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ON ASYMPTOTICALLY NONEXPANSIVE SEMIGROUPS OF MAPPINGS

BY

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1. **Introduction.** A selfmapping f of a metric space (X, d) is *nonexpansive* (*ε-nonexpansive*) if $d(f(x), f(y)) \leq d(x, y)$ for all $x, y \in X$ (respectively if $d(x, y) < \varepsilon$). In [1], M. Edelstein proved that a nonexpansive mapping f of E^n admits a fixed point provided the f-closure of E^n (i.e. the set of all points which are cluster points of $\{f^n(x)\}$ for some x) is nonempty. R. D. Holmes [2] considered commutative semigroups of selfmappings of a metric space and obtained fixed point theorems for such semigroups under certain contractivity conditions.

In this note, we consider asymptotically nonexpansive semigroups G which are defined analogously to the asymptotic contractions considered in [2]. For a general metric space X, we obtain results parallel to those of [1], and in the special case where $X = E^n$, we prove, under certain conditions, that there exists a point z such that G acts on the orbit of z as a semigroup of translations. As a corollary, it follows that if some orbit is bounded, G has a common fixed point.

2. **Definitions and Notation.** Throughout, (X, d) (or simply X) will denote a metric space and E^n the n-dimensional Euclidean space. Let $G: X \to X$ denote the collection of mappings $g: X \to X$, $g \in G$, where G is a commutative semigroup of mappings with identity(2). Set $G(x) = \{g(x): g \in G\}$ (the *orbit* of x), $G(A) = \{g(x): g \in G, x \in A\}$ and $G \mid A = \{g \mid A: g \in G\}$ for $A \subseteq X$. A *fixed point* for G will be a point $z \in X$ such that $g(z) = z$ for all g in G.

If $G: X \to X$, we say that G is *asymptotically nonexpansive* if for all x, y in X

(2.1) there exists $g \in G$ such that for all $f \in G$, $d(fg(x), fg(y)) \leq d(x, y)$.

If there is an $\varepsilon > 0$ such that whenever $d(x, y) < \varepsilon$, (2.1) holds, then G is called *ε-asymptotically nonexpansive*. G is said to be *asymptotically* (*ε-asymptotically*) *isometric* if

(2.2) for all $x, y \in X$ (with $d(x, y) < \varepsilon$) there exists $g \in G$ such that for all f in G, $d(fg(x), fg(y)) = d(x, y)$.

By the *G-closure* of X (denoted by X^G) we shall mean the set

(2.3) $\{z \in X \mid \exists x \in X$ such that $\forall f \in G, \varepsilon > 0, \exists g \in G$ with $d(fg(x), z) < \varepsilon\}$.

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(2) The presence of an identity is not essential for our results, and is assumed for convenience only.

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In the case when G is generated by a single mapping f, the G-closure of X is precisely the f-closure of X as defined in [1].

If $A \subseteq E^n$, co $A(\overline{\text{co}}\, A)$ denotes the convex hull (closed convex hull) of A. If $G: E^n \to E^n$ and $z \in E^n$ is such that $G \mid G(z)$ is a semigroup of translations, then z will be called a *T-point* (*translational point*) for G.

3. Asymptotically nonexpansive semigroups in general metric spaces.

PROPOSITION 1. *Let $G: X \to X$ be ε-asymptotically nonexpansive and $z \in X^G$. Then*

(3.1) $\forall f \in G, \varepsilon > 0\ \exists g \in G$ *such that* $d(fg(z), z) < \varepsilon$ (i.e. *in condition* (2.3), x *can be replaced by* z).

Proof. Let x be as in (2.3) and suppose that $f \in G$, $\varepsilon > 0$. It clearly follows from (2.3) that $\exists\ g_0 \in G$ so that $d(g_0(x), z) < \varepsilon/2$, and G being ε-asymptotically nonexpansive, $\exists g_1 \in G$ such that $\forall h \in G$, $d(hg_1g_0(x), hg_1(z)) \leq d(g_0(x), z) < \varepsilon/2$. Replacing f in (2.3) by fg_1g_0, we can find a $g_2 \in G$ such that $d(fg_2g_1g_0(x), z) < \varepsilon/2$. Setting $g = g_2g_1$ and $h = fg_2$, we have $d(gf(z), z) \leq d(fg_2g_1(z), fg_2g_1g_0(x)) + d(fg_2g_1g_0(x), z) < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

LEMMA 1. *Under the hypotheses of Proposition 1, $G \mid G(z)$ is an ε-asymptotic isometry.*

Proof. Suppose that contrary to the assumption $\exists f_1, f_2 \in G$ with $d(f_1(z), f_2(z)) < \varepsilon$ and such that $\forall g \in G$, $\exists f \in G$ for which $d(fgf_1(z), fgf_2(z)) \neq d(f_1(z), f_2(z))$. Since G is ε-asymptotically nonexpansive, there is a $g_0 \in G$ such that

$$\delta = d(f_1(z), f_2(z)) - d(g_0f_1(z), g_0f_2(z)) > 0.$$

Also replacing x and y in (2.1) by $g_0f_1(z)$, $g_0f_2(z)$ respectively, we obtain a $g_1 \in G$ such that

$$d(fg_1g_0f_1(z), fg_1g_0f_2(z)) \leq d(g_0f_1(z), g_0f_2(z)), \quad \forall f \in G.$$

Now by Proposition 1, $d(h_0g_1g_0(z), z)$ can be made so small that the continuity of f_1 and f_2 will imply the existence of an $h_0 \in G$ for which

$$d(h_0g_1g_0f_1(z), f_1(z)) < \delta/2 \quad \text{and} \quad d(h_0g_1g_0f_2(z), f_2(z)) < \delta/2.$$

Hence

$$\begin{aligned} d(f_1(z), f_2(z)) &\leq d(f_1(z), h_0g_1g_0f_1(z)) + d(h_0g_1g_0f_1(z), h_0g_1g_0f_2(z)) + d(h_0g_1g_0f_2(z), f_2(z)) \\ &< \delta/2 + d(g_0f_1(z), g_0f_2(z)) + \delta/2, \end{aligned}$$

which is impossible.

Now it follows from Lemma 1 that if G is asymptotically nonexpansive and $z \in X^G$, then $G \mid G(z)$ is an asymptotic isometry. An even stronger conclusion is obtained in the following

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PROPOSITION 2. *If $G: X \to X$ is asymptotically nonexpansive and $z \in X^G$, then $G \mid G(z)$ is an isometry.*

Proof. We have to show that $\forall f_1, f_2, g_0 \in G$, $d(g_0 f_1(z), g_0 f_2(z)) = d(f_1(z), f_2(z))$. By Lemma 1, $\exists g_1, g_2 \in G$ such that for all f, we have

$$d(fgf_1(z), fgf_2(z)) = d(f_1(z), f_2(z))$$

and

$$d(fg_2g_0f_1(z), fg_2g_0f_2(z)) = d(g_0f_1(z), g_0f_2(z)).$$

Substituting $g_2 g_0$ and g_1 for f in the first and second equalities respectively, and applying commutativity of G, the result follows.

REMARKS. (1) If G is asymptotically contractive and $x \in X^G$ then it is readily seen that x is a common fixed point for G. Thus Lemma 1 generalizes its counterpart (Theorem 2 in [2, p. 10]).

(2) If the asymptotically nonexpansive semigroup G has a fixed point $\omega \in X$, then for each $z \in X^G$, $G(z)$ lies on a sphere centered at ω. Indeed, if not, then $\exists g \in G$ with $d(g(z), \omega) \neq d(z, \omega)$. In case $d(g(z), \omega) < d(z, \omega)$ we set $\delta = d(z, \omega) - d(g(z), \omega) > 0$. Now there is a $g_1 \in G$ such that $\forall f \in G$, $d(fg_1g(z), \omega) \leq d(g(z), \omega)$ and $\exists f_1 \in G$ with $d(f_1g_1g(z), z) < \delta$. Now,

$$\delta = d(z, \omega) - d(g(z), \omega) \leq d(z, \omega) - d(f_1g_1g(z), \omega) \leq d(f_1g_1g(z), z) < \delta$$

which is absurd. The case where $d(g(z), \omega) > d(z, \omega)$ can be treated similarly.

4. Asymptotically nonexpansive semigroups of mappings in Euclidean spaces.

THEOREM 1. *Let $G: E^n \to E^n$ be asymptotically nonexpansive. If $x \in (E^n)^G$, then there exists a T-point $z \in \overline{\text{co}}\, G(x)$ for G.*

COROLLARY 1. *If $G: A \to A$ (where $A \subseteq E^n$) is asymptotically nonexpansive, A contains no nontrivial linear variety and there exists an $x \in A^G$ with $\overline{\text{co}}\, G(x) \subseteq A$, then $\overline{\text{co}}\, G(x)$ contains a unique fixed point.*

A number of properties of isometric and asymptotically nonexpansive semigroups of mappings in E^n are needed in the proof of Theorem 1. These are furnished in the following sequence of lemmas.

LEMMA 2. *Let $G: E^n \to E^n$ be asymptotically nonexpansive and $A \subseteq E^n$. If $G \mid A$ is an isometry, then $G \mid \text{co}\, A$ is again an isometry. If in addition, $G(A) \subseteq A$, then $G(\text{co}\, A) \subseteq \text{co}\, A$.*

The proof is similar to the corresponding Proposition 2 of [1]. It is also clear that in Lemma 2, we can replace $\text{co}\, A$ by $\overline{\text{co}}\, A$.

LEMMA 3. *If $G: E^n \to E^n$ is isometric, maps the closed convex set C of E^n into itself, $C^G \neq \phi$ and G has a T-point in E^n, then G has a T-point in C.*

Proof. Let $A = \overline{\text{co}}\, G(a)$ for some $a \in C^G$. If $x, \omega \in G(a)$, $\exists g_x, g_\omega \in G$, such that $x = g_x(a)$, $\omega = g_\omega(a)$. Also since $a \in C^G$, given f, $\varepsilon > 0$, $\exists g \in G$ with $\|a - gf(a)\| < \varepsilon$ by Proposition 1. For $z = \lambda x + (1-\lambda)\omega$, $0 \leq \lambda \leq 1$, using the facts that $f \mid A$ is affine, and that G is isometric, we obtain

$$\begin{aligned}
\|z - gf(z)\| &= \|\lambda x + (1-\lambda)\omega - gf(\lambda x + (1-\lambda)\omega\| \\
&\leq \lambda \|x - gf(x)\| + (1-\lambda)\|\omega - gf(\omega)\| \\
&= \lambda \|g_x(a) - gfg_x(a)\| + (1-\lambda)\|g_\omega(a) - gfg_\omega(a)\| \\
&= \lambda \|a - gf(a)\| + (1-\lambda)\|a - gf(a)\| \\
&= \|a - gf(a)\| < \varepsilon,
\end{aligned}$$

implying that $z \in C^G$. It is easy to check that C^G is closed, and hence we conclude that $A \subseteq C^G$.

Let x be a T-point of G. If $x \notin A$, let y be the (unique) point of A nearest to x and W be the supporting hyperplane at y of the closed ball with center x and radius $\|x - y\|$. We claim that y is a T-point of G.

If $f(x) = x$ for some $f \in G$, then $f(y) \in A$, $\|x - f(y)\| = \|f(x) - f(y)\| = \|x - y\|$, and the uniqueness property of y imply $f(y) = y$. If $f(x) \neq x$ and the line L containing $f(x)$ and x meets W, say at z, then since $f \mid L$ is a translation, it is true that for $g = f$ or $g = f^{-1}$, $x \in [z, g(x)]$ and so $\|g(x) - g(y)\| > \inf\{\|g(x) - u\| : u \in W\} > \|x - y\|$ which is impossible since g is an isometry. Hence $L \cap W = \phi$ so that L is parallel to W. We note that the line joining x and y is perpendicular to W, and clearly so is the line joining $f(x)$ and $f(y)$. Thus $f(y) - y = f(x) - x$. Hence for $f \in G$, either $f(y) = y$ or $f(y) = y - x + f(x)$. If now, $f, g \in G$ then $fg(y)$ is a translate of y. Since $g(y)$ is a translate of y, $fg(y)$ can be expressed as a translate of $g(y)$. In other words, y is a T-point of G.

LEMMA 4. *If $G: E^n \to E^n$ is isometric, then G has a T-point.*

Proof. Let g be a fixed element of G. We first show that the set $A_g = \{z \mid \exists a$ such that $\forall y \in G(z), g(y) = y + a\}$ is nonempty. If g has a fixed point z, then clearly $z \in A_g$. So assume that g has no fixed point. Since g is a rigid motion, we can introduce an orthonormal system of coordinates in E^n so that if

$$x = (x_1, x_2, \ldots, x_n), \qquad g(x) = xM + a,$$

where M is an orthogonal matrix in normal form. In particular suppose

$$M = \begin{pmatrix} I_k & & \\ & -I_l & \\ & & R \end{pmatrix}$$

where I_k, I_l are ($k\times k$, resp. $l\times l$) identity matrices and

$$R = \begin{pmatrix} R_1 & & & & \\ & R_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & R_{\frac{n-k-l}{2}} \end{pmatrix}$$

with
$$R_i = \begin{pmatrix} \cos\phi_i & -\sin\phi_i \\ \sin\phi_i & \cos\phi_i \end{pmatrix},$$

$\phi_i \neq m\pi$, $m=0,\pm 1,\pm 2,\ldots$.

It is fairly easy to show (cf. [1, Lemma 2]) that there exist numbers $\xi_{k+1}, \xi_{k+2},\ldots,$ ξ_n so that

$$g(x_1, x_2,\ldots, x_k, \xi_{k+1}, \xi_{k+2},\ldots, \xi_n) = (x_1+a_1, x_2+a_2,\ldots, x_k+a_k, \xi_{k+1}, \xi_{k+2},\ldots, \xi_n)$$

and note that the ξ_i are unique with the property of remaining fixed.

Suppose $f\in G$ is of the form $f(x)=xN+b$, where

$$N = \begin{array}{c} \\ k \\ n-k \end{array} \begin{array}{c} \begin{array}{cc} k & n-k \end{array} \\ \left(\begin{array}{c|c} D_1 & D_2 \\ \hline D_3 & D_4 \end{array}\right) \end{array}$$

Now as $fg=gf$, it is easy to see that we must have $D_2=0$, $D_3=0$. Thus if P is the projection defined by $P(x)=(0,0,\ldots,0,x_{k+1},\ldots,x_n)$ then we have $PfP=Pf$. Clearly $PgP=Pg$. Hence $PgPf=Pgf=Pfg=PfPg$ and so Pg and Pf commute. Pg has $\xi=(0,0,\ldots,0,\xi_{k+1},\ldots,\xi_n)$ as a unique fixed point, and so $Pf(\xi)=\xi$. In particular, if $B=\{(x_1,x_2,\ldots,x_k,\xi_{k+1},\ldots,\xi_n)\}$, $f(B)\subseteq B$, and as f was arbitrary, $G(B)\subseteq B$. Since $g\mid B$ is a translation, $B\subseteq A_g$.

Note that A_g is a linear variety and that $G(A_g)\subseteq A_g$. Now if V is any linear variety with $G(V)\subseteq V$, we must have $V\cap A_g\neq\phi$, for if we consider $G\mid V$, then the above argument shows that $A_g^*=\{z\in V \mid \exists a \text{ such that } \forall y\in G(z),\, g(y)=y+a\}$ is nonempty. But $A_g^*\subseteq V$ and $A_g^*\subseteq A_g$. This implies that for any $f_1,f_2,\ldots,f_{n+1}\in G$, $\bigcap_{i=1}^{n+1} A_{f_i}\neq\phi$. Consider all finite intersections of the A_g's. Each such intersection is a linear variety of dimension ≥ 0. If d denotes the smallest dimension of such intersections, $d\geq 0$ and there exists a finite family $\{A_{g_i}\}$, $i=1,2,\ldots,m$ so that $\dim\{\bigcap_{i=1}^m A_{g_i}\}=d$. It is clear that if an $A_g\neq A_{g_i}$, $i=1,2,\ldots,m$, then A_g must contain $\bigcap_{i=1}^m A_{g_i}$, as otherwise $\dim\{A_g\cap(\bigcap_{i=1}^m A_{g_i})\}<d$, a contradiction. This shows that $\bigcap_{g\in G} A_g\neq\phi$, and clearly any point of the intersection is a T-point.

Proof of Theorem 1. As $x\in(E^n)^G$, we know by Proposition 2 that $G\mid G(x)$ is isometric and thus $F=G\mid \overline{\text{co}}\, G(x)$ is isometric (Lemma 2). As in 3.4, Lemma 1

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of [1], F can be extended to H on all of E^n such that H is isometric and commutative. By Lemma 4, H has a T-point ω and as $H(\overline{\text{co}}\, G(x)) \subseteq \overline{\text{co}}\, G(x)$, Lemma 3 implies that H has a T-point $z \in \overline{\text{co}}\, G(x)$. But $H \mid \overline{\text{co}}\, G(x) = G \mid \overline{\text{co}}\, G(x)$ and z is thus a T-point of G.

Proof of Corollary 1. As in the proof of Theorem 1, $G \mid \overline{\text{co}}\, G(x)$ is isometric and thus we can conclude that $\overline{\text{co}}\, G(x)$ contains a T-point z. As A contains no nontrivial linear variety, the only translations of $G(z)$ are the trivial ones, i.e. $g(z) = z\ \forall g \in G$. Uniqueness of z follows from Remark (2) (in the same manner as the corresponding assertion in [1] follows from Remark 2.3 there).

In Corollary 1, we cannot relax the requirement that A contains no nontrivial linear variety. This is exhibited by the example where $n=1$, $A=E^1$ and G is the group of all translations on E^1. Clearly $A^G = E^1$, and $\overline{\text{co}}\, G(x) = E^1$ contains no fixed point of G.

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