if mL is a bona fide line bundle, for $m \in \mathbb{Z}_+$, then $m\phi$ is a bona fide metric on mL.

Remark 2.1. The normalization of ϕ_U used here coincides with the one in [Ber16, BB13], but it is twice the one employed in [BB10].

2.1.2 *Metrics on $-K_X$ versus volume forms on X.* First consider the case when X is smooth. Then any smooth metric $\|\cdot\|$ on $-K_X$ corresponds to a volume form on X, defined as follows. Given local holomorphic coordinates z on $U \subset X$, denote by e_U the corresponding trivialization of $-K_X$, i.e. $e_U = \partial/\partial z_1 \wedge \cdots \wedge \partial/\partial z_n$. The metric on $-K_X$ induces, as in § 2.1.1, a function ϕ_U on U, and the volume form in question is locally defined by

$$e^{-\phi_U}\left(\frac{i}{2}\right)^{n^2} dz \wedge d\bar{z}, \quad dz := dz_1 \wedge \cdots \wedge dz_n \tag{2.3}$$

on U, which glues to define a global volume form on X. In other words, $e^{-\phi_U}$ is the density of the volume form with respect to the local Euclidean volume form. Accordingly, we will simply denote the volume form in question by $e^{-\phi}$, abusing notation slightly. When X is singular any continuous metric ϕ on $-K_X$ induces a measure on X, symbolically denoted by $e^{-\phi}$, defined as before on the regular locus X_{reg} of X and then extended by zero to all of X. We will say that a measure dV on X is a continuous volume form if it corresponds to a continuous metric on $-K_X$.

A Fano variety has log terminal singularities if and only if it admits a continuous volume form dV with finite total volume [BBEGZ19, § 3.1].

2.1.3 *K-semistability.* We briefly recall the notion of K-semistability (see [Don02, RT07, Wan12, Oda13a] for more background). A polarized complex projective variety (X, L) is said to be *K-semistable* if the Donaldson–Futaki invariant $\mathrm{DF}(\mathscr{X}, \mathscr{L})$ of any test configuration $(\mathscr{X}, \mathscr{L})$ for (X, L) is non-negative. A test configuration $(\mathscr{X}, \mathscr{L})$ is defined as a $\mathbb{C}^*$-equivariant normal model for (X, L) over the complex affine line $\mathbb{C}$. More precisely, $\mathscr{X}$ is a normal complex variety endowed with a $\mathbb{C}^*$-action ρ, a $\mathbb{C}^*$-equivariant holomorphic projection π to $\mathbb{C}$ and a relatively ample $\mathbb{C}^*$-equivariant $\mathbb{Q}$-line bundle $\mathscr{L}$ (endowed with a lift of ρ):

$$\pi : \mathscr{X} \to \mathbb{C}, \quad \mathscr{L} \to \mathscr{X}, \quad \rho : \mathscr{X} \times \mathbb{C}^* \to \mathscr{X} \tag{2.4}$$

such that the fiber of $\mathscr{X}$ over $1 \in \mathbb{C}$ is equal to (X, L). Its *Donaldson–Futaki invariant* $\mathrm{DF}(\mathscr{X}, \mathscr{L}) \in \mathbb{R}$ may be defined as a normalized limit, as $k \to \infty$, of Chow weights of a sequence of one-parameter subgroups of $GL(H^0(X, kL))$ induced by $(\mathscr{X}, \mathscr{L})$ (in the sense of geometric invariant theory). As a consequence, (X, L) is K-semistable if, for example, (X, kL) is Chow semistable, for k sufficiently large [RT07]. However, for the purpose of the present paper it will be more convenient to employ the intersection-theoretic formula for $\mathrm{DF}(\mathscr{X}, \mathscr{L})$ established in [Wan12, Oda13a]:

$$\mathrm{DF}(\mathscr{X}, \mathscr{L}) = \frac{a}{(n+1)!}\overline{\mathscr{L}}^{n+1} + \frac{1}{n!}\mathscr{K}_{\overline{\mathscr{X}}/\mathbb{P}^1} \cdot \overline{\mathscr{L}}^n, \quad a = -n(K_X \cdot L^{n-1})/L^n,$$

where $\overline{\mathscr{L}}$ denotes the $\mathbb{C}^*$-equivariant extension of $\mathscr{L}$ to the $\mathbb{C}^*$-equivariant compactification $\overline{\mathscr{X}}$ of $\mathscr{X}$ over $\mathbb{P}^1$ and $\mathscr{K}_{\overline{\mathscr{X}}/\mathbb{P}^1}$ denotes the relative canonical divisor.

Remark 2.2. Usually the definition of $\mathrm{DF}(\mathscr{X}, \mathscr{L})$ involves a factor of $1/L^n$, but the present definition will be more convenient here (since the factor L^n is positive it does not alter the definition of K-stability). It is made so that $-(n+1)!\mathrm{DF}(\mathscr{X}, \mathscr{L}) = \overline{\mathscr{L}}^{n+1}$ when $\mathscr{L} = -\mathscr{K}_{\overline{\mathscr{X}}/\mathbb{P}^1}$.

2.2 Arithmetic setup

Let $\mathcal{X}$ be a projective flat scheme $\mathcal{X} \to \mathrm{Spec}\mathbb{Z}$ of relative dimension n, with the property that $\mathcal{X}$ is reduced and satisfies Serre's conditions S_2 (this is, for example, the case if $\mathcal{X}$ is normal). Denote by X the complex points of $\mathcal{X}$ and assume that X is a normal projective variety over $\mathbb{C}$. Such a scheme $\mathcal{X}$ will be called an *arithmetic variety.* A *polarized arithmetic variety* $(\mathcal{X}, \mathcal{L})$ is an arithmetic variety endowed with a relatively ample $\mathbb{Q}$-line bundle $\mathcal{L}$. We will denote by L the ample line bundle over X induced by $\mathcal{L}$; the polarized arithmetic variety $(\mathcal{X}, \mathcal{L})$ will be called a *model* for (X, L) *over* $\mathbb{Z}$ (or an *integral model* for (X, L)). We will use the following simple lemma.

Lemma 2.3. *Under the assumptions above on $\mathcal{X}$ the canonical embedding of $\mathbb{Z}$ in $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is an isomorphism. In other words, 1 generates the $\mathbb{Z}$-module $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$.*

Proof. We have injections $\mathbb{Z} \hookrightarrow H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \hookrightarrow H^0(X_{\mathbb{Q}}, \mathcal{O}_{X_{\mathbb{Q}}}) \simeq \mathbb{Q}$ (using flatness in the second injection and, in the isomorphism, that $X_{\mathbb{Q}}$ is geometrically connected and geometrically reduced). But, since $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a finitely generated $\mathbb{Z}$-module and $\mathbb{Z}$ is an integrally closed domain this implies that $\mathbb{Z} \hookrightarrow H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is an isomorphism. □

For any positive integer k we may identify the free $\mathbb{Z}$-module $H^0(\mathcal{X}, k\mathcal{L})$ with a lattice in $H^0(X, kL)$:

$$H^0(\mathcal{X}, k\mathcal{L}) \otimes \mathbb{C} = H^0(X, kL).$$

By definition a *metrized line bundle* $\overline{\mathcal{L}}$ is a line bundle $\mathcal{L} \to \mathcal{X}$ such that the corresponding line bundle $L \to X$ is endowed with a metric $\|\cdot\|$. We will use the additive notation ϕ for metrics $\|\cdot\|$ on L discussed in the previous section:

$$\overline{\mathcal{L}} := (\mathcal{L}, \phi).$$

2.2.1 *Arithmetic Fano varieties.* We will say that the relative canonical line bundle of an arithmetic variety $\mathcal{X}$ is defined as a $\mathbb{Q}$-line bundle, denoted by $\mathcal{K}$, if there exists a positive integer m such that the mth reflexive power $\omega^{[m]}_{X/\mathrm{Spec}\mathbb{Z}}$ of the dualizing sheaf $\omega_{X/\mathrm{Spec}\mathbb{Z}}$ of $\mathcal{X}$ is locally free. Then the line bundle $m\mathcal{K}$ over $\mathcal{X}$ may be identified with $\omega^{[m]}_{X/\mathrm{Spec}\mathbb{Z}}$ (see [Kol13, § 1.1] for a more general setup of canonical line bundles attached to schemes over regular excellent rings). An arithmetic variety $\mathcal{X} \to \mathrm{Spec}\mathbb{Z}$ will be called an *arithmetic Fano variety* if:

- the canonical line bundle $\mathcal{K}$ of $\mathcal{X}$ is well defined as a $\mathbb{Q}$-line bundle and its dual $-\mathcal{K}$ is relatively ample;
- the complexification X of $\mathcal{X}$ is normal and thus defines a complex Fano variety (i.e. $-K_X$ is ample)

Example 2.4. If $\mathcal{X}$ is locally a complete intersection, then $\mathcal{K}$ is defined as a line bundle (i.e. $m = 1$) [Kol13, § 1.1]. In particular, if $\mathcal{X}$ is the subscheme of $\mathbb{P}^{n+1}_{\mathbb{Z}}$ cut out by an irreducible homogeneous polynomial of degree d with integer coefficients, then $\mathcal{K}$ is well defined as a line bundle and $\mathcal{X}$ is an arithmetic Fano variety if and only if $d \leq n+1$.

2.2.2 *The χ-arithmetic volume, heights and arithmetic intersection numbers.* In the arithmetic setup there are different analogs of the volume $\mathrm{vol}(L)$ of an ample line bundle L. Here we shall focus on the one defined by the following asymptotic arithmetic Euler characteristic originating in [Fal84] (called the *χ-arithmetic volume* in [BPS14, BMPS16] and the *sectional capacity* in [RLV00]):

$$\widehat{\mathrm{vol}}_\chi(\overline{\mathcal{L}}) := \lim_{k\to\infty} k^{-(n+1)} \log \mathrm{Vol}\left\{ s_k \in H^0(\mathcal{X}, k\mathcal{L}) \otimes \mathbb{R} : \sup_X \|s_k\|_\phi \leq 1 \right\}, \tag{2.5}$$

where the volume is computed with respect to the Lebesgue measure, normalized such that a fundamental domain of the lattice $H^0(\mathcal{X}, k\mathcal{L})$ has unit volume. Here $H^0(\mathcal{X}, k\mathcal{L}) \otimes \mathbb{R}$ may be identified with the subspace of real sections in $H^0(X, kL)$. If the metric on L has positive curvature current, then, by the arithmetic Hilbert–Samuel theorem [GS92, Zha95],

$$\widehat{\mathrm{vol}}_\chi(\overline{\mathcal{L}}) = \frac{\overline{\mathcal{L}}^{n+1}}{(n+1)!}, \tag{2.6}$$

where $\overline{\mathcal{L}}^{n+1}$ denotes the top arithmetic intersection number in the sense of Gillet and Soulé [GS90], which, defines the *height* of $\mathcal{X}$ with respect to $\overline{\mathcal{L}}$ [Fal91, BGS94]. For the purpose of the present paper, formula (2.5) may be taken as the definition of $\overline{\mathcal{L}}^{n+1}$ (arithmetic intersections between $n+1$ metrized line bundles could then be defined by polarization). More generally, $\widehat{\mathrm{vol}}_\chi(\overline{\mathcal{L}})$ is naturally defined for $\mathbb{Q}$-line bundles, since it is homogeneous with respect to tensor products of $\overline{\mathcal{L}}$:

$$\widehat{\mathrm{vol}}_\chi(m\overline{\mathcal{L}}) = m^{n+1}\widehat{\mathrm{vol}}_\chi(\overline{\mathcal{L}}), \quad \text{if } m \in \mathbb{Z}_+. \tag{2.7}$$

https://doi.org/10.1112/S0010437X2400736X Published online by Cambridge University Press

Moreover, $\widehat{\mathrm{vol}}_\chi(\overline{\mathcal{L}})$ is additively equivariant with respect to scalings of the metric:

$$\widehat{\mathrm{vol}}_\chi(\mathcal{L},\phi+\lambda)=\widehat{\mathrm{vol}}_\chi(\overline{\mathcal{L}})+\frac{\lambda}{2}\mathrm{vol}(L),\quad \text{if } \lambda\in\mathbb{R}, \tag{2.8}$$

as follows directly from the definition.

2.3 Upper bounds on the χ-arithmetic volume versus K-semistability of Fano varieties

We are now ready to prove the following theorem, relating upper bounds on the χ-arithmetic volume of a metrized integral model of $(X,-K_X)$ to K-semistability.

Theorem 2.5. *Let $(\mathcal{X},\mathcal{L})$ be a polarized arithmetic variety such that X is a Fano variety and $L=-K_X$. Then the following statements are equivalent.*

(1) *$(X,-K_X)$ is K-semistable.*
(2) *The supremum of $\widehat{\mathrm{vol}}_\chi(\mathcal{L},\phi)$ over all continuous volume-normalized metrics ϕ on $-K_X$ is finite.*
(3) *The supremum of $\widehat{\mathrm{vol}}_\chi(\mathcal{L},\phi)$ over all continuous volume-normalized metrics ϕ on $-K_X$, which are invariant under complex conjugation, is finite.*

Moreover, $(X,-K_X)$ is K-polystable if and only if the supremum in item (2) above is attained at some locally bounded metric ψ in $\mathrm{PSH}(-K_X)$. In general, a locally bounded metric ψ in $\mathrm{PSH}(-K_X)$ attains the supremum in item (2) above if and only if it is a Kähler–Einstein metric.

Recall that on any complex projective variety X which is defined over $\mathbb{R}$ there is a globally defined complex conjugation map (whose orbits on X correspond to the maximal ideals of the scheme $X_{\mathbb{R}}$) and in Arakelov geometry it is often assumed that the metrics are invariant under complex conjugation [SABK92].

Before embarking on the proof we recall the definition of the (normalized) *Ding functional* on $\mathcal{C}^0(-K_X)\Cap\mathrm{PSH}(-K_X)$, introduced in [Din88], which depends on the choice of a reference metric ψ_0 in $\mathcal{C}^0(-K_X)\Cap\mathrm{PSH}(-K_X)$:

$$\hat{\mathcal{D}}_{\psi_0}(\psi):=-\frac{1}{\mathrm{vol}(-K_X)}\mathcal{E}_{\psi_0}(\psi)-\log\int_X e^{-\psi}, \tag{2.9}$$

where the functional $\mathcal{E}_{\psi_0}$ is a primitive of $(dd^c\psi)^n/n!$ (see formula (2.12)). More generally, as shown in [BBEGZ19] using the monotonicity of $\mathcal{E}_{\psi_0}$, $\hat{\mathcal{D}}_{\psi_0}(\psi)$ can be extended to the space $\mathcal{E}^1(-K_X)$ of all metrics in $\mathrm{PSH}(-K_X)$ with finite energy and a finite energy metric ψ minimizes $\hat{\mathcal{D}}_{\psi_0}(\psi)$ if and only if ψ is a Kähler–Einstein metric, i.e. $dd^c\psi$ defines a Kähler metric on the regular locus of X with constant positive Ricci curvature. When ψ is volume-normalized this equivalently means that

$$\frac{(dd^c\psi)^n}{\mathrm{vol}(-K_X)n!}=e^{-\psi}$$

on the regular locus of X. Identity (2.6) was extended to finite energy metrics in [BF14]. But for our purposes it will be enough to work with continuous metrics.

Remark 2.6. In general, any Kähler–Einstein metric ψ in $\mathcal{E}^1(-K_X)$ is locally bounded [BBEGZ19]. In the toric case this implies that ψ is, in fact, continuous [CGS19, Proposition 4.1].

By introducing an arithmetic version of the Ding functional we show that item (2) in the previous theorem is equivalent to the normalized Ding functional $\hat{\mathcal{D}}_{\psi_0}$ being bounded from below on $\mathcal{C}^0(-K_X)\Cap\mathrm{PSH}(-K_X)$ (which is equivalent to lower boundedness of the Mabuchi functional;

https://doi.org/10.1112/S0010437X2400736X Published online by Cambridge University Press

see (4.7)). By [Li17] this is equivalent to K-semistability when X is non-singular. In the proof of Theorem 2.5 we explain how to extend this result to general Fano varieties, leveraging the very recent solution of the Yau–Tian–Donaldson conjecture for singular Fano varieties [Li22, LXZ22]. The equivalence with item (3) leverages the recent result [Zhu21].

2.3.1 *Proof of Theorem 2.5.* We start with two lemmas. First, to a given continuous metric ϕ on L we associate, following [BB10], a continuous psh metric ψ on L defined as the following pointwise envelope:

$$P\phi := \sup\{\psi : \psi \text{ psh}, \psi \leq \phi\}. \tag{2.10}$$

Remark 2.7. More generally, when L is big the envelope above has to be replaced by its upper semicontinuous regularization in order to obtain a psh metric. However, when L is an ample line bundle over a normal variety X, as we assume here, the envelope $P\phi$ is already continuous (see [BE21, Lemma 7.9]).

Lemma 2.8. *Assume that $\mathcal{L}$ is relatively ample and let ϕ be a continuous metric on L. Then the arithmetic χ-volume may be expressed as the following top arithmetic intersection number:*

$$\widehat{\mathrm{vol}}_\chi(\mathcal{L},\phi) = \frac{(\mathcal{L}, P\phi)^{n+1}}{(n+1)!}.$$

Proof. When ϕ is psh the lemma follows directly from [Zha95, Theorem 1.4] (the latter proof reduces to the original arithmetic Hilbert–Samuel theorem in [GS92], where X is assumed non-singular, using a perturbation argument on a resolution of X). In fact, the result [Zha95, Theorem 1.4] applies more generally when $\mathcal{L}$ is merely assumed to be relatively nef over the closed points of Spec$\mathbb{Z}$. Next, the general case follows from the case when ϕ is psh (applied to $P\phi$) by the following simple observation:

$$\sup_X \|s\|_\phi = \sup_X \|s\|_{P\phi}, \quad \text{if } s \in H^0(X, kL),$$

as follows directly from the definition (2.10) of $P\phi$ (see [BB10, Proposition 1.8]). □

In order to state the next lemma consider the following functional on $\mathcal{C}^0(L)\cap \mathrm{PSH}(L)$, defined with respect to a given reference $\psi_0 \in \mathcal{C}^0(L)\cap\mathrm{PSH}(L)$:

$$\mathcal{E}_{\psi_0}(\psi) := \frac{1}{(n+1)!}\int_X (\psi-\psi_0)\sum_{j=0}^n (dd^c\psi)^j\wedge(dd^c\psi_0)^{n-j}. \tag{2.11}$$

Alternatively, the functional $\mathcal{E}_{\psi_0}$ may be characterized as the primitive of the 1-form on $\mathcal{C}^0(L)\cap \mathrm{PSH}(L)$ defined by the measure $(dd^c\psi)^n/n!$:

$$d\mathcal{E}_{\psi_0}(\psi) = \frac{1}{n!}(dd^c\psi)^n, \quad \mathcal{E}_{\psi_0}(\psi_0) = 0. \tag{2.12}$$

It follows directly from the definition of $\mathcal{E}_{\psi_0}(\psi)$ and the classical Hilbert–Samuel formula (2.2) that

$$\mathcal{E}_{\psi_0}(\psi + c) = \mathcal{E}_{\psi_0}(\psi) + c\mathrm{vol}(L), \quad \forall c\in\mathbb{R}. \tag{2.13}$$

The following lemma is an arithmetic refinement of the previous formula.

Lemma 2.9 (Change of metrics formula). *For any two continuous metrics on L, which are invariant under complex conjugation,*

$$\widehat{\mathrm{vol}}_\chi(\mathcal{L},\phi_1) - \widehat{\mathrm{vol}}_\chi(\mathcal{L},\phi_2) = \tfrac{1}{2}(\mathcal{E}_{\psi_0}(P\phi_1) - \mathcal{E}_{\psi_0}(P\phi_2)). \tag{2.14}$$

https://doi.org/10.1112/S0010437X2400736X Published online by Cambridge University Press

Proof. When ϕ_i are psh this is well known and follows from basic properties of arithmetic intersection numbers; see formula (5.2) or [Oda18, Proposition 2.2]). Alternatively, the result follows from the previous lemma combined with [BB10, Theorem A]. In order to check that the multiplicative normalizations adopted here are compatible, note that the scaling relations (2.8) and (2.13) are indeed compatible. □

2.3.2 *Conclusion of the proof of Theorem 2.5.* Consider the following functional on the space $\mathcal{C}^0(-K_X)$ of continuous metrics on $-K_X$:

$$\hat{\mathcal{D}}_{\mathbb{Z}}(\phi) := -2\frac{\widehat{\mathrm{vol}}_X(\mathcal{L},\phi)}{\mathrm{vol}(-K_X)} - \log\int_X e^{-\phi}. \tag{2.15}$$

Since this functional is invariant under scalings of the metric, $\phi \mapsto \phi + c$, the finiteness statement in the second point of the theorem amounts to showing that the infimum of $\hat{\mathcal{D}}_{\mathbb{Z}}(\phi)$ over $\mathcal{C}^0(-K_X)$ is finite. Now fix a continuous psh metric ψ_0 on $-K_X$ and consider the following extension of the normalized Ding functional (2.9) to all of $\mathcal{C}^0(-K_X)$:

$$\hat{\mathcal{D}}_{\psi_0}(\phi) := -\frac{1}{\mathrm{vol}(-K_X)}\mathcal{E}_{\psi_0}(P\phi) - \log\int_X e^{-\phi}. \tag{2.16}$$

Combining the previous two lemmas reveals that

$$\hat{\mathcal{D}}_{\mathbb{Z}}(\phi) = \hat{\mathcal{D}}_{\psi_0}(\phi) + C_0, \quad C_0 := -\frac{2\big(\mathcal{L},\psi_0\big)^{n+1}}{\mathrm{vol}(-K_X)(n+1)!}. \tag{2.17}$$

Next, observe that

$$\inf_{\mathcal{C}^0(-K_X)} \hat{\mathcal{D}}_{\psi_0} = \inf_{\mathcal{C}^0(-K_X)\Cap \mathrm{PSH}(-K_X)} \hat{\mathcal{D}}_{\psi_0}. \tag{2.18}$$

Indeed, this follows directly from the fact that the operator $\phi \mapsto P\phi$ from $\mathcal{C}^0(L)$ to $\mathcal{C}^0(L) \Cap \mathrm{PSH}(L)$ is increasing and satisfies $P^2 = P$.

(3) $\Longrightarrow$ (1). Let us first recall how item (2) implies item (1). Item (2) implies, thanks to the identities (2.17) and (2.18), that the infimum of $\hat{\mathcal{D}}_{\psi_0}$ over $\mathcal{C}^0(-K_X) \Cap \mathrm{PSH}(-K_X)$ is finite. Thus it follows from results in [Ber16] that $(X, -K_X)$ is K-semistable. Let us next show how to refine the proof in [Ber16] to show the stronger statement (3) $\Longrightarrow$ (1). More generally, we will show that when X is defined over the real field $\mathbb{R}$, X is K-semistable if the infimum of $\hat{\mathcal{D}}_{\psi_0}$ over the space $\overline{\mathcal{C}^0(-K_X)} \Cap \mathrm{PSH}(-K_X)$ is finite, where $\overline{\mathcal{C}^0(L)}$ denotes the subspace of $\mathcal{C}^0(L)$ consisting of metrics which are invariant under complex conjugation. To this end, let us first summarize the main steps in the proof in [Ber16]. A test configuration $(\mathscr{X}, \mathscr{L})$ for $(X, -K_X)$ and a given metric ϕ for $-K_X$ in $\mathcal{C}^0(-K_X) \Cap \mathrm{PSH}(-K_X)$ determine a ray ϕ_t in $\mathrm{PSH}(-K_X)$ emanating from ϕ parametrized by $t \in [0, \infty[$ (i.e. $\phi_0 = \phi$). Using the notation in formula (2.4), the ray ϕ_t is defined by

$$\phi_{-\log|\tau|} = \rho(\tau)^*(\Phi_{|\mathscr{X}_\tau}), \quad \tau \in \mathbb{C}^*,$$

where Φ is the S^1-invariant metric on the restriction of $\mathcal{L}$ to the inverse image $\pi^{-1}(\mathbb{D})$ in $\mathcal{X}$ of the unit disc $\mathbb{D} \subset \mathbb{C}$ defined by

$$\Phi := \sup\big\{\Psi : \Psi_{|\pi^{-1}(\partial\mathbb{D})} = \phi,\ \Psi \in \mathcal{C}^0(\mathcal{L}) \Cap \mathrm{PSH}(\mathcal{L}_{|\pi^{-1}(\mathbb{D})})\big\}, \tag{2.19}$$

where we have used the $\mathbb{C}^*$-action ρ to identify X with X_τ for any τ in the unit circle $\partial\mathbb{D}$. By [Ber16, Theorem 1.3],

$$\mathrm{vol}(-K_X)^{-1}\mathrm{DF}(\mathscr{X},\mathscr{L}) \geq \lim_{t\to\infty}\left(t^{-1}\hat{\mathcal{D}}_{\phi_0}(\phi_t)\right).$$

When $\hat{\mathcal{D}}_{\phi_0}(\phi_t)$ is bounded from below this means that $\mathrm{DF}(\mathscr{X},\mathscr{L})\geq 0$, showing that X is K-semistable. Now assume that X is defined over the real field $\mathbb{R}$. Then it follows from [Zhu21, Theorem 1.1] that in order to check K-semistability of $(X,-K_X)$ it is enough to consider test configurations $(\mathscr{X},\mathscr{L})$ defined over $\mathbb{R}$. Thus, we just have to verify that for such test configurations, if the given metric ϕ is taken to be in $\overline{\mathcal{C}^0(-K_X)}\Cap\mathrm{PSH}(-K_X)$, then the ray ϕ_t remains in $\overline{\mathcal{C}^0(-K_X)}\Cap\mathrm{PSH}(-K_X)$, for all $t>0$. Since $(\mathscr{X},\mathscr{L})$ is defined over $\mathbb{R}$ there is a complex conjugation map F from $\mathscr{X}$ to $\mathscr{X}$ (which lifts to $\mathscr{L}$) and thus it is enough to show that $F^*\phi=\phi$ implies that $F^*\Phi=\Phi$. But this follows from the definition (2.19) of Φ only using that F^* preserves the psh property of a metric (as follows from a direct local calculation that reduces to the fact that the Laplacian $i\partial_z\partial_{\bar z}$ in $\mathbb{C}$ is invariant under $z\mapsto\bar z$).

(1) $\implies$ (2). Recall that any K-semistable normal Fano variety (i.e. such that $(X,-K_X)$ is K-semistable) has log terminal singularities [Oda13b, Theorem 1.3]. In the case when X is non-singular it was shown in [Li17] that if X is K-semistable, then the infimum of the Ding functional $\hat{\mathcal{D}}_{\psi_0}$ over $\mathcal{C}^0(-K_X)\Cap\mathrm{PSH}(-K_X)$ is finite. Thus, by formula (2.18), so is the infimum of $\hat{\mathcal{D}}_{\psi_0}$ over $\mathcal{C}^0(-K_X)$. The proof in [Li17] relied, in particular, on the resolution of the Yau–Tian–Donaldson conjecture in [CDS15] for Fano manifolds. But thanks to the recent resolution of the Yau–Tian–Donaldson conjecture for singular Fano varieties the proof in [Li17] can be extended to singular Fano varieties, mutatis mutandis. We briefly summarize the argument, using Deligne pairings as in [Ber16] (rather than the Bott–Chern classes used in [Li17]). The starting point is the result [LWX21, Theorem 1.3], saying that if X is K-semistable then there exists a test configuration $(\mathscr{X},\mathscr{L})$ for $(X,-K_X)$ whose central fiber X_0 is given by a K-polystable Fano variety. More precisely, the test configuration is *special* in the sense that $\mathscr{L}$ is the relative anti-canonical line bundle. Since the central fiber X_0 of $\mathscr{X}$ is K-polystable it admits, by the solution of the Yau–Tian–Donaldson conjecture for singular Fano varieties [LXZ22] (building on [Li22]), a Kähler–Einstein metric ϕ_{KE}. It thus follows from [BBEGZ19, Theorem 4.8] that the Ding functional is bounded from below on $\mathcal{C}^0(-K_{X_0})\Cap\mathrm{PSH}(-K_{X_0})$. More precisely, its infimum is attained at the Kähler–Einstein metric ϕ_{KE}:

$$\inf_{\mathcal{C}^0(-K_{X_0})\Cap\mathrm{PSH}(-K_{X_0})}\hat{\mathcal{D}}=\hat{\mathcal{D}}(\phi_{\mathrm{KE}})>-\infty. \tag{2.20}$$

Now, given a metric ϕ in $\mathcal{C}^0(-K_X)\Cap\mathrm{PSH}(-K_X)$, let Φ be the corresponding metric on $\mathscr{L}\to\pi^{-1}(\mathbb{D})$ defined by formula (2.19). It induces a metric on the $(n+1)$-fold Deligne pairing $\langle\mathscr{L},\mathscr{L},\ldots,\mathscr{L}\rangle\to\mathbb{D}$ that we denote by $\langle\Phi\rangle$ (see [Ber16, § 2.3]). Consider the corresponding twisted metric on $-\langle\mathscr{L},\mathscr{L},\ldots,\mathscr{L}\rangle\to\mathbb{D}$ defined by

$$-\langle\Phi\rangle-\log\int_{X_\tau}e^{-\Phi_{|X_\tau}},$$

dubbed the *Ding metric* in [Ber16]. Fixing a trivialization $S(\tau)$ of $\langle\mathscr{L},\mathscr{L},\ldots,\mathscr{L}\rangle\to\mathbb{D}$, we may identify this metric with a function $\psi(\tau)$ on $\mathbb{D}$:

$$\psi(\tau):=\log(\|S(\tau)\|^2_{\langle\Phi\rangle})-\log\int_{X_\tau}e^{-\Phi_{|X_\tau}}.$$

https://doi.org/10.1112/S0010437X2400736X Published online by Cambridge University Press

For a fixed τ this metric coincides with the normalized Ding functional $\hat{\mathcal{D}}(\phi_\tau)$ up to an additive constant depending on τ (by the 'change of metrics formula' for Deligne pairing; see [Ber16, § 2.3]). In particular, there exists $a \in \mathbb{R}$ such that

$$\psi(1) := \hat{\mathcal{D}}_{\psi_0}(\phi) + a, \quad \psi(0) \geq b := \log(\|S(0)\|^2_{\langle\phi_{\mathrm{KE}}\rangle}) - \log \int_{X_0} e^{-\phi_{\mathrm{KE}}}, \tag{2.21}$$

using (2.20) in the inequality. As shown in [Ber16, Proposition 3.5], $\psi(\tau)$ is subharmonic on $\mathbb{D}$ and the first term $\langle\Phi\rangle$ is continuous on $\mathbb{D}$ (as follows from [Mor99, Theorem A]; see the proof of [Ber16, Proposition 3.6]). Moreover, the second term is also continuous on $\mathbb{D}$, as shown when X is non-singular in [Li17, Lemma 1.9] and in general in [LWX18, Lemma 7.1]. As a consequence,

$$\psi(0) \leq \int_{\partial D} \psi \, d\theta = \psi(1),$$

using that $\psi(\tau)$ is S^1-invariant in the last equality. Finally, invoking formula (2.21) shows that $\hat{\mathcal{D}}_{\psi_0}(\phi)$ is uniformly bounded from below, as desired.

2.4 Compatibility of Conjecture 1.1 with taking products

The previous theorem shows, in particular, that the K-semistability assumption in Conjecture 1.1 is necessary. We next show that the conjecture is compatible with taking products.

Proposition 2.10. *Let $m \geq 2$ and $\mathcal{X}_1, \ldots, \mathcal{X}_m$ be arithmetic Fano varieties which are K-semistable over $\mathbb{C}$. Assume that the inequality in Conjecture 1.1 holds for all $\mathcal{X}_i$ (for any volume-normalized metrics on $-K_{X_i}$ with positive curvature current). Then the inequality holds for $\mathcal{X}_1 \times \cdots \times \mathcal{X}_m$ with strict inequality (for any volume-normalized metric on $-K_{X_1 \times \cdots \times X_m}$ with positive curvature current). More precisely,*

$$\widehat{\mathrm{vol}}_\chi\big(\overline{-\mathcal{K}_{\mathcal{X}_1 \times \cdots \times \mathcal{X}_m}}\big) < \widehat{\mathrm{vol}}_\chi(\overline{-\mathcal{K}_{\mathbb{P}^n_\mathbb{Z}}}).$$

Proof. By a simple induction argument it is enough to consider the case when $m = 2$. Note that, in general, given two polarized metrized arithmetic varieties $(\mathcal{X}_i, \overline{\mathcal{L}_i})$ of relative dimension n_i

$$\frac{\widehat{\mathrm{vol}}_\chi\big(\rho_1^*\overline{\mathcal{L}_1} \otimes \rho_2^*\overline{\mathcal{L}_2}\big)}{\mathrm{vol}\big(\rho_1^*\overline{\mathcal{L}_1} \otimes \rho_2^*\overline{\mathcal{L}_2}\big)} = \frac{\widehat{\mathrm{vol}}_\chi(\overline{\mathcal{L}_1})}{\mathrm{vol}(\overline{\mathcal{L}_1})} + \frac{\widehat{\mathrm{vol}}_\chi(\overline{\mathcal{L}_2})}{\mathrm{vol}(\overline{\mathcal{L}_2})}, \tag{2.22}$$

where ρ_1 and ρ_2 denote the natural morphisms from $\mathcal{X}_1 \times \mathcal{X}_2$ to $\mathcal{X}_1$ and $\mathcal{X}_2$, respectively (as follows readily from formula (2.5)).

Assume now that the inequality in Conjecture 1.1 holds for $\overline{-\mathcal{K}_{\mathcal{X}_1}}$ and $\overline{-\mathcal{K}_{\mathcal{X}_2}}$. Endow $-K_{X_1 \times X_2}$ with the induced product metric (which is volume-normalized, since the metrics on $-K_{X_i}$ are assumed to be volume-normalized). Identity (2.22) yields

$$\widehat{\mathrm{vol}}_\chi\big(\overline{-\mathcal{K}_{\mathcal{X}_1 \times \mathcal{X}_2}}\big) = \widehat{\mathrm{vol}}_\chi\big(\overline{-\mathcal{K}_{\mathcal{X}_1}}\big)\mathrm{vol}(-K_{X_2}) + \widehat{\mathrm{vol}}_\chi\big(\overline{-\mathcal{K}_{\mathcal{X}_1}}\big)\mathrm{vol}(-K_{X_1}).$$

Accordingly, by assumption,

$$\widehat{\mathrm{vol}}_\chi\big(\overline{-\mathcal{K}_{\mathcal{X}_1 \times \mathcal{X}_2}}\big) \leq \widehat{\mathrm{vol}}_\chi\big(\overline{-\mathcal{K}_{\mathbb{P}^{n_1}_\mathbb{Z}}}\big)\mathrm{vol}(-K_{X_2}) + \widehat{\mathrm{vol}}_\chi\big(\overline{-\mathcal{K}_{\mathbb{P}^{n_2}_\mathbb{Z}}}\big)\mathrm{vol}(-K_{X_1}),$$

where the projective spaces have been induced by the volume-normalized Fubini–Study metric and we have used that $\widehat{\mathrm{vol}}_\chi(\overline{-\mathcal{K}_{\mathbb{P}^n_\mathbb{Z}}})$ is positive for any n (as shown in Lemma 3.6). Hence, applying Fujita's inequality (1.1) yields

$$\widehat{\mathrm{vol}}_\chi\big(\overline{-\mathcal{K}_{\mathcal{X}_1 \times \mathcal{X}_2}}\big) \leq \widehat{\mathrm{vol}}_\chi\big(\overline{-\mathcal{K}_{\mathbb{P}^{n_1}_\mathbb{Z}}}\big)\mathrm{vol}(-K_{\mathbb{P}^{n_1}_\mathbb{C}}) + \widehat{\mathrm{vol}}_\chi\big(\overline{-\mathcal{K}_{\mathbb{P}^{n_2}_\mathbb{Z}}}\big)\mathrm{vol}(-K_{\mathbb{P}^{n_2}_\mathbb{C}}).$$

https://doi.org/10.1112/S0010437X2400736X Published online by Cambridge University Press

But, the right-hand side above equals $\widehat{\mathrm{vol}}_\chi(\overline{-\mathcal{K}_{\mathbb{P}^{n_1}_{\mathbb{Z}}\times\mathbb{P}^{n_2}_{\mathbb{Z}}}})$ (by identity (2.22)), which is strictly smaller than $\widehat{\mathrm{vol}}_\chi(\overline{-\mathcal{K}_{\mathbb{P}^{n_1+n_2}_{\mathbb{Z}}}})$, by the toric case, considered in § 3.2.2.

All that remains is thus to show that the supremum of $\widehat{\mathrm{vol}}_\chi(\overline{-\mathcal{K}_{\mathcal{X}_1\times\mathcal{X}_2}})$ over all continuous volume-normalized metrics coincides with the supremum restricted to those which have positive curvature current and are product metrics. First, as shown in the proof of Theorem 2.5 we may restrict to those with positive curvature current. To prove that we may restrict to product metrics first consider the case when $(X_i,-K_{X_i})$ are both K-polystable. They thus admit Kähler–Einstein metrics and the corresponding product metric is Kähler–Einstein on $X_1\times X_2$ and, as a consequence, realizes the supremum of $(\overline{-\mathcal{K}_{\mathcal{X}_1\times\mathcal{X}_2}})^{n+1}$, by Theorem 2.5 (strictly speaking, in the singular case the Kähler–Einstein metric is merely known to be locally bounded, but it can, in a standard way, be approximated by continuous ones). Finally, in the case when $(X_i,-K_{X_i})$ are merely K-semistable we will use the following general observation. If X_1 and X_2 are K-semistable Fano varieties over $\mathbb{C}$, then the infimum of the Ding functional (formula (2.9)) corresponding to $X_1\times X_2$ coincides with the infimum over product metrics. To prove this, first recall the definition of the twisted Ding normalized functional $\hat{\mathcal{D}}_{\psi_0,\gamma}$ corresponding to a given locally bounded psh metric ψ_0 and $\gamma\in]0,1]$:

$$\hat{\mathcal{D}}_{\psi_0,\gamma}(\psi)=-\frac{1}{\mathrm{vol}(-K_X)}\mathcal{E}_{\psi_0}(\psi)-\log\int_X e^{-(\gamma\psi+(1-\gamma)\psi_0)}.$$

By Hölder's inequality $\hat{\mathcal{D}}_{\psi_0,\gamma}(\psi)$ is decreasing in γ. Since, as shown in the proof of Theorem 2.5, $\hat{\mathcal{D}}_{\psi_0,1}$ is bounded from below when X is K-semistable, so is $\hat{\mathcal{D}}_{\psi_0,\gamma}(\psi)$ for any $\gamma\in]0,1[$. More precisely, $\hat{\mathcal{D}}_{\psi_0,\gamma}(\psi)$ is coercive for any given $\gamma\in]0,1[$ (see the proof of [Ber13, Corollary 3.6]) and thus $\hat{\mathcal{D}}_{\psi_0,\gamma}$ admits a minimizer ψ_γ and the minimizers are precisely the solutions to the twisted Kähler–Einstein equation

$$\frac{(dd^c\psi)^n/n!}{\mathrm{vol}(-K_X)}=\frac{e^{-(\gamma\psi+(1-\gamma)\psi_0)}}{\int_X e^{-(\gamma\psi+(1-\gamma)\psi_0)}}$$

(see [BBEGZ19]). Thus, given two K-semistable Fano varieties X_1 and X_2 and $\gamma\in]0,1[$, we may take two twisted Kähler–Einstein metrics $\psi_\gamma^{(1)}$ and $\psi_\gamma^{(2)}$ on $-K_{X_1}$ and $-K_{X_2}$, respectively. The corresponding product metric ψ_γ on $-K_{X_1\times X_2}$ is a twisted Kähler–Einstein metric and thus minimizes the twisted normalized Ding functional $\hat{\mathcal{D}}_{\psi_0,\gamma}$ on $X_1\times X_2$. Moreover, as $\gamma\to 1$,

$$\hat{\mathcal{D}}_{\psi_0}(\psi_\gamma)\to\inf\hat{\mathcal{D}}_{\psi_0}. \tag{2.23}$$

Indeed, $\gamma\to\hat{\mathcal{D}}_{\psi_0,\gamma}(\psi)$ is continuous and concave on $]0,1]$ for a fixed continuous metric ψ, by Hölder's inequality. The convergence (2.23) thus follows from Lemma 2.11 below. Finally, since in our setup ψ_γ is a product metric it follows that the infimum of $\hat{\mathcal{D}}_{\psi_0}$ coincides with the infimum restricted to product metrics, as desired. □

In the proof we used the following elementary result about convex functions (applied to $-f$).

Lemma 2.11. *Let $f(t)$ be a function $[0,1]\to]-\infty,\infty]$ of the form*

$$f(t)=\sup_{p\in\mathcal{P}}(f_p(t)),$$

where $f_p(t)$ is a family of continuous convex functions on $[0,1]$, parametrized by a set $\mathcal{P}$. Then $f(t)$ is continuous on $[0,1]$.

Proof. This is standard, but for completeness we provide a proof. Recall that the supremum of a family of continuous functions is lower semicontinuous. Hence, it will be enough to show that $f(t)$ is upper semicontinuous. To this end, observe that since $t \mapsto f_p(t)$ is convex it follows that $f(t)$ is also convex. But any convex function on $[0,1]$ is upper semicontinuous. Indeed, f is (Lipschitz) continuous on $]0,1[$, since it is convex there. By symmetry, it is thus enough to prove upper continuity at $t = 1$. Now, since $f(t)$ is convex we have, given $t \in]0,1[$, that

$$f(1) \geq f(t) + (1-t)\partial f(t)$$

for any subgradient $\partial f(t)$ at t, i.e. any one-sided derivative at t. But since $f(t)$ is convex, $\partial f(t) \geq \partial f(t_0)$ for any fixed t_0 such that $t_0 \leq t$. Hence, $f(1) \geq f(t) + (1-t)\partial f(t_0)$ and letting $t \to 1$ thus shows that $f(1)$ is greater than or equal to the limit supremum of $f(t)$ as $t \to 1$, as desired. □

3. Sharp height inequalities in the toric case

We now specialize to the case when X is a toric Fano variety.

3.1 The toric setup

We start by recalling the notation for toric metrics employed in [BB13] and the relation to the canonical toric integral model.

3.1.1 *The moment polytope $P(L)$.* Let X be an n-dimensional complex projective toric variety, i.e. a complex projective variety endowed with an action of the n-dimensional complex torus $\mathbb{C}^{*n}$ with an open dense orbit. We shall denote by T_c the complex torus and by T the real maximal compact subtorus of T_c, i.e. $T = (S^1)^n$. Let L be a toric ample line bundle, i.e. an ample line bundle over X endowed with a T_c-action covering the action of T_c on X. It induces a bounded convex polytope $P(L)$ in $\mathbb{R}^d$ with non-empty interior, defined as follows. Consider the induced action of the group T_c on the space $H^0(X, kL)$ of global holomorphic sections of $kL \to X$ (for k a given positive integer). Decomposing the action of T_c according to the corresponding one-dimensional representations e^m, labeled by $m \in \mathbb{Z}^n$,

$$H^0(X, kL) = \bigoplus_{m \in B_k} \mathbb{C}e^m, \tag{3.1}$$

the lattice polytope $P_{(X,L)}$ may be defined as the convex hull of $k^{-1}B_k$ in $\mathbb{R}^n$, for k sufficiently large. More generally, by homogeneity, $P_{(X,L)}$ is defined for any ample $\mathbb{Q}$-line bundle.

In particular, if X is Fano, then the polytope $P(-K_X)$ has vertices in $\mathbb{Q}^n$ and may be represented as follows:

$$P(-K_X) = \{p \in \mathbb{R}^n : \langle l_F, p\rangle \geq -1,\ \forall F\}, \tag{3.2}$$

where F ranges over all facets of $P(-K_X)$ and l_F denotes the unique primitive element in $\mathbb{Z}^n$ which is an interior normal to the facet F (i.e. $P(-K_X)$ is the dual of the polytope with primitive vertices l_F). Conversely, any such polytope corresponds to a Fano variety X [CLS11, BB13].

Example 3.1. When $X = \mathbb{P}^n$ the polytope $P(-K_X)$ is $(n+1)(\Sigma_n - (1,\dots,1))$ where Σ_n denotes the n-dimensional unit simplex. An infinite family of two-dimensional toric Fano varieties $X_{p,q}$, parametrized by two prime numbers p and q, is obtained by letting $P(-K_{X_{p,q}})$ be the polytope which is dual to the polytope with the four primitive vertices $(\pm p, \pm q)$. In particular, $\mathrm{vol}(-K_{X_{p,q}}) = 2/(pq)$ tends to zero when pq tends to infinity.

Remark 3.2. From an invariant point of view, the real vector space $\mathbb{R}^n$ above arises as $M \otimes_{\mathbb{Z}} \mathbb{R}$, where M is the lattice $\mathrm{Hom}(T_c, \mathbb{C}^*)$ of characters of the group T_c (cf. [CLS11]).

https://doi.org/10.1112/S0010437X2400736X Published online by Cambridge University Press

3.1.2 *Logarithmic coordinates and the Legendre transform ϕ^* of a metric ϕ on L.* Since X is toric we can identify T_c with its open orbit in X. Let Log be the map from T_c to $\mathbb{R}^n$ defined by

$$\mathrm{Log}: T_c \to \mathbb{R}^n, \quad \mathrm{Log}(z) := x := (\log(|z_1|^2), \dots, \log(|z_n|^2)).$$

The real compact torus T acts transitively on its fibers. We will refer to x as the (real) *logarithmic coordinates* on T_c. Let L be a toric ample line bundle over X and assume that P contains the origin, $0 \in P$, and denote by e^0 the corresponding T-invariant element in $H^0(X, kL)$. Any continuous T-invariant metric $\|\cdot\|$ on L induces a continuous function on $\mathbb{R}^n$ which we shall denote by $\phi(x)$, defined as

$$\phi(x) := -\log\left(\|e^0\|^2(z)\right), \quad z \in T_c \Subset X, \quad x := \mathrm{Log}\ z.$$

Thus, in the present additive notation ϕ for metrics we have $\phi(x) = \phi_U(z)$, when $U = T_c$, abusing notation slightly. The Legendre transform of $\phi(x)$, which defines a lower-semicontinuous convex function on $\mathbb{R}^n$ (taking values in $]-\infty, \infty]$) will be denoted by ϕ^*:

$$\phi^*(p) := \sup_{x \in \mathbb{R}^n} \langle p, x\rangle - \phi(x).$$

A T-invariant continuous metric ψ on L is psh if and only if the corresponding function $\psi(x)$ on $\mathbb{R}^n$ is convex (if and only if $\psi(x) = \psi^{**}(x)$). We will denote by $\psi_{P(L)}$ the unique continuous convex function on $\mathbb{R}^n$ whose Legendre transform is equal to 0 on $P(L)$ and equal to ∞ on the complement of $P(L)$:

$$\psi_{P(L)}(x) := \sup_{p \in P(L)} \langle p, x\rangle \quad (\psi^*_{P(L)} = 0 \text{ on } P,\ \psi^*_{P(L)} = \infty \text{ on } P(L)^c). \tag{3.3}$$

It corresponds to a continuous psh metric on L (see the proof of [BB13, Proposition 3.3]) and it will be used as a canonical reference metric in the present toric setup. It follows that for any other continuous metric ϕ on L,

$$\phi - \psi_{P(L)} \in L^\infty(\mathbb{R}^n), \quad P(L) = \overline{\{\phi^* < \infty\}}. \tag{3.4}$$

Remark 3.3. From an invariant point of view the logarithm coordinates take values in $N \otimes \mathbb{R}$, where N is the lattice $\mathrm{Hom}(\mathbb{C}^*, T_c)$ of one-parameter subgroups of T_c, i.e. the dual of the lattice $\mathrm{Hom}(T_c, \mathbb{C}^*)$ of characters of T_c.

3.1.3 *Pushing forward measures from X to $\mathbb{R}^n$.* For any T-invariant continuous psh metric ψ on L the push-forward of the measure $(dd^c\psi)^n/n!$ on L under the map Log is given by

$$\mathrm{Log}\left(\frac{(dd^c\psi)^n}{n!}\right) = \det(\nabla^2\phi)\, dx$$

(since the integral along the T^n-fibers equals $(2\pi)^n$). The measure on the right-hand side is defined in the weak sense of Alexandrov. Since the closure of the image of $\mathbb{R}^n$ under the subgradient map of ϕ equals P it follows that

$$\mathrm{vol}(L) = \int_P dy := \mathrm{Vol}(P),$$

where dy is Lebesgue measure. Next consider the case when $L = -K_X$. Then

$$e_0 := z_1 \frac{\partial}{\partial z_1} \wedge \cdots \wedge z_n \frac{\partial}{\partial z_n} \tag{3.5}$$

defines a T_c-invariant global holomorphic section of $-K_X$, trivializing $-K_X$ over $U := \mathbb{C}^{*n}$. We can thus identify a continuous metric ϕ on $-K_X$ with the corresponding function ϕ_U on $\mathbb{C}^{*n}$

https://doi.org/10.1112/S0010437X2400736X Published online by Cambridge University Press

(formula (2.1)) and volume form on X (formula (2.3)) expressed as follows on $\mathbb{C}^{*n}$, with respect to the local holomorphic coordinate $\log z$:

$$e^{-\phi_U}\left(\frac{i}{2}\right)^n d(\log z_1)\wedge d(\log \overline{z}_1)\wedge\cdots\wedge d(\log z_n)\wedge d(\log \overline{z}_n),$$

symbolically denoted by $e^{-\phi}$. Using again that the integral along the T^n-fibers equals $(2\pi)^n$ yields

$$\int_X e^{-\phi} = \pi^n \int_{\mathbb{R}^n} e^{-\phi(x)}\,dx. \tag{3.6}$$

3.1.4 *K-semistability and toric Kähler–Einstein metrics.* We recall the following result, which is a combination of [BB13, Theorem 1.2] and [Ber16, Corollary 1.2] (which are formulated in terms of T_c-equivariant K-polystability and K-polystability, respectively).

Proposition 3.4. *Let X be a toric Fano variety. The following statements are equivalent.*

- *X is K-semistable.*
- *X is K-polystable.*
- *X admits a T-invariant Kähler–Einstein metric.*
- *The barycenter of $P(-K_X)$ coincides with the origin* 0.

3.1.5 *The arithmetic χ-volume of a toric metric.* Any toric ample line bundle $L\to X$ admits a canonical integral model $\mathcal{L}\to\mathcal{X}$ over $\mathbb{Z}$ with $\mathcal{X}$ normal (see [Mai00, § 2] and [BPS14, Def 3.5.6]).

The following result is a special case of the main result of [BPS14, Theorem 3] (combined with Lemma 2.8):

Proposition 3.5. *Let $L\to X$ be an ample toric line bundle and denote by $(\mathcal{X},\mathcal{L})$ its canonical toric model over $\mathbb{Z}$. Assume that ϕ is a continuous T-invariant metric on L. Then*

$$2\widehat{\mathrm{vol}}_\chi(\mathcal{L},\phi) = -\int_{P(L)}\phi^*\,dy.$$

An alternative analytic proof of this formula can also be given, using that the integral lattice $H^0(\mathcal{X},k\mathcal{L})$ in $H^0(X,kL)$ is generated by the T_c-equivariant bases e^m appearing in the decomposition (3.1) [Mai00]. Since this basis is orthonormal with respect to the L^2-norm on $H^0(X,kL)$ induced by the metric $\psi_{P(L)}$ on L, defined by formula (3.3) and the Haar measure on the unit torus $T\Subset X$, applying [BB10, Theorem A] yields

$$\widehat{2\mathrm{vol}}(\mathcal{L},\phi) = \mathcal{E}_{\psi_{P(L)}}(\phi). \tag{3.7}$$

When ϕ is toric the right-hand side above coincides, by [BB13, Proposition 2.9], with the right-hand side of the formula in the previous proposition.

3.1.6 *Arithmetic toric Fano varieties.* Now assume that X is a toric Fano variety, so that $-K_X$ defines an ample $\mathbb{Q}$-line bundle. Then the canonical integral model $\mathcal{X}$ of X over $\mathbb{Z}$ is a normal arithmetic Fano variety, i.e. the relative anti-canonical divisor $-\mathcal{K}$ on $\mathcal{X}$ defines a relatively ample $\mathbb{Q}$-line bundle on $\mathcal{X}$. Indeed, $-\mathcal{K}$ coincides with the canonical integral model $\mathcal{L}$ of $-K_X$. This follows (just as in the function field case considered in [Ber16, Lemma 2.2]) from the fact that the fibers of the structure morphism $\mathcal{X}\to\mathrm{Spec}\mathbb{Z}$ are reduced and irreducible.

3.2 Proof of Theorem 1.2

Given a Fano variety X, let ϕ be a continuous metric on $-K_X$ which is volume-normalized. We will prove the following more general formulation of the inequality in Theorem 1.2 (where the

https://doi.org/10.1112/S0010437X2400736X Published online by Cambridge University Press

psh assumption on ϕ has been dispensed with):

$$\widehat{\mathrm{vol}}_\chi(-\mathcal{K},\phi) \leq \widehat{\mathrm{vol}}_\chi\big(-\overline{\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}}\big),$$

where the metric on $-K_{\mathbb{P}^n}$ is the one induced by the volume-normalized Fubini–Study metric.

A T-invariant continuous metric ϕ will, as above, be identified with a function $\phi(x)$ on $\mathbb{R}^n$. Moreover, if ϕ is volume-normalized then Proposition 3.5 gives

$$\begin{aligned} 2\widehat{\mathrm{vol}}_\chi(-\mathcal{K},\phi)/\mathrm{vol}(-K_X) &= -\hat{\mathcal{D}}_{\mathbb{Z}}(\phi) = -\hat{\mathcal{D}}_{\psi_P}(\phi) \\ &= -\int_P \phi^*\,dy/\mathrm{vol}(-K_X) + \log\int_{\mathbb{R}^n} e^{-\phi(x)}\,dx + n\log\pi, \end{aligned} \tag{3.8}$$

where $\hat{\mathcal{D}}_{\mathbb{Z}}(\phi)$ and $\hat{\mathcal{D}}_{\psi_P}(\phi)$ are the Ding type functionals defined by formula (2.15) and formula (2.16), respectively, and we have used formula (3.6).

We start by recording the following explicit formula for the arithmetic volume of projective space $\mathbb{P}^n$, endowed with a volume-normalized Kähler–Einstein metric (which may be assumed to be the metric induced by the Fubini–Study metric).

LEMMA 3.6. *The following formulas hold for the metrics ϕ_{KE} on the anti-canonical line bundles of $\mathbb{P}^n_{\mathbb{C}}$ induced by a volume-normalized toric Kähler–Einstein metric:*

$$X=\mathbb{P}^n_{\mathbb{C}} \implies 2\widehat{\mathrm{vol}}_\chi(-\mathcal{K},\phi_{\mathrm{KE}}) = \frac{(n+1)^n}{n!}\left((n+1)\sum_{k=1}^n k^{-1} - n + \log\left(\frac{\pi^n}{n!}\right)\right) > 0.$$

Proof. Consider the case when $X=\mathbb{P}^n_{\mathbb{C}}$, whose canonical integral model is given by $\mathcal{X}=\mathbb{P}^n_{\mathbb{Z}}$. The canonical model of the anti-canonical line bundle of $\mathbb{P}^n_{\mathbb{C}}$ is given by $\mathcal{O}(1)^{\otimes n+1}\to\mathbb{P}^n_{\mathbb{Z}}$. As shown in [GS90, § 5.4] (using the induction formula for the height; see also [Sou21, Proposition 3.10]) the height h_{FS} of $\mathcal{O}(1)\to\mathbb{P}^n_{\mathbb{Z}}$ endowed with the Fubini–Study metric ϕ_{FS} is given by

$$h_{\mathrm{FS}} = \frac{1}{2}\sum_{k=1}^n\sum_{m=1}^k m^{-1}.$$

Since $(n+1)\phi_{\mathrm{FS}}$ defines a Kähler–Einstein metric on $-K_{\mathbb{P}^n}$ and $\pi^{-n}\int_{\mathbb{P}^n} e^{-(n+1)\phi_{\mathrm{FS}}} = 1/n!$ this gives

$$\begin{aligned} 2\widehat{\mathrm{vol}}_\chi(-\mathcal{K},\phi_{\mathrm{KE}}) - n\log\pi &= (n+1)^{n+1}\frac{h_{\mathrm{FS}}}{(n+1)!} + \frac{(n+1)^n}{n!}\log\left(\frac{1}{n!}\right) \\ &= \frac{(n+1)^n}{n!}\left(h_{\mathrm{FS}} + \log\left(\frac{1}{n!}\right)\right), \end{aligned}$$

using formula (2.6) in the first term, combined with the homogeneity property (2.7) and, in the second term, the scaling property (2.8). Rewriting the formula for h_{FS} above as a triangle sum and changing the order of summation then concludes the proof of the formula of the lemma. The last positivity statement will be shown in the course of the proof of Lemma 3.8. □

The key ingredient in the proof of Theorem 1.2 is the following universal bound on the arithmetic volume, in terms of the ordinary volume.

PROPOSITION 3.7. *For any n-dimensional toric Fano variety X which is K-semistable, the following bound holds for any volume-normalized continuous metric ϕ on $-K_X$:*

$$2\widehat{\mathrm{vol}}_\chi(-\mathcal{K},\phi) \leq -\mathrm{vol}(X)\log\left(\frac{\mathrm{vol}(X)}{(2\pi^2)^n}\right), \quad \mathrm{vol}(X) := \mathrm{vol}(-K_X).$$

Proof. Recall that, as shown at the beginning of the proof of Theorem 2.5, it is equivalent to establish the upper bound for $-\hat{\mathcal{D}}_{\psi_P}(\phi)$ when ϕ is a continuous psh metric on L. Since X is assumed K-semistable, it follows from Proposition 3.4 that X admits a T-invariant Kähler–Einstein metric. In general, a Kähler–Einstein metric ϕ on $-K_X$ minimizes the normalized Ding functional $\hat{\mathcal{D}}_{\psi_0}$ [BBEGZ19]. Thus in the toric case the infimum of $\hat{\mathcal{D}}_{\psi_0}$ coincides with the infimum over all continuous T-invariant psh metrics. As explained in § 3.1.2, such a metric may be identified with a convex function $\phi(x)$ on $\mathbb{R}^n$ satisfying $\phi - \psi_P \in L^\infty(\mathbb{R}^n)$. By formula (3.8) it will be enough to show that for such convex functions

$$-\int_P \phi^* \, dy/V + \log \int_{\mathbb{R}^n} e^{-\phi(x)} \, dx \leq -\log V + n\log(2\pi), \quad V := \operatorname{vol}(-K_X). \tag{3.9}$$

Since 0 is contained in the interior of P the measure $e^{-\phi}dx$ on $\mathbb{R}^n$ has finite moments. Recall that by Proposition 3.4 the barycenter of P coincides with $0 \in \mathbb{R}^n$ and, as a consequence, the left-hand side in inequality (3.9) is invariant under translations of ϕ, $\phi(x) \mapsto \phi(x+a)$ for any given $a \in \mathbb{R}^n$ [BB13, Lemma 2.14]. As a consequence, in order to prove inequality (3.9) we may as well assume that

$$\int_{\mathbb{R}^n} x e^{-\phi} \, dx = 0.$$

By the functional form of Santaló's inequality [AKM04, Lemma 2.14] this implies that

$$\int_{\mathbb{R}^n} e^{-\phi^*(y)} \, dy \cdot \int_{\mathbb{R}^n} e^{-\phi(x)} \, dx \leq (2\pi)^n$$

(where equality holds if $\phi = \phi^*$, i.e. if $\phi(x) = |x|^2/2$). Moreover, by Jensen's inequality,

$$-\int_P \phi^* \, d\lambda/V \leq \log\left(\int_P e^{-\phi^*(y)} \, dy/V\right) = \log\left(\int_{\mathbb{R}^n} e^{-\phi^*(y)} \, dy/V\right),$$

using in the last equality that $\phi^* = \infty$ on the complement of P (see formula (3.4)). Combining the latter two inequalities yields the desired inequality (3.9). □

Recall that $\mathbb{P}^n$ has maximal volume among all K-semistable n-dimensional Fano varieties (as shown in [BB17] in the toric case and in [Fuj18] in general). We next show that it will be enough to prove that, in the toric case, the next to largest volume is attained by $\mathbb{P}^{n-1} \times \mathbb{P}^1$.

LEMMA 3.8. *For any n-dimensional toric Fano variety X which is K-semistable,*

$$\operatorname{vol}(X) \leq \operatorname{vol}(\mathbb{P}^{n-1} \times \mathbb{P}^1) \implies \widehat{\operatorname{vol}}_\chi(-\mathcal{K}, \phi) < \widehat{\operatorname{vol}}_\chi\big(-\overline{\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}}\big),$$

where $-K_{\mathbb{P}^n}$ is endowed with the volume-normalized Fubini–Study metric.

Proof. Observe that the function of $\operatorname{vol}(X)$ appearing on the right-hand side of the inequality in the previous proposition is increasing when $\operatorname{vol}(X) \leq (2\pi^2)^n/e$. This bound is, in fact, satisfied for any K-semistable X. Indeed, by [BB17],

$$\operatorname{vol}(X) \leq \operatorname{vol}(\mathbb{P}^n) = \frac{(n+1)^n}{n!} < (2\pi^2)^n/e \tag{3.10}$$

(using a simple induction argument in the last inequality). Thus, by the previous proposition, it will be enough to show that

$$-\operatorname{vol}(\mathbb{P}^{n-1} \times \mathbb{P}^1)\log(\operatorname{vol}(\mathbb{P}^{n-1} \times \mathbb{P}^1)/(2\pi^2)^n) < 2\widehat{\operatorname{vol}}_\chi\big(-\overline{\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}}\big). \tag{3.11}$$

for any $n \geq 2$. To this end, note that

$$-\mathrm{vol}(\mathbb{P}^{n-1}\times\mathbb{P}^1)\log(\mathrm{vol}(\mathbb{P}^{n-1}\times\mathbb{P}^1)/(2\pi^2)^n) = -\frac{2n^{n-1}}{(n-1)!}\left(\log\left(\frac{2n^{n-1}}{(n-1)!}\right) - n\log(2\pi^2)\right).$$

We check that the inequality holds for $n=2$, and with induction in mind we simplify the right-hand side of (3.11) by $n+1$ for n and get

$$\begin{aligned}
2\widehat{\mathrm{vol}}_\chi(-\overline{\mathcal{K}_{\mathbb{P}^{n+1}_{\mathbb{Z}}}}) &= -\frac{(n+2)^{n+1}}{(n+1)!}\left(n+1-(n+2)\sum_{k=1}^{n+1}\frac{1}{k} + \log((n+1)!) - (n+1)\log(\pi)\right)\\
&= -\left(\frac{n+2}{n+1}\right)^{n+1}\frac{(n+1)^n}{n!}\left(\left(n-(n+1)\sum_{k=1}^{n}\frac{1}{k} + \log(n!) - n\log(\pi)\right)\right.\\
&\quad \left. + \left(1-(n+2)\sum_{k=1}^{n+1}\frac{1}{k} + (n+1)\sum_{k=1}^{n}\frac{1}{k} + \log(n+1) - \log(\pi)\right)\right)\\
&= \left(\frac{n+2}{n+1}\right)^{n+1} 2\widehat{\mathrm{vol}}_\chi(-\overline{\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}}) - \left(\frac{n+2}{n+1}\right)^{n+1}\left(1-\log(\pi)+\log(n+1)\right.\\
&\quad \left. -\frac{n+2}{n+1} - \sum_{k=1}^{n}\frac{1}{k}\right).
\end{aligned}$$

Here we observe for later use that $\widehat{\mathrm{vol}}_\chi(-\overline{\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}}}) > 0$ for all $n \geq 1$ by evaluating it at $n=1$ and then using the above to perform induction and noting that

$$-\left(1-\log(\pi)+\log(n+1)-\frac{n+2}{n+1}-\sum_{k=1}^{n}\frac{1}{k}\right) > -(-\log(\pi)+\log(2)) = \log\left(\frac{\pi}{2}\right) > 0$$

for $n \geq 1$. We have used that $-\log(n+1)+\sum_{k=1}^{n}(1/k)$ is increasing and can thus be estimated from below by putting $n=1$. We also simplify the left-hand side of (3.11),

$$\begin{aligned}
&-\mathrm{vol}(\mathbb{P}^n\times\mathbb{P}^1)\log(\mathrm{vol}(\mathbb{P}^n\times\mathbb{P}^1)/(2\pi^2)^{n+1})\\
&\quad = -\frac{2(n+1)^n}{n!}\left(\log\left(\frac{2(n+1)^n}{n!}\right) - (n+1)\log(2\pi^2)\right)\\
&\quad = -\left(\frac{n+1}{n}\right)^n \frac{2n^n}{n!}\left(\log\left(\frac{2n^n}{n!}\right) - n\log(2\pi^2)\right.\\
&\qquad \left. + \left(\log\left(\left(\frac{n+1}{n}\right)^n\right) - \log(2\pi^2)\right)\right.\\
&\quad = -\left(\frac{n+1}{n}\right)^n \mathrm{vol}(\mathbb{P}^{n-1}\times\mathbb{P}^1)\log(\mathrm{vol}(\mathbb{P}^{n-1}\times\mathbb{P}^1)/(2\pi^2)^n)\\
&\qquad - 2\frac{(n+1)^n}{n!}\left(-\log\left(\left(\frac{n+1}{n}\right)^n\right) - \log(2\pi^2)\right).
\end{aligned}$$

https://doi.org/10.1112/S0010437X2400736X Published online by Cambridge University Press

Fix $n \geq 2$ and assume $-\mathrm{vol}(\mathbb{P}^{n-1}\times\mathbb{P}^1)\log(\mathrm{vol}(\mathbb{P}^{n-1}\times\mathbb{P}^1)/(2\pi^2)^n) \leq 2\widehat{\mathrm{vol}}_\chi(-\overline{\mathcal{K}_{\mathbb{P}^n_\mathbb{Z}}})$. Define for brevity $e_n = (1+1/n)^n$ and estimate

$$\begin{aligned}
&2\widehat{\mathrm{vol}}_\chi(-\overline{\mathcal{K}_{\mathbb{P}^{n+1}_\mathbb{Z}}}) - (-\mathrm{vol}(\mathbb{P}^n\times\mathbb{P}^1)\log(\mathrm{vol}(\mathbb{P}^n\times\mathbb{P}^1)/(2\pi^2)^{n+1}))\\
&\quad= e_{n+1}\widehat{\mathrm{vol}}_\chi(-\overline{\mathcal{K}_{\mathbb{P}^n_\mathbb{Z}}}) - (-e_n\mathrm{vol}(\mathbb{P}^n\times\mathbb{P}^1)\log(\mathrm{vol}(\mathbb{P}^n\times\mathbb{P}^1)/(2\pi^2)^n))\\
&\qquad+2\frac{(n+1)^n}{n!}\bigg(\log\bigg(\frac{(n+1)^n}{n}\bigg)-\log(2\pi^2)\bigg)\\
&\qquad-\frac{(n+2)^{n+1}}{(n+1)!}\bigg(1-\log(\pi)+\log(n+1)-\frac{n+2}{n+1}-\sum_{k=1}^n\frac{1}{k}\bigg)\\
&\quad> 2\frac{(n+1)^n}{n!}\bigg(\log\bigg(\frac{(n+1)^n}{n}\bigg)-\log(2\pi^2)\bigg)\\
&\qquad-\frac{(n+2)^{n+1}}{(n+1)!}\bigg(1-\log(\pi)+\log(n+1)-\frac{n+2}{n+1}-\sum_{k=1}^n\frac{1}{k}\bigg)\\
&\quad=\frac{(n+2)^{n+1}}{(n+1)!}\bigg(\frac{(n+1)^n}{n!}\bigg/\frac{(n+2)^{n+1}}{(n+1)!}2\bigg(\log\bigg(\bigg(\frac{n+1}{n}\bigg)^n\bigg)-\log(2\pi^2)\bigg)\\
&\qquad-1+\log(\pi)-\log(n+1)+\frac{n+2}{n+1}+\sum_{k=1}^n\frac{1}{k}\bigg)\\
&\quad=\frac{(n+2)^{n+1}}{(n+1)!}\bigg[\frac{2}{e_n}\big(\log(e_n)-\log(2\pi^2)\big)+\log(\pi)+\sum_{k=1}^{n+1}\frac{1}{k}-\log(n+1)\bigg].
\end{aligned}$$

In the inequality above we have used $\widehat{\mathrm{vol}}_\chi(-\overline{\mathcal{K}_{\mathbb{P}^n_\mathbb{Z}}}) > 0$ for all $n \geq 1$ and $e_n < e_{n+1}$ and the induction hypothesis. Next check numerically that this last expression is positive for $n = 2, 3$. For $n \geq 4$ we have

$$\begin{aligned}
&\frac{2}{e_n}\big(\log(e_n)-\log(2\pi^2)\big)+\log(\pi)+\sum_{k=1}^{n+1}\frac{1}{k}-\log(n+1)\\
&\quad>\frac{2}{e_4}(\log(e_4)-\log(2\pi^2))+\log(\pi)+\gamma>0.
\end{aligned}$$

We used again that $e_n < e_{n+1}$ and the fact that $\sum_{k=1}^{n+1}(1/k)-\log(n+1) > \gamma$ [TT71], where γ is the Euler–Mascheroni constant. The last inequality is checked numerically. □

We expect that any K-semistable toric Fano variety X, not equal to $\mathbb{P}^n$, satisfies the volume bound in the previous lemma (see § 3.2.1). Here we will show that this is the case under the conditions of Theorem 1.2. First, the singular cases are handled using the following bound.

Lemma 3.9. *Let X be a singular K-semistable toric Fano variety. Then*

$$\mathrm{vol}(-K_X) \leq \tfrac{1}{2}(n+1)^n/n!$$

if any one of the following conditions holds.

- *X is $\mathbb{Q}$-factorial (or equivalently, X has abelian quotient singularities).*
- *X is not Gorenstein.*

https://doi.org/10.1112/S0010437X2400736X Published online by Cambridge University Press

In particular, by the first point, when $n = 2$ this inequality holds for any singular K-semistable toric Fano variety X.

Proof. The result concerning the first point is the toric case of [Liu18, Theorem 3] concerning quotient singularities, but in the toric case it also follows from the proof of [BB17, Theorem 1.2]. For future reference we recall the argument in [BB17]. Let P be a given polytope with rational vertices and represent P as the intersection of hyperplanes $\{p \in \mathbb{R}^n : \langle l_F, p\rangle \geq -a_F\}$, where the index F ranges over the facets of P, l_F is a primitive vector in $\mathbb{Z}^n$ and a_F is a positive number. In the present Fano case $a_F = 1$. Moreover, since X is assumed to be $\mathbb{Q}$-factorial for any vertex p_0 of P there are precisely n facets $F_1, \dots, F_n$ of P intersecting p_0, numbered so that the corresponding normals define a positively oriented basis in $\mathbb{R}^n$ [CLS11]. Fixing a vertex p_0 of P, we denote by P' the image of P under the map

$$p \mapsto \left(\frac{\langle l_{F_1}, p\rangle + a_{F_1}}{a_{F_1}}, \dots, \frac{\langle l_{F_n}, p\rangle + a_{F_n}}{a_{F_n}}\right), \tag{3.12}$$

which is a polytope in $[0, \infty[^n$. Moreover, assuming that 0 is the barycenter of P, the barycenter of P' is $(1, \dots, 1)$. By [BB17, Theorem 1.5] the volume $\mathrm{Vol}(P')$ of any such polytope is maximal when P' is $(n + 1)$ times the unit simplex in $[0, \infty[^n$ with vertex at $(0, \dots, 0)$. Hence,

$$\mathrm{Vol}(P') \leq (n + 1)^n/n!, \quad \mathrm{Vol}(P') = \frac{\delta}{a_{F_1} \cdots a_{F_n}} \mathrm{Vol}(P), \tag{3.13}$$

where δ is the determinant of the map $p \mapsto (\langle l_{F_1}, p\rangle, \dots, \langle l_{F_n}, p\rangle)$. Thus δ is a positive integer and $\delta = 1$ if and only if the map is invertible, i.e. if and only if $l_{F_1}, \dots, l_{F_n}$ generate $\mathbb{Z}^n$, which is equivalent to the T_c-invariant neighborhood U_0 corresponding to the vertex p_0 being biholomorphic to $\mathbb{C}^n$ [CLS11]. Hence, if X is singular (i.e. X is not non-singular), then there must be some vertex p_0 with $\delta \geq 2$. Since $a_{F_i} = 1$, this concludes the proof.

To prove the second point we employ a similar argument. This time, for X possibly not $\mathbb{Q}$-factorial, there might be more than n facets intersecting a vertex p_0. Still, there are at least n facets intersecting at p_0, and we can construct the map (3.12) by choosing any n of them. Next note that if $\delta = 1$, the map and its inverse have integer coefficients (since $a_{F_i} = 1$ when X is Fano) and since p_0 is mapped to 0, $p_0 \in \mathbb{Z}^n$. Since p_0 was arbitrary, it follows that P is a lattice polytope and hence X is Gorenstein. Thus $\delta \geq 2$ and we are done. □

The volume bound in the previous lemma implies the volume bound in Lemma 3.8 is satisfied:

$$\frac{(n + 1)^n}{2n!} \leq \frac{2n^{n-1}}{(n - 1)!} \iff (1 + 1/n)^n \leq 4. \tag{3.14}$$

The left-hand side in the latter inequality increases to e, which is, indeed, smaller than 4. This proves Theorem 1.2 in the singular cases. Finally, in the case that X is non-singular there are, for any given dimension n, only a finite number of cases to check in order to verify the volume bound in Lemma 3.8. When $n \leq 6$ we may apply the database [Øbr07] of all non-singular Fano varieties of dimension n. The condition that the barycenter of P vanishes corresponds in the database to the condition 'zero dual barycenter'. Adding the condition $(-K_X)^n \geq n!\mathrm{vol}(\mathbb{P}^{n-1} \times \mathbb{P}^1)$, the database only furnishes $\mathbb{P}^n$ and $\mathbb{P}^{n-1} \times \mathbb{P}^1$, as desired.

3.2.1 *Remarks on the 'gap hypothesis'.* In order to extend the proof of Theorem 1.2 to any dimension n one would need to establish the following conjecture (established above under the conditions in Theorem 1.2).

https://doi.org/10.1112/S0010437X2400736X Published online by Cambridge University Press

Conjecture 3.10 (The 'gap hypothesis'). For any n-dimensional toric K-semistable Fano manifold X different from $\mathbb{P}^n$, $\mathrm{vol}(X) \leq \mathrm{vol}(\mathbb{P}^{n-1} \times \mathbb{P}^1)$.

This conjecture might even hold without the toric assumptions in any dimension (as pointed out to us by Ziquan Zhuang, this appears to be a folklore conjecture). For example, when $n = 3$ and X is non-singular it follows from the well-known classification of three-dimensional Fano manifolds (see the 'big table' in [Ara$^+$, § 6]) that the only Fano manifolds X, different from $\mathbb{P}^3$, which do not satisfy the inequality in question are $\mathbb{P}^3$ blown up at one point and $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2))$. But both of these are K-unstable, i.e. they are not K-semistable. Indeed, these two Fano manifolds are toric and if they were K-semistable they would satisfy the gap hypothesis, by the toric case ($n \leq 6$) applied to $n = 3$. Let us also point out that in the toric case it is only $\mathbb{P}^{n-1} \times \mathbb{P}^1$ that saturates the inequality in the 'gap hypothesis' when $n \leq 6$ and it thus seems natural to ask if this is also the case when $n > 6$. However, in the general non-toric case the inequality is also saturated by the non-singular quadratic hypersurface X_2 in $\mathbb{P}^{n+1}$, i.e. the base of the ordinary double point (ODP). Moreover, as pointed out to us by Yuji Odaka, in the general case our 'gap hypothesis' is reminiscent of the ODP conjecture in [SS17], very recently settled in the toric case [MS24]. More precisely, in our setup, the ODP conjecture implies that

$$\mathrm{vol}(X) \leq \mathrm{vol}(\mathbb{P}^{n-1} \times \mathbb{P}^1)(n/I(X)), \tag{3.15}$$

where $I(X)$ denotes the Fano index of X (i.e. the largest positive integer such that $K_X/I(X)$ is a line bundle). However, $I(X) \leq n$ when $X \neq \mathbb{P}^n$ (with equality if and only if $X = X_2$) and hence inequality (3.15) is weaker than our 'gap hypothesis'.

3.2.2 *The case of products in any dimension.*

Lemma 3.11. *The 'gap hypothesis' holds when X is the product of K-semistable Fano varieties $X_1, \ldots, X_M$ (not necessarily assumed toric), for $M \geq 2$.*

Proof. By a simple induction argument we may as well assume that $M = 2$. Without loss of generality, let $n := \dim(X_1) \geq \dim(X_2) =: m > 1$. Note that if $m = 1$ we are done, since then $\mathrm{vol}(X) = \mathrm{vol}(X_1)\mathrm{vol}(X_2) \leq \mathrm{vol}(\mathbb{P}^{N-1})\mathrm{vol}(\mathbb{P}^1) = \mathrm{vol}(\mathbb{P}^{N-1} \times \mathbb{P}^1)$ using that, by Fujita's inequality (1.1), the complex projective space maximizes the volume among K-semistable Fano varieties in each dimension. Using again that the complex projective space maximizes the volume in each given dimension and defining for brevity $e_k := (1 + 1/k)^k$, we get

$$\begin{aligned}
\mathrm{vol}(X) &= \mathrm{vol}(X_1)\mathrm{vol}(X_2) \leq \mathrm{vol}(\mathbb{P}^n)\mathrm{vol}(\mathbb{P}^m) \\
&= \frac{(n+1)^n}{n!}\frac{(m+1)^m}{m!} = \frac{(n+2)^{n+1}}{(n+1)!}\frac{m^{m-1}}{(m-1)!}\left(\frac{n+1}{n+2}\right)^{n+1}\left(\frac{m+1}{m}\right)^m \\
&= \frac{(n+2)^{n+1}}{(n+1)!}\frac{m^{m-1}}{(m-1)!}\frac{e_m}{e_{n+1}} < \frac{(n+2)^{n+1}}{(n+1)!}\frac{m^{m-1}}{(m-1)!} = \mathrm{vol}(\mathbb{P}^{n+1})\mathrm{vol}(\mathbb{P}^{m-1}),
\end{aligned}$$

where in the last inequality we have used that e_k is increasing in k. We may continue in similar manner until we have $\mathrm{vol}(\mathbb{P}^{N-1} \times \mathbb{P}^1)$ in the right-hand side and we are done. □

As explained in the previous section, it follows from the previous lemma that Conjecture 1.1 holds when $\mathcal{X}$ is a product of toric arithmetic Fano varieties, i.e. $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_M$, where $\mathcal{X}_i$ is endowed with its canonical integral structure.

3.3 The height of toric Kähler–Einstein metrics; proof of Theorem 1.3

By Proposition 3.7 it only remains to prove the lower bound. Using the notation in the proof of Proposition 3.7 we have that, for any continuous convex function ψ on $\mathbb{R}^n$ such that $\psi - \psi_P$ is bounded,

$$2\big(\overline{-\mathcal{K}_{\mathcal{X}}}\big)^{n+1}/\mathrm{vol}(-K_X) \geq -\int_P \psi^* \, dy/\mathrm{Vol}(P) + \log \int_{\mathbb{R}^n} e^{-\psi}\, dx + n \log \pi.$$

In particular, taking $\psi = \psi_P$, the first term on the right-hand side vanishes. Moreover,

$$I := \int_{\mathbb{R}^n} e^{-\psi_P} dx = n!\mathrm{Vol}(P^*),$$

where P^* denotes the polar dual of P, i.e. P^* consists of all $x \in \mathbb{R}^n$ such that $x \cdot p \leq 1$ for all $p \in P$. Indeed,

$$I = \int_{[0,\infty[} e^{-t} (\psi_P)_* \, dx = \int e^{-t} \frac{dV(t)}{dt}\, dt = \int e^{-t} V(t)\, dt = \int_0^\infty e^{-t} t^n \, dt \mathrm{Vol}(P^*),$$

where $V(t)$ is the Lebesgue volume of $\{\psi_P < t\}$, i.e. of tP^*. Hence,

$$2\big(\overline{-\mathcal{K}_{\mathcal{X}}}\big)^{n+1} \geq \mathrm{Vol}(P)\big(\log(n!\mathrm{Vol}(P^*)) + n \log \pi\big).$$

Since, by definition, $\mathrm{Vol}(P^*)\mathrm{Vol}(P) \geq m_n$ this concludes the proof of the lower bound in the theorem. Next, by [Kup08, Corollary 1.8] (see also [Ber21]),

$$m_n \geq \left(\frac{\pi}{2e}\right)^{n-1} (n+1)^{n+1}/(n!)^2 = \left(\frac{\pi}{2e}\right)^{n-1} \frac{(n+1)}{n!} \sigma_n,$$

where $\sigma_n = \mathrm{vol}(\mathbb{P}^n)$. Since $\mathrm{Vol}(P) \leq \sigma_n$ (by (3.10)) this means that

$$n!\pi^n m_n \mathrm{Vol}(P)^{-1} \geq n!\pi^n m_n \sigma_n^{-1} = \pi \left(\frac{\pi^2}{2e}\right)^{n-1} (n+1) > 1,$$

proving the positivity in the theorem.

3.4 Examples

We next provide examples of families of toric varieties X for which the height of the corresponding Kähler–Einstein can be explicitly computed as a function of $\mathrm{vol}(X)$ of the same form as in Theorem 1.3. The examples are based on the following proposition.

Proposition 3.12. *Let X_1 and X_2 be two K-semistable toric Fano varieties of dimension n with moment polytopes P_1 and P_2 such that $P_2 = AP_1$ for an invertible linear transformation A (the polytopes are linearly equivalent). Denote the canonical integral models of X_1 and X_2 by $\mathcal{X}_1$ and $\mathcal{X}_2$, respectively. Then, with heights taken with respect to the volume-normalized Kähler–Einstein metrics,*

$$\frac{\big(\overline{-\mathcal{K}_{\mathcal{X}_2}}\big)^{n+1}/(n+1)!}{(-K_{X_2})^n/n!} = \frac{\big(\overline{-\mathcal{K}_{\mathcal{X}_1}}\big)^{n+1}/(n+1)!}{(-K_{X_1})^n/n!} - \frac{1}{2}\log \det A.$$

As a consequence, for X a K-semistable toric Fano variety of dimension n,

$$\big(\overline{-\mathcal{K}_{\mathcal{X}}}\big)^{n+1} = \frac{(n+1)!}{2} \mathrm{vol}(X) \log\left(\frac{a}{\mathrm{vol}(X)}\right), \tag{3.16}$$

https://doi.org/10.1112/S0010437X2400736X Published online by Cambridge University Press

where a is a constant independent of the choice of X within a class of toric varieties with linearly equivalent moment polytopes. More precisely,

$$a = \mathrm{vol}(X) \exp\left(\frac{2\left(\overline{-\mathcal{K}_{\mathcal{X}}}\right)^{n+1}/(n+1)!}{\mathrm{vol}(X)} \right) \tag{3.17}$$

and Proposition 3.12 ensures the claimed independence.

Proof of Proposition 3.12. Recall that, with heights taken with respect to Kähler–Einstein metrics,

$$\frac{\left(\overline{-\mathcal{K}_{\mathcal{X}_2}}\right)^{n+1}/(n+1)!}{(-K_{X_2})^n/n!} = -\frac{1}{2} \sup_{\phi} - \frac{1}{\mathrm{vol}(P_2)} \int_{P_2} \phi^*(p)\, dp + \log \int_{\mathbb{R}^n} \exp(-\phi(x))\, dx.$$

Changing variables in the integrals, $p \mapsto A^t p'$ and $x \mapsto Ax'$, we get

$$\frac{(\overline{-\mathcal{K}_{\mathcal{X}_2}})^{n+1}/(n+1)!}{(-K_{X_2})^n/n!} \\ = -\frac{1}{2} \left(\sup_{\phi(A\cdot)} - \frac{1}{\mathrm{vol}(P_1)} \int_{P_1} \phi^*(A^t p')\, dp' + \log \int_{\mathbb{R}^n} \exp(-\phi(Ax'))\, dx' + \log \det \mathrm{A} \right).$$

Now we rename $\phi' = \phi(A\cdot)$ and use that then $\phi'^* = \phi^*(A^t \cdot)$ to get the result. $\square$

Example 3.13. Recall the K-semistable toric Fano varieties $X_{q,p}$ parametrized with two prime numbers from Example 3.1. The corresponding polytope $P(-K_{X_{p,q}})$ is the image of the polytope $P(-K_{\mathbb{P}^1 \times \mathbb{P}^1}) = \mathrm{conv}\{(1,1),(1,-1),(-1,1),(-1,-1)\}$ under the linear map A given in matrix form by $\begin{bmatrix} 1/2p & 1/2p \\ -1/2q & 1/2q \end{bmatrix}$. Thus the family $\mathcal{F} = \{\mathbb{P}^1 \times \mathbb{P}^1, X_{p,q} : p, q \text{ prime}\}$ comprises an example of a family of K-semistable toric Fano varieties with linearly equivalent moment polytopes. Thus by (3.16), for $X \in \mathcal{F}$,

$$\left(\overline{-\mathcal{K}_{\mathcal{X}}}\right)^{n+1} = \frac{(n+1)!}{2} \mathrm{vol}(X) \log\left(\frac{a}{\mathrm{vol}(X)} \right)$$

with, by (3.17), Lemma 3.6 and a simple computation,

$$a = \mathrm{vol}(\mathbb{P}^1 \times \mathbb{P}^1) \exp\left(\frac{2\left(\overline{-\mathcal{K}_{\mathbb{P}^1 \times \mathbb{P}^1}}\right)^{n+1}/(n+1)!}{\mathrm{vol}(\mathbb{P}^1 \times \mathbb{P}^1)} \right) = 4 \exp(2 - \log \pi^2).$$

Recall also that $\mathrm{vol}(-K_{X_{p,q}}) = 2/(pq)$, so that in this family the heights with respect to the Kähler–Einstein metrics are explicitly computed by the previous formula.

4. Sharp bounds on Donaldson's toric Mabuchi functional

Let (X, L) be a polarized complex manifold and denote by $\mathcal{H}(X, L)$ the space of all smooth metrics ψ on L whose curvature form $dd^c\psi$ is positive, $dd^c\psi > 0$.

4.1 The Mabuchi functional (recap)

The Mabuchi functional $\mathcal{M}$ on $\mathcal{H}(X, L)$ is defined, up to addition by a constant, by declaring that its differential on $\mathcal{H}(X, L)$ at a given point ψ is represented by the following measure on X:

$$d\mathcal{M}_{|\psi} := (-S(\psi) + a) \frac{(dd^c\psi)^n}{n!}, \quad a := n(-K_X) \cdot L^{n-1}/L^n, \tag{4.1}$$

https://doi.org/10.1112/S0010437X2400736X Published online by Cambridge University Press

where $S(\psi)$ denotes the scalar curvature of the Kähler form $(dd^c\psi)$, i.e. the trace of the Ricci curvature,

$$S(\psi)\frac{(dd^c\psi)^n}{n!} := \mathrm{Ric}(dd^c\psi) \wedge \frac{(dd^c\psi)^{n-1}}{(n-1)!}.$$

Recall that the Ricci curvature $\mathrm{Ric}(dd^c\psi)$ of the Kähler form $dd^c\psi$ is the $(1,1)$-form defined as the curvature of the metric on $-K_X$ induced by the volume form of $dd^c\psi$. We have followed Donaldson's multiplicative normalizations in [Don02, formula (3.2.1)], which differ from the original definition in [Mab86], where the measure $(dd^c\psi)^n/n!$ on X is volume-normalized. At any rate, formula (4.1) only determines the Mabuchi functional $\mathcal{M}$ up to an additive constant.

4.1.1 *The case when X is a Fano manifold and $L = -K_X$.* We now specialize to the case when $L = -K_X$ and note that a choice of reference metric ψ_0 in $\mathcal{C}^0(L) \cap \mathrm{PSH}(L)$ induces a particular choice of Mabuchi functional, i.e. a functional whose differential satisfies formula (4.1), which we shall denote by $\mathcal{M}_{\psi_0}$. This is a consequence of the thermodynamical formalism introduced in [Ber13], which expresses

$$\mathcal{M}_{\psi_0}(\psi) := \mathrm{vol}(-K_X)F_{\psi_0}(MA(\psi)), \tag{4.2}$$

where $MA(\psi)$ is the probability measure on X defined by the normalized volume form of the Kähler metric $dd^c\psi$,

$$MA(\psi) := \frac{1}{n!}(dd^c\psi)^n/\mathrm{vol}(L), \tag{4.3}$$

and $F_{\psi_0}(\mu)$ denotes the *free energy functional* on the space $\mathcal{P}(X)$ of all probability measures on X, defined as

$$F_{\psi_0}(\mu) := -E_{\psi_0}(\mu) + \mathrm{Ent}_{dV_0}(\mu) \in]-\infty, \infty]. \tag{4.4}$$

Here $\mathrm{Ent}_{dV_0}(\mu)$ denotes the *entropy* of μ relative to the volume form dV_0 on X induced by ψ_0 (i.e. $dV_0 = e^{-\psi_0}$ in the notation of § 2.1.2) defined by

$$\mathrm{Ent}_{dV_0}(\mu) := \int \log \frac{\mu}{dV_0}\mu$$

when $\mu \in L^1(X, dV_0)$ and otherwise $\mathrm{Ent}_{dV_0}(\mu) := \infty$. Furthermore, $E_{\psi_0}(\mu)$ is the *pluricomplex energy* of μ, relative to ψ_0, introduced in [BBGZ13], which may be defined as a Legendre–Fenchel transform of the functional $\mathcal{E}_{\psi_0}/\mathrm{vol}(L)$ (defined by formula (2.11)). For our purposes it will be enough to define $E_{\psi_0}(\mu)$ when μ is of the form $\mu = MA(\psi)$ for ψ in $\mathcal{C}^0(L) \cap \mathrm{PSH}(L)$:

$$E_{\psi_0}(MA(\psi)) = \frac{\mathcal{E}_{\psi_0}(\psi)}{\mathrm{vol}(L)} - \int_X (\psi - \psi_0)MA(\psi). \tag{4.5}$$

We recall that formula (4.2) follows readily from the fact that on the subspace of all volume forms μ in $\mathcal{P}(X)$ the differential of E_{ψ_0} at $\mu \in \mathcal{P}(X)$ is represented by the function $\psi_0 - \psi_\mu$:

$$dE_{\psi_0|\mu} = -(\psi_\mu - \psi_0)$$

(this formula is dual to formula (2.12) in the sense of Legendre transforms; see [Ber13]).

Remark 4.1. Formula (4.2) defines $\mathcal{M}_{\psi_0}(\psi)$ on the space $\mathcal{C}^0(L) \cap \mathrm{PSH}(L)$ as a function taking values in $]-\infty, \infty]$. More generally, the functional $\mathcal{M}_{\psi_0}(\psi)$ is well defined as soon as $E(MA(\psi)) < \infty$ (see [Ber13, BBEGZ19]). For ψ smooth, formula (4.2) is essentially equivalent to a formula for the Mabuchi functional appearing in [Tia96] and [Che00].

https://doi.org/10.1112/S0010437X2400736X Published online by Cambridge University Press

4.1.2 *The case when X is a singular Fano variety.* In the case when X is a singular Fano variety we will denote by $\mathcal{H}(X, -K_X)$ the space of all continuous metrics ψ on L such that ψ is smooth on the regular locus X_{reg} of X and $dd^c\psi > 0$ on X_{reg}.

4.2 Proof of Theorem 1.5

Recall the basic inequality that holds on any Fano variety [BBEGZ19, Lemma 4.4],

$$F_{\psi_0}(MA(\psi)) \geq \hat{\mathcal{D}}_{\psi_0}(\psi), \tag{4.6}$$

as follows from the non-negativity of the relative entropy between two probability measures (or Jensen's inequality). In fact, the following identity holds [BBEGZ19, Lemma 4.4]:

$$\inf_{\mathcal{C}^0(L)\cap \mathrm{PSH}(L)} F_{\psi_0}(MA(\psi)) = \inf_{\mathcal{C}^0(L)\cap \mathrm{PSH}(L)} \hat{\mathcal{D}}_{\psi_0}(\psi) \tag{4.7}$$

(the two infima above may, equivalently, be restricted to $\mathcal{H}(X, L)$; see the regularization result in [BDL17]).

Combining Theorem 1.2 with inequality (4.6), the proof is concluded by invoking the following formula relating $\mathcal{M}_{\psi_P}$(where ψ_P is the canonical toric reference defined by formula (3.3)) to Donaldson's toric Mabuchi functional

$$\mathcal{M}_{-K_X}(\psi) := \int_{\partial P} \psi^* \, d\sigma - n \int_P \psi^* \, dx - \int_P \log\det(\nabla^2\psi^*) \, dx, \tag{4.8}$$

where ψ^* denotes the Legendre transform of the T-invariant metric $\psi \in \mathcal{H}(X, -K_X)$ and $d\sigma$ is the measure on ∂P, absolutely continuous with respect to the $(n-1)$-dimensional Lebesgue measure $d\lambda_{\partial P}$, defined by $d\sigma = d\lambda_{\partial P}/\|l_F\|$ on a facet F of ∂P, where $\|l_F\|$ denotes the Euclidean norm of a primitive normal vector to F.

Lemma 4.2. *Let X be an n-dimensional toric Fano variety. The following identity holds on the space of all T-invariant metrics in $\mathcal{H}(X, -K_X)$:*

$$\mathcal{M}_{\psi_P} = \mathcal{M}_{-K_X} - \mathrm{vol}(-K_X) \log \mathrm{vol}(-K_X).$$

Proof. This formula is essentially the content of [BB13, Proposition 4.6], but since the normalizations are a bit different we recall the proof. Identifying a toric metric ψ with a convex function on $\mathbb{R}^n$ (as in § 3.1.2), formula (4.2), combined with formula (4.5), yields

$$\begin{aligned}\mathcal{M}_{\psi_P}(\psi) &= -\mathcal{E}_{\psi_P}(\psi) + \int_{\mathbb{R}^n} (\psi - \psi_P)(dd^c\psi)^n/n! + \int_{\mathbb{R}^n} \log\left(\frac{MA(\psi)}{e^{-\psi_P}\, dx}\right) \mathrm{vol}(-K_X) MA(\psi) \\ &= \int_P \psi^* \, d\lambda + \int_{\mathbb{R}^n} \psi (dd^c\psi)^n/n! + \int_{\mathbb{R}^n} \log\det(\nabla^2\psi)\det(\nabla^2\psi) \\ &\quad - \mathrm{vol}(-K_X)\log\mathrm{vol}(-K_X).\end{aligned}$$

By [BB13, Lemma 4.7], making the change of variables $y = \nabla\psi$, the second term above may be expressed as

$$\int_{\mathbb{R}^n} \psi(dd^c\psi)^n/n! = \int_{\partial P} \psi^* \, d\sigma - (n+1) \int u \, dp, \tag{4.9}$$

giving

$$\mathcal{M}_{\psi_P}(\psi) = \int_{\partial P} \psi^* \, d\sigma - n \int_P \psi^* \, d\lambda + \int_{\mathbb{R}^n} \log\det(\nabla^2\psi)\det(\nabla^2\psi) - \mathrm{vol}(-K_X)\log\mathrm{vol}(-K_X).$$

Again making the change of variables $y = \nabla\psi$ in the remaining integral over $\mathbb{R}^n$ concludes the proof, using the standard relation $\det(\nabla^2\psi)(x)\det(\nabla^2\psi^*)(\nabla\psi(x)) = 1$ (which follows from the fact that the map $y \mapsto \nabla\psi^*(y)$ is the inverse of $x \mapsto \nabla\psi(x)$). ☐

https://doi.org/10.1112/S0010437X2400736X Published online by Cambridge University Press

5. Relations to the arithmetic Mabuchi functional

Given an integral model $(\mathcal{X},\mathcal{L})$ of a polarized variety (X,L), consider the *arithmetic Mabuchi functional* $\mathcal{M}_{(\mathcal{X},\mathcal{L})}$ on $\mathcal{H}(X,L)$ defined by

$$\mathcal{M}_{(\mathcal{X},\mathcal{L})}(\psi) := \frac{a}{(n+1)!}\overline{\mathcal{L}}^{n+1} + \frac{1}{n!}\overline{\mathcal{K}}_{\mathcal{X}}\cdot\overline{\mathcal{L}}^{n}, \quad a = -n(K_X\cdot L^{n-1})/L^n, \tag{5.1}$$

where $\overline{\mathcal{L}} = (\mathcal{L},\psi)$ and $\overline{\mathcal{K}}_{\mathcal{X}}$ is endowed with the metric induced by the measure $MA(\psi)$ on X, i.e. the normalized volume form of the Kähler form $dd^c\psi$. As discussed in § 1.4, this functional coincides, up to additive and multiplicative normalizations, with the arithmetic Mabuchi functional introduced in [Oda18].

Lemma 5.1. *The differential of the functional $\psi\mapsto 2\mathcal{M}_{(\mathcal{X},\mathcal{L})}(\mathcal{L},\psi)$ on $\mathcal{H}(X,L)$ satisfies the defining formula (4.1) of the Mabuchi functional.*

Proof. As pointed out in [Oda18], this formula can be deduced from the formula for the Mabuchi functional in [Tia96, Che00]. But for completeness and to check the normalizations we provide a simple direct proof. Recall the following property of arithmetic intersection numbers which holds if $\mathcal{L}_0\to\mathcal{X}$ is the trivial line bundle (which is a consequence of the restriction formula [BGS94, Proposition 2.3.1] and Lemma 2.3):

$$(\mathcal{L}_0,\phi_0)\cdot(\mathcal{L}_1,\phi_0)\cdot\ldots\cdot(\mathcal{L}_n,\phi_n) = \frac{1}{2}\int_X \phi_0 dd^c\phi_1\wedge\cdots\wedge dd^c\phi_n, \tag{5.2}$$

where ϕ_0 is the globally well-defined function on X defined by formula (2.1) when e_U is the standard global trivialization 1 of the trivial line bundle over X, i.e. $\phi_0/2 = -\log\|s\|_{\phi_0}$, in which s is a global trivialization of $\mathcal{L}$. In particular, differentiating along a curve $t\mapsto\psi_t$ in $\mathcal{H}(X,L)$ and using the symmetry of arithmetic intersection numbers gives

$$\frac{d}{dt}\big((\mathcal{L},\psi_t)^{n+1}\big) = (n+1)\left(\mathcal{L}_0,\frac{d\psi_t}{dt}\right)\cdot(\mathcal{L},\psi_t)^n = \frac{1}{2}\int_X\frac{d\psi_t}{dt}(dd^c\psi)^n,$$

where $d\psi_t/dt$ is a globally well-defined function on X and can thus be identified with a metric on the trivial line bundle that we denote by $\mathcal{L}_0$. Likewise, denoting by ρ_t a local density for $MA(\psi_t)$ with respect to the Euclidean measure defined by local holomorphic coordinates,

$$\frac{d}{dt}\big((\mathcal{K}_{\mathcal{X}},\log\rho_t)(\mathcal{L},\psi_t)^n\big) = (\mathcal{K}_{\mathcal{X}},\log\rho_t)n\left(\mathcal{L},\frac{d\psi_t}{dt}\right)\cdot(\mathcal{L},\psi_t)^{n-1} + \left(\left(\mathcal{L}_0,\frac{d}{dt}\log\rho_t\right)\cdot(\mathcal{L},\psi_t)^n\right), \tag{5.3}$$

where we have used Leibniz's rule. Applying formula (5.2), the second term above may, after multiplication by 2, be expressed as

$$= \int_X\frac{d}{dt}\log\rho_t(dd^c\psi_t)^n = n!\mathrm{vol}(L)\int_X\frac{d}{dt}\log\rho_t\rho_t = n!\mathrm{vol}(L)\frac{d}{dt}\int_X\rho_t = 0,$$

using in the last equality that $\int_X\rho_t = \mathrm{vol}(L)$ for any t. Likewise, applying formula (5.2) to the first term in formula (5.3) yields

$$2\big(\mathcal{K}_{\mathcal{X}},\log\rho_t\big)\left(\mathcal{L},\frac{d\psi_t}{dt}\right)\Big/ n = \int_X\frac{d\psi_t}{dt}dd^c(\log\rho_t)\wedge(dd\psi_t)^{n-1} = -\int_X\frac{d\psi_t}{dt}\mathrm{Ric}(dd^c\psi_t)\wedge(dd\psi_t)^{n-1}.$$

This concludes the proof. □

The following proposition relates the arithmetic Mabuchi functional $\mathcal{M}_{(\mathcal{X},-\mathcal{K}_{\mathcal{X}})}$ to Donaldson's toric Mabuchi functional $\mathcal{M}_{-K_X}$ (formula (4.8)).

Proposition 5.2. *Given a toric Fano variety X, denote by $\mathcal{X}$ its canonical integral model. Then the following formula holds for any T-invariant metric in $\mathcal{H}(X,-K_X)$:*

$$2\mathcal{M}_{(\mathcal{X},-\mathcal{K}_{\mathcal{X}})}=\mathcal{M}_{-K_X}-\operatorname{vol}(-K_X)\log\operatorname{vol}(-K_X).$$

Proof. In this case $a=n$ and we can thus decompose $\mathcal{M}_{(\mathcal{X},\mathcal{L})}(\psi)$ as

$$\frac{1}{(n+1)!}\overline{\mathcal{L}}^{n+1}+\frac{1}{n!}(\overline{\mathcal{L}}+\overline{\mathcal{K}}_{\mathcal{X}})\cdot\overline{\mathcal{L}}^{n}=-\frac{1}{(n+1)!}\overline{\mathcal{L}}^{n+1}+\frac{1}{2}\int\log\left(\frac{MA(\psi)}{e^{-\psi}}\right)(dd^c\psi)^n/n!,\tag{5.4}$$

where, in the last equality, we have exploited that $\mathcal{L}+\mathcal{K}_{\mathcal{X}}$ is trivial so that formula (5.2) applies. Applying formula (3.7) to the first term on the right-hand side above thus gives

$$\begin{aligned}2\mathcal{M}_{(\mathcal{X},\mathcal{L})}(\psi)&:=-\mathcal{E}_{\psi_P}(\psi)+\int\log\left(\frac{MA(\psi)}{e^{-\psi}}\right)(dd^c\psi)^n/n!\\&=\operatorname{vol}(-K_X)\left(-\frac{1}{V(X)}\mathcal{E}_{\psi_P}(\psi)+\langle\psi-\psi_P,MA(\psi)\rangle+\int\log\left(\frac{MA(\psi)}{e^{-\psi_P}}\right)MA(\psi)\right).\end{aligned}$$

The right-hand side in the last equation above equals $\mathcal{M}_{\psi_P}(\psi)$ (by definition (4.2)). Invoking Lemma 4.2 thus concludes the proof. □

Next, consider an arithmetic Fano variety $\mathcal{X}$ (defined in § 2.2.1). Denote by $\hat{\mathcal{D}}_{\mathbb{Z}}(\psi)$ the functional defined by formula (2.15), corresponding to the integral model $\mathcal{L}=-\mathcal{K}_{\mathcal{X}}$. In this arithmetic setup the following variants of inequality (4.6) and identity (4.7) hold.

Proposition 5.3. *When $\mathcal{L}=-\mathcal{K}_{\mathcal{X}}$ the following relations hold:*

$$2\mathcal{M}_{(\mathcal{X},-\mathcal{K}_{\mathcal{X}})}\geq\operatorname{vol}(-K_X)\hat{\mathcal{D}}_{\mathbb{Z}}$$

and

$$\inf_{\mathcal{C}^0(L)\cap\mathrm{PSH}(L)}2\mathcal{M}_{(\mathcal{X},\mathcal{L})}=\operatorname{vol}(-K_X)\inf_{\mathcal{C}^0(L)\cap\mathrm{PSH}(L)}\hat{\mathcal{D}}_{\mathbb{Z}}.$$

Proof. First, note that the second term in the decomposition (5.4) of $\mathcal{M}_{(\mathcal{X},-\mathcal{K}_{\mathcal{X}})}(\psi)$ is precisely the entropy of $(dd^c\psi)^n/n!$ relative to $e^{-\psi}$:

$$\mathcal{M}_{(\mathcal{X},-\mathcal{K}_{\mathcal{X}})}(\psi)=-\frac{(\mathcal{L},\psi)^{n+1}}{(n+1)!}+\operatorname{Ent}_{e^{-\psi}}((dd^c\psi)^n/n!).$$

Since the entropy between two probability measure is non-negative (by Jensen's inequality) this proves the inequality in the proposition when the measure $e^{-\psi}$ has unit total volume. The general case then follows from a simple scaling argument.

Next, to prove the identity in the proposition fix a reference metric ψ_0 in $\mathcal{H}(X,-K_X)$ and rewrite the previous formula as

$$\frac{\mathcal{M}_{(\mathcal{X},-\mathcal{K}_{\mathcal{X}})}(\psi)}{\operatorname{vol}(-K_X)}=-\left(\frac{(\mathcal{L},\psi)^{n+1}}{(n+1)!\operatorname{vol}(-K_X)}+\langle\psi-\psi_0,MA(\psi)\rangle\right)+\frac{1}{2}\operatorname{Ent}_{e^{-\psi_0}}(MA(\psi)).\tag{5.5}$$

Accordingly, expressing $(\mathcal{L},\psi)^{n+1}=(\mathcal{L},\psi_0)^{n+1}+(n+1)!\mathcal{E}_{\psi_0}(\psi)/2$, using Lemma 2.9, gives

$$\frac{\mathcal{M}_{(\mathcal{X},-\mathcal{K}_{\mathcal{X}})}(\psi)}{\operatorname{vol}(-K_X)}=-\frac{1}{2}F_{\psi_0}(MA(\psi))-\frac{1}{(n+1)!}(\mathcal{L},\psi_0)^{n+1},$$

where $F_{\psi_0}(\mu)$ is the free energy functional (4.4). The proof is thus concluded by invoking the identity (4.7) and using Lemma 2.9 again. □

Remark 5.4. When $-K_X$ admits a Kähler–Einstein metric ϕ_{KE} both infima in the previous proposition are attained at ϕ_{KE} [BBEGZ19]. The identity then follows directly from the Kähler–Einstein equation, giving $MA(\phi_{\mathrm{KE}}) = e^{-\phi_{\mathrm{KE}}}$, when ϕ_{KE} is volume-normalized.

In § 6.2 the inequality in the previous proposition will be generalized to any model $(\mathcal{X}, \mathcal{L})$ of $(X, -K_X)$, by introducing an arithmetic Ding functional $\mathcal{D}_{(\mathcal{X},\mathcal{L})}$, coinciding (up to normalization) with the functional $\hat{\mathcal{D}}_{\mathbb{Z}}$ under the conditions in the previous proposition.

6. Discussion and outlook

6.1 The function field analog

Recall that, according to the philosophy of Arakelov geometry, the function field analog of a metrized arithmetic variety $\mathcal{X} \to \mathrm{Spec}\mathbb{Z}$ is a flat projective morphism

$$\mathscr{X} \to \mathscr{B}$$

from a normal complex projective variety $\mathscr{X}$ to a fixed regular complex projective curve $\mathscr{B}$. In particular, the analog of the setup of arithmetic Fano varieties in Conjecture 1.1 appears when $\mathscr{X}$ is normal, the relative anti-canonical divisor $-\mathscr{K}_{\mathscr{X}/\mathscr{B}}$ defines a relatively ample $\mathbb{Q}$-line bundle and the generic fiber is K-semistable. The analog of the inequality in Conjecture 1.1 does hold in this situation, but not the uniqueness statement. More precisely, if $(X, -K_X)$ is assumed K-semistable then it follows from [CP21] (see the beginning of [CP21, § 1.7.1]) that

$$(-\mathscr{K}_{\mathscr{X}/\mathscr{B}})^{n+1} \leq 0. \tag{6.1}$$

Equality holds for the trivial fibrations $\mathscr{X} = X \times \mathscr{B}$ for any K-semistable X. In particular,

$$(-\mathscr{K}_{\mathscr{X}/\mathscr{B}})^{n+1} \leq \left(-\mathscr{K}_{\mathbb{P}^n \times \mathscr{B}/\mathscr{B}}\right)^{n+1} (= 0) \tag{6.2}$$

which is the function field analog of the inequality in Conjecture 1.1. Note that when $\mathscr{B} = \mathbb{P}^1$ and the standard $\mathbb{C}^*$-action on $\mathbb{P}^1$ lifts to $\mathscr{X}$, inequality (6.1) follows directly from the definition of K-semistability.

Remark 6.1. The analog of the volume normalization (appearing in Conjecture 1.1) is automatically satisfied in the function field case. Indeed, the second term in the corresponding Ding functional $\mathcal{D}_{(\mathcal{X}_{\mathscr{X}/\mathscr{B}}, -\mathscr{K}_{\mathscr{X}/\mathscr{B}})}$, discussed in the following section, then vanishes.

In contrast to Conjecture 1.1, projective space thus plays no special role in the function field case (since equality holds in the inequality (6.2) for *any* product $\mathscr{X} = X \times \mathscr{B}$). Conversely, it should be stressed that the analog of inequality (6.1) *fails* in the arithmetic situation (by the strict positivity in Lemma 3.6). Hence, the function field analogy is somewhat deceptive. Our general motivation for Conjecture 1.1 is rather the analogy with the corresponding result over $\mathbb{C}$ (corresponding to the trivial morphism $X \to \mathrm{Spec}\mathbb{C}$) and the fact that projective space maximizes the degree of $-K_X$ [Fuj18], among K-semistable X of a given dimension (as well as a range of other positivity properties of $-K_X$; see, for example, the discussion and references in the introduction to [LZ18]).

6.2 A generalization of Conjecture 1.1

Consider a Fano variety X_F defined over a number field F, i.e. a field extension F of $\mathbb{Q}$ of finite degree $[F : \mathbb{Q}]$. Let $(\mathcal{X}, \mathcal{L})$ be a normal polarized model of $(X_F, -K_{X_F})$ over the ring of integers $\mathcal{O}_F$ of F such that $\mathcal{K}_{\mathcal{X}/\mathrm{Spec}\mathcal{O}_F}$ is defined as a $\mathbb{Q}$-line bundle. We will denote by ψ a collection of continuous psh ψ_σ metrics on $-K_{X_\sigma}$, where σ ranges over all embeddings of the field F into $\mathbb{C}$ and X_σ denotes the corresponding complex projective varieties. To the model $(\mathcal{X}, \mathcal{L})$ we attach an *arithmetic Ding functional,* defined as follows. First consider a model $(\mathcal{X},\mathcal{L})$ of $(X_F,-K_{X_F})$ such that $\mathcal{L}+\mathcal{K}_{\mathcal{X}/\mathrm{Spec}\mathcal{O}_F}$ defines a bona fide line bundle. Then

$$\mathcal{D}_{(\mathcal{X},\mathcal{L})} := \frac{[F:\mathbb{Q}](-K_X)^n}{n!}\hat{\mathcal{D}}_{(\mathcal{X},\mathcal{L})}(\psi),$$

in which $\hat{\mathcal{D}}_{(\mathcal{X},\mathcal{L})}(\psi)$ is the *normalized arithmetic Ding functional* defined by

$$\hat{\mathcal{D}}_{(\mathcal{X},\mathcal{L})}(\psi) := -\frac{(\mathcal{L},\psi)^{n+1}}{[F:\mathbb{Q}](n+1)(-K_X)^n}+\frac{1}{[F:\mathbb{Q}]}\widehat{\deg}\pi_*(\mathcal{L}+\mathcal{K}_{\mathcal{X}/\mathrm{Spec}\mathcal{O}_F}),$$

where the second term denotes the arithmetic (Arakelov) degree of the line bundle $\pi_*(\mathcal{L}+\mathcal{K}_{\mathcal{X}/\mathrm{Spec}\mathcal{O}_F})\to\mathrm{Spec}\mathcal{O}_F$, endowed with the L^2-metric induced by the metric ψ on $\mathcal{L}$ (i.e. on $-K_X$). More generally, when $\mathcal{K}_{\mathcal{X}/\mathrm{Spec}\mathcal{O}_F}$ is merely defined as a $\mathbb{Q}$-line bundle we fix a positive integer r such that $r(\mathcal{L}+\mathcal{K}_{\mathcal{X}/\mathrm{Spec}\mathcal{O}_F})$ is defined as a line bundle and replace $\widehat{\deg}\pi_*(\mathcal{X},(\mathcal{L}+\mathcal{K}_{\mathcal{X}/\mathrm{Spec}\mathcal{O}_F})$ with $r^{-1}\widehat{\deg}\pi_*(\mathcal{X},(r(\mathcal{L}+\mathcal{K}_{\mathcal{X}/\mathrm{Spec}\mathcal{O}_F}))$, where now $\pi_*(r(\mathcal{L}+\mathcal{K}_{\mathcal{X}/\mathrm{Spec}\mathcal{O}_F}))$ is endowed with the $L^{2/r}$-metric induced by ψ. Concretely, given a rational global section s_r of $\pi_*(r(\mathcal{L}+\mathcal{K}_{\mathcal{X}/\mathrm{Spec}\mathcal{O}_F}))$, one may express

$$\widehat{\deg}\pi_*(r(\mathcal{L}+\mathcal{K}_{\mathcal{X}/\mathrm{Spec}\mathcal{O}_F})) = -\frac{1}{2}\sum_\sigma \log\int_{X_\sigma}|s_r|^{2/r}e^{-\psi_\sigma}+\sum_{\mathfrak{p}}\mathrm{ord}_{\mathfrak{p}}(s_r)\log|\mathfrak{p}|, \tag{6.3}$$

where $|s_r|^{2/r}e^{-\psi_\sigma}$ denotes corresponding measure on X_σ, $\mathrm{ord}_{\mathfrak{p}}(s)$ denotes the order of vanishing of s_r at the closed point $\mathfrak{p}$ in $\mathrm{Spec}\mathcal{O}_F$ and $|\mathfrak{p}|$ denotes the norm of the prime ideal in $\mathcal{O}_F$ defined by $\mathfrak{p}$ (i.e. the cardinality of the corresponding residue field $\mathcal{O}_F/\mathfrak{p}$). The functional $\hat{\mathcal{D}}_{(\mathcal{X},\mathcal{L})}$ thus coincides with the functional $\hat{\mathcal{D}}_{\mathbb{Z}}$, defined in formula (2.15), up to an additive constant and a factor of 2. Note that when $F=\mathbb{Q}$ and $\mathcal{L}=-\mathcal{K}_{\mathcal{X}}$ we have $2\widehat{\deg}\pi_*(\mathcal{L}+\mathcal{K}_{\mathcal{X}/\mathrm{Spec}\mathcal{O}_F})=-\log\int_X e^{-\phi}$. Indeed, in this we can take $s_r=1\in H^0(\mathcal{X},\mathcal{O}_{\mathcal{X}})$, which is globally non-vanishing, by Lemma 2.3.

Remark 6.2. The functional $\mathcal{D}_{(\mathcal{X},\mathcal{L})}(\psi)$ is the arithmetic analog of the degree of the Ding line bundle of a test configuration $(\mathscr{X},\mathscr{L})$ for $(X,-K_X)$ introduced in [Ber16]. As shown in [Fuj19], a Fano variety X is K-semistable if and only if the degree of the Ding line bundle is non-negative for any test configuration $(\mathscr{X},\mathscr{L})$.

Now consider the following invariant of the Fano variety X_F:

$$\mathcal{D}(X_F) := \inf\big([F:\mathbb{Q}]^{-1}\mathcal{D}_{(\mathcal{X},\mathcal{L})}\big),$$

where the infimum runs over all integral models $(\mathcal{X},\mathcal{L})$ of $(X,-K_X)$ and metrics ψ as above. We propose the following generalization of Conjecture 1.1.

Conjecture 6.3. Let X_F be a K-semistable Fano variety defined over a number field F. Then

$$\mathcal{D}(X_F)\geq \mathcal{D}_{(\mathbb{P}^n_{\mathbb{Z}},-\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}})}(\psi_{\mathrm{FS}}),$$

where ψ_{FS} denotes the volume-normalized Fubini–Study metric ψ_{FS} on $-K_{\mathbb{P}^n}$. Equivalently, for any model $(\mathcal{X},\mathcal{L})$ and continuous psh metric ψ, normalized so that $\widehat{\deg}\pi_*(\mathcal{L}+\mathcal{K}_{\mathcal{X}/\mathrm{Spec}\mathcal{O}_F})=0$,

$$\frac{1}{[F:\mathbb{Q}]}(\mathcal{L},\psi)^{n+1}\leq(-\mathcal{K}_{\mathbb{P}^n_{\mathbb{Z}}},\psi_{\mathrm{FS}})^{n+1}.$$

Moreover, equality holds if and only if $(\mathcal{X},\mathcal{L})$ is isomorphic to $(\mathbb{P}^n_{\mathcal{O}_F},-\mathcal{K}_{\mathbb{P}^n_{\mathcal{O}_F}}+\pi^*M)$ for some line bundle $M\to\mathrm{Spec}\mathcal{O}_F$ and ψ coincides with ψ_{FS}, up to the action of an automorphism of $\mathbb{P}^n$.

Note that, in general, $\mathcal{D}_{(\mathcal{X},\mathcal{L})}(\psi) = \mathcal{D}_{(\mathcal{X},\mathcal{L}+\pi^*M)}(\psi)$ for any line bundle $M \to \mathrm{Spec}\mathcal{O}_F$. We expect – inspired by Odaka's conjecture discussed in § 1.4 – that any integral model $(\mathcal{X},\mathcal{L})$ which is globally K-semistable realizes the infimum defining the invariant $\mathcal{D}(X_F)$.

Next, given a polarized scheme $(\mathcal{X},\mathcal{L})$ over a number field F, we will, as in the case $F=\mathbb{Q}$, denote by $\mathcal{M}_{(\mathcal{X},\mathcal{L})}(\psi)$ the arithmetic Mabuchi functional defined by the intersection-theoretic expression in formula (5.1). In general, the following inequality between the arithmetic Mabuchi functional and the arithmetic Ding functional holds, showing, in particular, that Conjecture 6.3 implies Conjecture 1.6 concerning Odaka's modular invariant. The inequality can be viewed as an arithmetic analogy of the inequality for test configurations in [Ber16, Lemma 3.10].

PROPOSITION 6.4. *If $(\mathcal{X},\mathcal{L})$ is a normal polarized model of $(X,-K_X)$ over $\mathrm{Spec}\mathcal{O}_F$ which is $\mathbb{Q}$-Gorenstein, then*

$$\mathcal{M}_{(\mathcal{X},\mathcal{L})}(\psi) \geq \mathcal{D}_{(\mathcal{X},\mathcal{L})}(\psi)$$

with equality if and only if ψ is a Kähler–Einstein metric and $\mathcal{L}$ is isomorphic to $-\mathcal{K}_{\mathcal{X}/\mathrm{Spec}\mathcal{O}_F} \otimes \pi^*M$ for some line bundle M over $\mathrm{Spec}\mathcal{O}_F$.

Proof. To simplify the notation we assume that $r=1$ (but the proof in the general case is essentially the same). It follows directly from the definitions that we need to prove that

$$\frac{1}{L^n}(\overline{\mathcal{L}}+\overline{\mathcal{K}})\cdot\overline{\mathcal{L}}^n - \widehat{\deg}\pi_*(\mathcal{L}+\mathcal{K}_{\mathcal{X}/\mathrm{Spec}\mathcal{O}_F}) \geq 0 \tag{6.4}$$

with equality if and only if the conditions in the proposition hold. Observe that the left-hand side above is invariant when $\mathcal{L}$ is replaced by $\mathcal{L}+\pi^*M$, where M is any line bundle over $\mathrm{Spec}\mathcal{O}_F$. Hence, we may as well assume that $\pi_*(\mathcal{L}+\mathcal{K}_{\mathcal{X}/\mathrm{Spec}\mathcal{O}_F})$ admits a global regular section s that is non-vanishing over the generic fiber. Now, by the restriction formula for arithmetic intersection numbers [BGS94, Proposition 2.3.1],

$$\frac{1}{L^n}(\overline{\mathcal{L}}+\overline{\mathcal{K}})\cdot\overline{\mathcal{L}}^n = \frac{1}{2}\int_{X(\mathbb{C})}\log\left(\frac{MA(\psi)}{|s|^2e^{-\psi}}\right)MA(\psi) + \frac{1}{L^n}(s=0)\cdot\overline{\mathcal{L}}^n, \tag{6.5}$$

where $(s=0)$ denotes the subscheme of $\mathcal{X}$ cut out by s. By Jensen's inequality,

$$\begin{aligned}\int_{X(\mathbb{C})}\log\left(\frac{MA(\psi)}{|s|^2e^{-\psi}}\right)MA(\psi) &\geq -\frac{1}{2}\sum_{\sigma}\log\int_{X_\sigma}|s|^2e^{-\psi}\\ &= \widehat{\deg}\pi_*(\mathcal{L}+\mathcal{K}_{\mathcal{X}/\mathrm{Spec}\mathcal{O}_F}) - \sum_{\mathfrak{p}}\mathrm{ord}_{\mathfrak{p}}(s)\log|\mathfrak{p}|.\end{aligned} \tag{6.6}$$

Hence, decomposing the subscheme $(s=0)$ of $\mathcal{X}$ as a sum of effective divisors $E_{\mathfrak{p}}$, where $E_{\mathfrak{p}}$ is supported on the fiber $\mathcal{X}_{\mathfrak{p}}$ of $\mathcal{X}$ over $\mathfrak{p}$,

$$\frac{1}{L^n}(\overline{\mathcal{L}}+\overline{\mathcal{K}})\cdot\overline{\mathcal{L}}^n \geq \frac{1}{L^n}(s=0)\cdot\overline{\mathcal{L}}^n - \sum_{\mathfrak{p}}\mathrm{ord}_{\mathfrak{p}}(s)\log|\mathfrak{p}| = \left(\frac{1}{L^n}\mathcal{L}^n_{|\mathcal{X}_{\mathfrak{p}}}\cdot E_{\mathfrak{p}} - \sum_{\mathfrak{p}}\mathrm{ord}_{\mathfrak{p}}(s)\right)\log|\mathfrak{p}|,$$

using, again, the restriction formula in the last equality. Since $\mathrm{ord}_{\mathfrak{p}}(s)\geq 0$, we can express $E_{\mathfrak{p}} = E'_{\mathfrak{p}} + \mathrm{ord}_{\mathfrak{p}}(s)\mathcal{X}_{\mathfrak{p}}$ for an effective divisor $E'_{\mathfrak{p}}$, giving

$$\frac{1}{L^n}(\overline{\mathcal{L}}+\overline{\mathcal{K}})\cdot\overline{\mathcal{L}}^n \geq \left(\frac{1}{L^n}\mathcal{L}^n_{|\mathcal{X}_{\mathfrak{p}}}\cdot E'_{\mathfrak{p}}\right)\log|\mathfrak{p}| \geq 0.$$

Finally, equality holds in inequality (6.6) if and only if $MA(\psi)$ is proportional to $|s|^2e^{-\psi}$ for all X_σ, i.e. if and only if ψ is Kähler–Einstein. Moreover, since $\mathcal{L}$ is relatively ample the right-hand side in the last inequality above vanishes if and only if $E'_{\mathfrak{p}}$ is the zero-divisor for all $\mathfrak{p}$, i.e. if and

https://doi.org/10.1112/S0010437X2400736X Published online by Cambridge University Press

only if $(s=0)$ is a linear combination of fibers $\mathcal{X}_{\mathfrak{p}}$ and thus $\mathcal{L}+\mathcal{K}$ is isomorphic to π^*M for some line bundle M over $\mathrm{Spec}\mathcal{O}_F$. □

6.3 Comparison with bounds on Bost–Zhang normalized heights

The normalized arithmetic Ding functional $\hat{\mathcal{D}}_{(\mathcal{X},\mathcal{L})}$ is reminiscent of Bost's normalized height h_{norm}, introduced in [Bos96] in the general setup of polarized variety (X_F,L_F) defined over a number field F:

$$h_{\mathrm{norm}}(\mathcal{L},\psi):=\frac{(\mathcal{L},\psi)^{n+1}}{[F:\mathbb{Q}](n+1)(L_F)^n}-\frac{1}{[F:\mathbb{Q}]N}\widehat{\deg}\pi_*\mathcal{X},$$

assuming that the rank N of the vector bundle $\pi_*\mathcal{L}\to\mathrm{Spec}\mathcal{O}_F$ is non-zero and $\pi_*(\mathcal{X},\mathcal{L})$ is endowed with the L^2-norm induced by the continuous psh metrics ψ_σ on L_σ and the volume forms $MA(\psi_\sigma)$ on X_σ (defined by formula (4.3)). When L_F is very ample it is shown in [Bos96] that the functional $h_{\mathrm{norm}}(\mathcal{L},\cdot)$ is bounded from below if and only if the Chow point of (X_F,L_F) is semistable with respect to the action of the group $GL(N,F)$ on the Chow variety (in the sense of geometric invariant theory). More precisely, it is shown in [Bos96] that the semistability in question is equivalent to a lower bound on Bost's intrinsic normalized height of (X_F,L_F):

$$\inf h_{\mathrm{norm}}>-\infty,$$

where the infimum runs over all models $(\mathcal{X},\mathcal{L})$ and metrics ψ as above. In fact, by [Bos96, Proposition 2.1] and [Zha96, Theorem 4.4] the Chow semistability in question is equivalent to the following explicit lower bound:

$$h_{\mathrm{norm}}(\mathcal{L},\psi)\geq-\frac{1}{2}\sum_{n=1}^{N+1}\sum_{m=1}^{n}\frac{1}{m}-\frac{1}{2}\log N \tag{6.7}$$

(moreover, it is conjectured in [Zha96] that the first term in the right-hand side above may be replaced by 0).

In this setup the role of the normalization $\widehat{\deg}\pi_*(\mathcal{L}+\mathcal{K}_{\mathcal{X}/\mathrm{Spec}\mathcal{O}_F})=0$ in Conjecture 6.3 is thus played by the normalization $\widehat{\deg}\pi_*\mathcal{L}=0$. However, in contrast to Conjecture 6.3, the lower bound (6.7) on $h_{\mathrm{norm}}(\mathcal{L},\psi)$ corresponds to a *lower* bound on $(\mathcal{L},\psi)^{n+1}$ for any normalized metric. Note also that one virtue of the normalization condition in Conjecture 6.3 is that it is comparatively explicit, since $\pi_*(\mathcal{L}+\mathcal{K}_{\mathcal{X}/\mathrm{Spec}\mathcal{O}_F})$ has rank 1 (so that formula (6.3) applies, showing that it is enough to assume that the volume forms $|s_r|^{2/r}e^{-\psi_\sigma}$ on X_σ are normalized). Another advantage of this normalization condition is that it applies to any continuous metric ψ (at the cost of replacing $(\mathcal{L},\psi)^{n+1}$ with the χ-arithmetic volume of $\mathcal{L}$, as in Theorem 2.5).

Finally, we recall that when $\mathcal{L}$ is replaced by $k\mathcal{L}$ for a large positive integer k it follows from [Oda18, Theorem 3.7] that there exist constants $a>0$ and b (depending only on (X_F,L_F)) such that

$$\mathcal{M}_{(\mathcal{X},\mathcal{L})}(\psi)/L^n=h_{\mathrm{norm}}(k\mathcal{L},\psi)-a\log N_k+b+o(1), \tag{6.8}$$

as $k\to\infty$, where N_k denotes the rank of $H^0(\mathcal{X},k\mathcal{L})$ which diverges as $k\to\infty$. Unfortunately, the diverging term $a\log N_k$ makes it impossible to infer lower bounds on $\mathcal{M}_{(\mathcal{X},\mathcal{L})}(\psi)$ from lower bounds on $h_{\mathrm{norm}}(k\mathcal{L})$. Since $\mathcal{M}_{(\mathcal{X},\mathcal{L})}(\psi)$ coincides with $\mathcal{D}_{(\mathcal{X},\mathcal{L})}(\psi)$ when $\mathcal{L}$ equals $-\mathcal{K}_{\mathcal{X}/\mathrm{Spec}\mathcal{O}_F}$ this means that Conjecture 6.3 cannot be deduced from bounds of the form (6.7) by letting k (and hence N) tend to infinity.

6.4 Comparison with Odaka's and Faltings's modular heights

Finally, let us compare our normalizations of the arithmetic Mabuchi functional with those of Odaka [Oda20] and Faltings [Fal83a]. First of all, our multiplicative normalization for the arithmetic Mabuchi functional $\mathcal{M}_{(\mathcal{X},\mathcal{L})}$ (formula (1.6)) is made so that $\pm\mathcal{M}_{(\mathcal{X},\pm K_{\mathcal{X}})} = (\pm\mathcal{K}_{\mathcal{X}})^{n+1}/(n+1)$. Moreover, as discussed in § 1.4.1, we are employing the metric on $-K_X$ induced by the *normalized* volume form ω^n/L^n of the Kähler form ω defined by a given metric ψ on $\mathcal{L}$ with positive curvature (i.e. $\omega = dd^c\psi$). Comparing with Odaka's arithmetic Mabuchi functional, which we shall denote by $\mathcal{M}^{(O)}_{(\mathcal{X},\mathcal{L})}(\psi)$, thus yields

$$\frac{1}{(n+1)!L^n}\mathcal{M}^{(O)}_{(\mathcal{X},\mathcal{L})} = \mathcal{M}_{(\mathcal{X},\mathcal{L})} + \frac{1}{2}\frac{L^n}{n!}\log(L^n/n!). \tag{6.9}$$

In the case that $\mathcal{X}$ is an abelian variety it was shown in [Oda20] that the infimum of Odaka's arithmetic Mabuchi functional over all metrics on $\mathcal{L}$ with positive curvature coincides with Faltings's (modular) height [Fal83a], up to a multiplicative and an additive constant depending on L^n. Here we note that our normalizations are consistent with those of Faltings.

Proposition 6.5. *Let $\mathcal{X}$ be a projective and flat scheme over $\mathbb{Z}$ and assume that $\mathcal{K}_{\mathcal{X}}$ is trivial. For any relatively ample line bundle $\mathcal{L}$ over $\mathcal{X}$,*

$$\inf_{\psi}\frac{1}{L^n/n!}\mathcal{M}_{(\mathcal{X},\mathcal{L})}(\psi) = -\frac{1}{2[\mathbb{F}:\mathbb{Q}]}\log\frac{1}{2^n}\left|\int_{X(\mathbb{C})}\Omega\wedge\bar{\Omega}\right|, \tag{6.10}$$

where Ω is a generator of $H^0(\mathcal{X},\mathcal{K}_{\mathcal{X}})$ and the infimum ranges over all psh metrics ψ on $\mathcal{L}$ and $V := L^n/n!$.

Proof. This is essentially equivalent to [Oda20, Theorem 2.11], using relation (6.9). In any case, in order to verify that all normalizations are consistent we provide a simple direct proof. Assume, to simplify the notation, that $\mathbb{F} = \mathbb{Q}$. Recall that Faltings's modular height [Fal83a] is defined as the arithmetic degree of $\pi_*(\mathcal{X}, K_{\mathcal{X}})$, with respect to the L^2-metric on $H^0(X, K_X)$ defined by $\|\Omega\|^2 := (1/2^n)|\int_{X(\mathbb{C})}\Omega\wedge\bar{\Omega}|$. This is precisely the right-hand side in formula (6.10). As for the left-hand side, it is given by

$$\int_X \log\left(\frac{(dd^c\psi)^n/Vn!}{(i^{n^2/2}/2^n)\Omega\wedge\bar{\Omega}/\|\Omega\|^2}\right)\frac{(dd^c\psi)^n}{Vn!} = \int_X \log\left(\frac{(dd^c\psi)^n/Vn!}{(i^{n^2/2}/2^n)\Omega\wedge\bar{\Omega}/\|\Omega\|^2}\right)\frac{(dd^c\psi)^n}{Vn!} - \log\|\Omega\|^2$$

(as follows readily from the definitions, just as in formula (6.5)). Now, by Jensen's inequality this expression is minimal precisely when the two probability measures $(dd^c\psi)^n/Vn!$ and $2^{-n}i^{n^2/2}\Omega\wedge\bar{\Omega}/\|\Omega\|^2$ coincide, which, equivalently, means that $dd^c\psi$ is a Kähler–Einstein metric. By the Calabi–Yau theorem such a metric exists for any given ample L, which concludes the proof. □

The previous proposition has the following consequence, when combined with well-known properties of Faltings's modular height of abelian varieties (cf. the discussion in relation to [Oda20, Theorem 2.11] and [Oda20, § 2.3.2]). Consider a polarized abelian variety $(X_{\mathbb{F}_0}, L_{\mathbb{F}_0})$ defined over a given number field $\mathbb{F}_0$. Then the infimum of $\mathrm{vol}(L)^{-1}\mathcal{M}_{(\mathcal{X},\mathcal{L})}$ over all metrics, finite field extensions $\mathbb{F}$, models over $\mathcal{O}_{\mathbb{F}}$ and positively curved metrics on $L \to X_{\mathbb{F}}(\mathbb{C})$ is attained at any semistable reduction of the Néron model $\mathcal{X}$ of $X_{\mathbb{F}}$, when L is endowed with a Kähler–Einstein metric. Moreover, in the particular case of elliptic curves it was observed in [Del85, p. 29] that the minimal value of the aforementioned infimum over all $X_{\mathbb{F}}$ is attained at the semistable reduction of the Néron model $\mathcal{X}_0$ of any elliptic curve with vanishing j-invariant ($\mathcal{X}_0$ is uniquely determined for any sufficiently large field extension). Thus the role of $\mathcal{X}_0$ among all models of elliptic curves is somewhat analogous to the role of $\mathbb{P}^n_{\mathbb{Z}}$ in Conjectures 1.1 and 1.6. However, it should be stressed

https://doi.org/10.1112/S0010437X2400736X Published online by Cambridge University Press

that in the setup of Fano varieties the choice of multiplicative normalization is crucial. Indeed, while $\mathbb{P}^n_{\mathbb{Z}}$ minimizes $\mathcal{M}_{(\mathcal{X},-\mathcal{K}_{\mathcal{X}})}(\psi_{\mathrm{KE}})$ over the canonical toric integral models of all K-semistable toric Fano varieties X (assuming that $n \leq 6$) it does not minimize $\mathrm{vol}(-K_X)^{-1}\mathcal{M}_{(\mathcal{X},-\mathcal{K}_{\mathcal{X}})}(\psi_{\mathrm{KE}})$. In fact, for all we know it could actually be the case that $\mathrm{vol}(-K_X)^{-1}\mathcal{M}_{(\mathcal{X},-\mathcal{K}_{\mathcal{X}})}(\psi_{\mathrm{KE}})$ is *maximal* on $\mathbb{P}^n_{\mathbb{Z}}$. For example, this turns out to be the case in the more general setup of Fano orbifolds (not assumed toric) when X has relative dimension 1 (a proof will appear in a separate publication).

Acknowledgements

We are grateful to Bo Berndtsson, Dennis Eriksson, Gerard Freixas i Montplet, Benjamin Nill, Yuji Odaka, Per Salberger, Chenyang Xu and Ziquan Zhuang for illuminating discussions/comments and, in particular, to Alexander Kasprzyk for updating the database [Øbr07]. We are also grateful to the refere for very helpful comments.

Conflicts of interest

None.

Financial support

This work was supported by grants from the Knut and Alice Wallenberg Foundation, the Göran Gustafsson Foundation and the Swedish Research Council.

Data availability

The database of smooth toric Fano varieties of dimension at most six (http://www.grdb.co.uk/forms/toricsmooth) is used in the proof of Theorem 1.2. This data is provided by Mikkel Obro.

Journal information

Compositio Mathematica is owned by the Foundation Compositio Mathematica and published by the London Mathematical Society in partnership with Cambridge University Press. All surplus income from the publication of *Compositio Mathematica* is returned to mathematics and higher education through the charitable activities of the Foundation, the London Mathematical Society and Cambridge University Press.

References

AB23 R. Andreasson and R. J. Berman, *Sharp bounds on the height of K-semistable Fano varieties II, The log case*, Preprint (2023), arXiv:2312.05064.

Ara+ C. Araujo, *et al.*, The Calabi problem for Fano threefolds. https://www.maths.ed.ac.uk/cheltsov/pdf/Fanos.pdf.

AKM04 S. Artstein, M. Klartag and V. Milman, *The Santaló point of a function and a functional form of Santaló inequality*, Mathematika **51** (2004), 33–48.

Ber13 R. J. Berman, *A thermodynamical formalism for Monge–Ampère equations, Moser–Trudinger inequalities and Kähler–Einstein metrics*, Adv. Math. **248** (2013), 1254.

Ber16 R. J. Berman, *K-polystability of Q-Fano varieties admitting Kähler–Einstein metrics*, Invent. Math. **203** (2016), 973–1025.

Ber23 R. J. Berman, *On K-stability, height bounds and the Manin-Peyre conjecture*, Pure Appl. Math. Q., to appear. Preprint (2023), arXiv:2305.07272.

BB13 R. J. Berman and B. Berndtsson, *Real Monge–Ampère equations and Kähler–Ricci solitons on toric log Fano varieties*, Ann. Fac. Sci. Toulouse Math. (6) **22** (2013), 649–711.

https://doi.org/10.1112/S0010437X2400736X Published online by Cambridge University Press

BB17 R. J. Berman and B. Berndtsson, *The volume of Kähler–Einstein Fano varieties and convex bodies*, J. Reine Angew. Math. **723** (2017), 127–152.

BB10 R. J. Berman and S. Boucksom, *Growth of balls of holomorphic sections and energy at equilibrium*, Invent. Math. **181** (2010), 337.

BBEGZ19 R. J. Berman, S. Boucksom, P. Eyssidieux, V. Guedj and A. Zeriahi, *Kähler–Einstein metrics and the Kähler–Ricci flow on log Fano varieties*, J. Reine Angew. Math. **751** (2019), 27–89.

BBGZ13 R. J. Berman, S. Boucksom, V. Guedj and A. Zeriahi, *A variational approach to complex Monge–Ampère equations*, Publ. Math. Inst. Hautes Études Sci. **117** (2013), 179–245.

BDL17 R. J. Berman, T. Darvas and C. H. Lu, *Convexity of the extended K-energy and the long time behavior of the Calabi flow*, Geom. Topol. **21** (2017), 2945–2988.

BF14 R. J. Berman and G. Freixas i Montplet, *An arithmetic Hilbert–Samuel theorem for singular hermitian line bundles and cusp forms*, Compos. Math. **150** (2014), 1703–1728.

Ber21 B. Berndtsson, *Complex integrals and Kuperberg's proof of the Bourgain–Milman theorem*, Adv. Math. **388** (2021), 107927.

BX19 H. Blum and C. Xu, *Uniqueness of K-polystable degenerations of Fano varieties*, Ann. of Math. (2) **190** (2019), 609–665.

Bos94 J.-B. Bost, *Semistability and height of cycles*, Invent. Math. **118** (1994), 223–253.

Bos96 J.-B. Bost, *Intrinsic heights of stable varieties and abelian varieties*, Duke Math. J. **82** (1996), 21–70.

BGS94 J.-B. Bost, H. Gillet and C. Soulé, *Heights of projective varieties and positive Green forms*, J. Amer. Math. Soc. **7** (1994), 903–102.

BE21 S. Boucksom and D. Eriksson, *Spaces of norms, determinant of cohomology and Fekete points in non-Archimedean geometry*, Adv. Math. **378** (2021), 107501.

BMPS16 J. I. Burgos Gil, A. Moriwaki, P. Philippon and M. Sombra, *Arithmetic positivity on toric varieties*, Algebr. Geom. **25** (2016), 201–272.

BPS14 J. I. Burgos Gil, P. Philippon and M. Sombra, *Arithmetic geometry of toric varieties. Metrics, measures and heights*, Astérisque **360** (2014).

CM00 J. Cassaigne and V. Maillot, *Hauter des hypersurfaces et fonctions zeta d'Igusa*, J. Number Theory **83** (2000), 226–255.

Che00 X. X. Chen, *On the lower bound of the Mabuchi energy and its application*, Int. Math. Res. Not. IMRN **2000** (2000), 607–623.

CDS15 X. X. Chen, S. Donaldson and S. Sun, *Kähler–Einstein metrics on Fano manifolds, I, II, III*, J. Amer. Math. Soc. **28** (2015), 183–278.

CP21 G. Codogni and Z. Patakfalvi, *Positivity of the CM line bundle for families of K-stable klt Fano varieties*, Invent. Math. **223** (2021), 811–894.

CGS19 D. Coman, V. Guedj and S. Sahin, *Toric pluripotential theory*, Ann. Polon. Math. **123** (2019), 215–242.

CLS11 D. A. Cox, J. B. Little and H. K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124 (American Mathematical Society, Providence, RI, 2011).

Deb03 O. Debarre, *Fano varieties*, in *Higher dimensional varieties and rational points (Budapest 2001)*, Bolyai Soc. Math. Stud., vol. 12 (Springer, Berlin, 2003), 93–13.

Del85 P. Deligne, *Preuve des conjectures de Tate et de Shafarevitch (d'après G. Faltings)*, Astérisque **121–122** (1985), 25–41.

Din88 W.-Y. Ding, *Remarks on the existence problem of positive Kähler–Einstein metrics*, Math. Ann. **282** (1988), 463–471.

DT92 W.-Y. Ding and G. Tian, *Kähler–Einstein metrics and the generalized Futaki invariant*, Invent. Math. **110** (1992), 315–335.

Don02 S. K. Donaldson, *Scalar curvature and stability of toric varieties*, J. Differential Geom. **62** (2002), 289–349.

Fal83a G. Faltings, *Endlichkeitssätze für abelsche Varietäten iiber Zahlkorpern*, Invent. Math. **73** (1983), 349–366.

Fal84 G. Faltings, *Calculus on arithmetic surfaces*, Ann. of Math. (2) **119** (1984), 387–424.

Fal91 G. Faltings, *Diophantine approximation on abelian varieties*, Ann. of Math. (2) **119** (1991), 549–576.

Fuj18 K. Fujita, *Optimal bounds for the volumes of Kähler–Einstein Fano manifolds*, Amer. J. Math. **140** (2018), 391–414.

Fuj19 K. Fujita, *A valuative criterion for uniform K-stability of Q-Fano varieties*, J. Reine Angew. Math. **751** (2019), 309–338.

GS90 H. Gillet and C. Soulé, *Characteristic classes for algebraic vector bundles with Hermitian metric. II*, Ann. of Math. (2) **131** (1990), 205–238.

GS92 H. Gillet and C. Soulé, *An arithmetic Riemann–Roch theorem*, Invent. Math. **110** (1992), 473–454.

HO22 M. Hattori and Y. Odaka, *Minimization of Arakelov K-energy for many cases*, Preprint (2022), arXiv:2211.03415.

KK02 C. Kaiser and K. Köhler, *A fixed point formula of Lefschetz type in Arakelov geometry III: Representations of Chevalley schemes and heights of flag varieties*, Invent. Math. **147** (2002), 633–669.

Kol13 J. Kollár, *Singularities of the minimal model program*, Cambridge Tracts in Mathematics, vol. 200 (Cambridge University Press, Cambridge, 2013). With the collaboration of Sándor Kovács.

Kup08 G. Kuperberg, *From the Mahler conjecture to Gauss linking integrals*, Geom. Funct. Anal. **18** (2008), 870–892.

Li17 C. Li, *Yau–Tian–Donaldson correspondence for K-semistable Fano manifolds*, J. Reine Angew. Math. **733** (2017), 55–85.

Li22 C. Li, *G-uniform stability and Kähler–Einstein metrics on Fano varieties*, Invent. Math. **227** (2022), 661–744.

LWX18 C. Li, X. Wang and C. Xu, *Quasi-projectivity of the moduli space of smooth Kähler–Einstein Fano manifolds*, Ann. Sci.Éc. Norm. Supér. (4) **51** (2018), 739–772.

LWX21 C. Li, X. Wang and C. Xu, *Algebraicity of the metric tangent cones and equivariant K-stability*, J. Amer. Math. Soc. **34** (2021), 1175–1214.

Liu18 Y. Liu, *The volume of singular Kähler–Einstein Fano varieties*, Compos. Math. **154** (2018), 1131–1158.

LXZ22 Y. Liu, C. Xu and Z. Zhuang, *Finite generation for valuations computing stability thresholds and applications to K-stability*, Ann. of Math. (2) **196** (2022), 507–566.

LZ18 Y. Liu and Z. Zhuang, *Characterization of projective spaces by Seshadri constants*, Math. Z. **289** (2018), 25–38.

Mab86 T. Mabuchi, *K-energy maps integrating Futaki invariants*, Tohoku Math. J. (2) **38** (1986), 575–593.

Mah39 K. Mahler, *EinÜbertragungsprinzip für konvexe Körper*, Časopis Pest Mat. Fys. **68** (1939), 93–102.

Mai95 V. Maillot, *Un calcul de Schubert arithmétique*, Duke Math. J. **80** (1995), 195–221.

Mai00 V. Maillot, *Géométrie d'Arakelov des variétés toriques et fibrés en droites intégrables*, in *Mém. Soc. Math. France*, vol. 80 (Société mathématique de France, 2000).

Man85 Y. Manin, *New dimensions in geometry*, in *Workshop at Bonn 1984 (Bonn, 1984)*, Lecture Notes in Mathematics, vol. 1111 (Springer, 1985), 59–101.

MS24 J. Moraga and H. Süß, *Bounding toric singularities with normalized volume*, Bull. London Math. Soc. **56** (2024), 2212–2229.

Mor99 A. Moriwaki, *The continuity of Deligne's pairing*, Int. Math. Res. Not. IMRN **19** (1999), 1057–1066.

NT96 T. Nakashima and Y. Takeda, *Chern classes of vector bundles on arithmetic varieties*, Pacific J. Math. **176** (1996), 205–216.

Øbr07 M. Øbro, *An algorithm for the classification of smooth Fano polytopes*, Preprint (2007), arXiv:0704.0049. Database at http://www.grdb.co.uk/forms/toricsmooth.

Oda13a Y. Odaka, *A generalization of the Ross–Thomas slope theory*, Osaka J. Math. **50** (2013), 171–185.

Oda13b Y. Odaka, *The GIT stability of polarized varieties via discrepancy*, Ann. of Math. **177** (2013), 645–666.

Oda18 Y. Odaka, *Canonical Kähler metrics and arithmetics: generalizing Faltings heights*, Kyoto J. Math. **58** (2018), 243–288.

Oda20 Y. Odaka, *On the K-stability and moduli of varieties – their connection with Kähler–Einstein metrics*, Sūgaku **72** (2020), 337–364.

Par88 A. N. Parshin, *The Bogomolov–Yau–Miyaoka inequality for arithmetical surfaces and its applications*, in *Seminaire de theorie des hombres de Paris*, Progr. Maths, vol. 75 (Birkhäuser, 1988), 299–312.

RT07 J. Ross and R. Thomas, *A study of the Hilbert–Mumford criterion for the stability of projective varieties*, J. Algebraic Geom. **16** (2007), 201–255.

RLV00 R. Rumely, C. F. Lau and R. Varley, *Existence of the sectional capacity*, Mem. Amer. Math. Soc. **145** (2000).

Sou94 C. Soulé, *A vanishing theorem on arithmetic surfaces*, Invent. Math. **116** (1994), 544–899.

Sou21 C. Soulé, *Chapter I: Arithmetic intersection*, in *Arakelov geometry and Diophantine applications* (Springer, 2021), 9–36.

SABK92 C. Soulé, D. Abramovich, J.-F. Burnol and J. Kramer, *Lectures on Arakelov geometry*, Cambridge Stud. Adv. Math., vol. 33 (Cambridge University Press, Cambridge, 1992).

SS17 C. Spotti and S. Sun, *Explicit Gromov–Hausdorff compactifications of moduli spaces of Kähler–Einstein Fano manifolds*, Pure Appl. Math. Q. **13** (2017), 477–515.

Tam99a H. Tamvakis, *Arakelov theory of the Lagrangian Grassmannian*, J. Reine Angew. Math. **516** (1999), 207–223.

Tam99b H. Tamvakis, *Schubert calculus on the arithmetic Grassmannian*, Duke Math. J. **98** (1999), 421–443.

Tam00 H. Tamvakis, *Height formulas for homogeneous varieties*, Michigan Math. J. **48** (2000), 593–610.

Tia96 G. Tian, *Kähler–Einstein metrics on algebraic manifolds*, in *Transcendental methods in algebraic geometry*, Lecture Notes in Mathematics, vol. 1646 (Springer, Berlin, 1996), 143–185.

Tia97 G. Tian, *Kähler–Einstein metrics with positive scalar curvature*, Invent. Math. **130** (1997), 1–37.

TT71 S. R. Tims and J. A. Tyrrell, *Approximate evaluation of Euler's constant*, Math. Gaz. **55** (1971), 65–67.

Voj88 P. Vojta, *Diophantine inequalities and Arakelov theory*, in *Introduction to Arakelov theory*, ed. S. Lang (Springer, 1988), 155–178.

Wan12 X. Wang, *Height and GIT weight*, Math. Res. Lett. **19** (2012), 909–926.

WZ04 X. Wang and X. Zhu, *Kähler-Ricci solitons on toric manifolds with positive first Chern class*, Adv. Math. **188** (2004), 87–103.

Xu21 C. Xu, *K-stability of Fano varieties: an algebro-geometric approach*, EMS Surv. Math. Sci. **8** (2021), 265–354.

Zha95 S. Zhang, *Positive line bundles on arithmetic varieties*, J. Amer. Math. Soc. **8** (1995), 187–221.

Zha96 S. Zhang, *Heights and reduction of semi-stable varieties*, Compos. Math. **104** (1996), 77–105.

Zhu21 Z. Zhuang, *Optimal destabilizing centers and equivariant K-stability*, Invent. Math. **226** (2021), 195–223.

Rolf Andreasson rolfan@chalmers.se
Chalmers University of Technology and the University of Gothenburg, Chalmers tvärgata 3, SE-412 96 Göteborg, Sweden

Robert J. Berman robertb@chalmers.se
Chalmers University of Technology and the University of Gothenburg, Chalmers tvärgata 3, SE-412 96 Göteborg, Sweden

https://doi.org/10.1112/S0010437X2400736X Published online by Cambridge University Press