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LOCAL MALCEV CONDITIONS

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ABSTRACT. Let p and q be polynomial symbols of a type of algebras having operations $\vee$, $\wedge$, and ; (interpreted as the join, meet, and product of congruence relations). If $\mathfrak{A}$ is an algebra, $L(\mathfrak{A})$, the local variety of $\mathfrak{A}$, is the class of all algebras $\mathfrak{B}$ such that for each finite subset G of $\mathfrak{B}$ there is a finite subset F of $\mathfrak{A}$ such that every identity of F is also an identity of G.

THEOREM. *There is an algorithm which, for each inequality*

$$p \leq q,$$

and pair of integers n, $k \geq 2$, determines a set $U_{n,k}$ of (Malcev) equations with the property:

For each algebra $\mathfrak{A}$, $p \leq q$ is true in the congruence lattice of $\mathfrak{B}$ for each $\mathfrak{B} \in L(\mathfrak{A})$ if and only if for each finite subset F of $\mathfrak{A}$ and integer $n \geq 2$ there is a $k = k(n, F)$ such that $U_{n,k}$ are identities of F.

This generalizes a corresponding result for varieties due to Wille (*Kongruenzklassengeometrien*, Lect. Notes in Math. Springer-Verlag, Berlin-Heidelberg, New York, 1970) and at the same time provides a more direct proof.

1. **Introduction.** Let p and q be polynomial symbols of a type of algebras having operation symbols $\vee$, $\wedge$, and ; (which we interpret as the join, meet, and relation product of congruence relations). In [3] Grätzer raised the following question: is the condition that the congruences of the algebras of a variety V satisfy $p = q$ equivalent to a Malcev conditions for V, that is to the existence of certain identities in V? Wille [10] (using a much more general concept of Malcev condition than that proposed by Grätzer) answered this question affirmatively by establishing the following theorem:

THEOREM 1.1. *There is an algorithm which, for each inequality*

$$p \leq q$$

and pair of integers n, $k \geq 2$, determines a finite set $U_{n,k}$ of equations (of polynomial symbols of unspecified type) with the property:

For each variety V of algebras of type τ, $p \leq q$ is true in $\Theta(\mathfrak{A})$(1) for all $\mathfrak{A} \in V$ if and only if for each $n \geq 2$ there is a $k = k(n)$ and a τ-realization $U^{\tau}_{n,k}$ of $U_{n,k}$ such that $U^{\tau}_{n,k}$ are identities of V.

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Applications of this result yield Malcev's classic theorem characterizing permutability [7], the author's characterization of joint permutability and distributivity [8], and, with modifications, Jónsson's conditions for distributivity [6], and Day's for modularity [1].

In the present paper we extend Wille's result to classes of algebras more general than varieties and, at the same time, provide a more direct proof than that appearing in [10]. Our extension is based on the following

DEFINITION. (Foster [2], Hu [4]): Let $\mathfrak{A}$ and $\mathfrak{B}$ be algebras of the same type τ. $\mathfrak{B}$ *locally satisfies the identities of* $\mathfrak{A}$ if for each finite subset G of (the universe of) $\mathfrak{B}$ there is a finite subset F of $\mathfrak{A}$ such that every identity $t_1(x, y, \ldots)=t_2(x, y, \ldots)$ (t_1 and t_2 polynomials) which holds for all $x, y, \ldots \in F$ also holds for all $x, y, \ldots \in G$. (The values of $t_i(x, y, \ldots)$ need not lie in F or G.)

$L(\mathfrak{A})$ denotes the *local variety* of $\mathfrak{A}$, i.e.: the class of all algebras of type τ which locally satisfy the identities of $\mathfrak{A}$. Evidently $L(\mathfrak{A})$ is a subclass of the variety $V(\mathfrak{A})$ of all algebras of type τ satisfying the identities of $\mathfrak{A}$. According to the following theorem varieties are special cases of local varieties.

THEOREM 1.2. (Hu and Kelenson [5].) *Let V be a variety and let $\mathfrak{F}_\omega$ be V-free with a denumerable set of free generators. Then $V=L(\mathfrak{F}_\omega)$.*

On the other hand, simple examples show that not every $L(\mathfrak{A})$ is a variety, though if $\mathfrak{A}$ is finite $L(\mathfrak{A})=V(\mathfrak{A})$ obviously.

Applications of local varieties appear in [2, 4, 9].

Our main result is the following:

THEOREM 1.3. *There is an algorithm*(2) *which, for each inequality*

$$p \leq q,$$

and pair of integers $n, k \geq 2$, determines a finite set $U_{n,k}$ of equations of unspecified type with the property:

For each algebra $\mathfrak{A}$ of type τ, $p \leq q$ is true in $\Theta(\mathfrak{B})$ for each $\mathfrak{B} \in L(\mathfrak{A})$ if and only if for each finite subset F of $\mathfrak{A}$ and integer $n \geq 2$ there is a $k=k(n, F)$ and a τ-realization $U^\tau_{n,k}$ of $U_{n,k}$ such that $U^\tau_{n,k}$ are identities of F.

The algorithm is essentially the same as Wille's [10]. Also, taking $\mathfrak{A}=\mathfrak{F}_\omega$ and applying Theorem 1.2, Theorem 1.3 specializes to Wille's result.

An example of our technique appears in [9, Lemma 4.2].

In order to state our result conveniently we have spoken above of polynomial symbols (or equations) of *unspecified type*. We consider such a polynomial symbol

(1) $\Theta(\mathfrak{A})$ denotes the lattice of congruence relations of $\mathfrak{A}$ (and is, of course, not generally closed under ;). By "$p \leq q$ is true in $\Theta(\mathfrak{A})$" we mean that the binary relations obtained by substituting elements of $\Theta(\mathfrak{A})$ for the variables of p are included in the corresponding binary relations obtained from q.

(2) Algorithm 2.3.

to be either one of the letters (variables) $x_1, x_2, \ldots$ or a primitive operation symbol t whose argument places have been filled with variables. A set U^τ of equations of type τ is called a *τ-realization* of a set U of equations of unspecified type if U^τ can be obtained from U by replacing each operation symbol, in all of its occurrences in U, by some fixed polynomial symbol of type τ.

2. **The Wille algorithm; varieties.** In this section we shall give a somewhat more direct proof of Theorem 1.1 than that appearing in [10]. By an adaptation of our method to local varieties we shall, in §3, then generalize the result to local varieties and establish Theorem 1.3.

As a preliminary we consider first the special case in which p (in the inequality $p \leq q$) is join free, i.e.: involves only the operations $\wedge$ and ;, while q is allowed to involve $\vee$ as well. In this case we shall obtain (Theorem 2.2) a Malcev condition (in the precise sense of [3]) characterizing $p \leq q$.

For $q = q(\theta_1, \ldots, \theta_r)$, in which all three of $\wedge$, ;, $\vee$ may occur, and for each integer $k \geq 2$, let q^k be the join free polynomial symbol obtained from q by replacing each occurence of $\vee$ in q by the k-fold relation product (;) of the operands. (For definiteness we associate these products from the left.) For example, if

$$q = ((\theta_1 \vee \theta_2) \wedge \theta_3) \vee \theta_4,$$

then

$$q^3 = ((((\theta_1;\theta_2);\theta_1) \wedge \theta_3);\theta_4);(((\theta_1;\theta_2);\theta_1) \wedge \theta_3).$$

With these conventions we can state the following algorithm (cf. Grätzer [3]) which is basis to the remainder of the paper:

ALGORITHM 2.1. *Let p be join free. Start with the left side of $p \leq q$*

$$(p = p(\theta_1, \ldots, \theta_r), q = q(\theta_1, \ldots, \theta_r))$$

and write the formula

$$x_1 p x_2 \quad (x_1, x_2 \textit{ new variables}). \tag{2.1}$$

Next, according as $p = p_1 \wedge p_2$ or $p_1 ; p_2$, we write either

$$x_1 p_1 x_2 \quad \textit{and} \quad x_1 p_2 x_2, \tag{2.2}$$

or

$$x_1 p_1 x_3 \quad \textit{and} \quad x_3 p_2 x_2, \tag{2.3}$$

respectively, where x_3 is a new variable. We repeat this process on each formula of the pair (2.2) *or* (2.3), *as appropriate, and continue in this fashion as far as possible, i.e.: until the formulas we obtain are of the form*

$$x_j \theta_i x_k, \tag{2.4}$$

where θ_i is one of the variables $\theta_1, \ldots, \theta_r$ which may occur in p, and at each step, any new variable x_i introduced in obtaining a formula of type (2.3) *is different from*

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any previously introduced variable. Let U^1 be the (finite) set of all formulas of type (2.4) *obtained by this process and let $x_1, \ldots, x_n$ be all of the variables occurring in the formulas of U^1 and which have been introduced by the construction of U^1.*

Similarly, start with the formula

$$x_1 q^k x_2 \tag{2.5}$$

and by the same process as above, construct another finite set U_k^2 of formulas of the form

$$t_u(x_1, \ldots, x_n)\theta_i t_v(x_1, \ldots, x_n), \tag{2.6}$$

where, instead of introducing a new variable x_i when a product is encountered, we introduce a new polynomial symbol $t_i = t_i(x_1, \ldots, x_n)$ of unspecified type and in the variables $x_1, \ldots, x_n$ introduced by the construction of U^1. Let $x_1, x_2, t_1, \ldots, t_{m_k}$ be all of the polynomial symbols occurring in U_k^2 and which have been introduced by this construction.

Now consider a particular formula (2.6) *of U_k^2. Let*

$$x_{j_1}\theta_i x_{k_1}, \ldots, x_{j_s}\theta_i x_{k_s} \tag{2.7}$$

be all of the formulas of U^1 in which this particular θ_i occurs. From the polynomial symbols $t_u = t_u(x_1, \ldots, x_n)$, $t_v = t_v(x_1, \ldots, x_n)$ occurring in (2.6) *obtain new polynomial symbols t_u' and t_v' by equating those variables among $x_1, \ldots, x_n$ which would, in consequence of* (2.7), *be deduced equivalent modulo θ_i if we considered θ_i to be an equivalence relation on the variables $x_{j_1}, x_{k_1}, \ldots, x_{j_s}, x_{k_s}$ occurring in* (2.7). *Then form the equation*

$$t_u' = t_v'. \tag{2.8}$$

For each k let U_k be the set of all equations (2.8) *obtained in this way, one for each formula in U_k^2 (deleting repetitions).*

EXAMPLE. Consider $p = \theta_1 \wedge (\theta_2; \theta_3) \le (\theta_1 \wedge \theta_2) \vee (\theta_1 \wedge \theta_3) = q$. Starting with $x_1\theta_1 \wedge (\theta_2;\theta_3)x_2$ we obtain for U^1 the formulas

$$x_1\theta_1 x_2, \qquad x_1\theta_2 x_3, \qquad x_3\theta_3 x_2. \tag{2.4$'$}$$

Starting with $x_1 q^2 x_2 = x_1(\theta_1 \wedge \theta_2);(\theta_1 \wedge \theta_3)x_2$ we obtain for U_2^2 the formulas

$$\begin{matrix} x_1\theta_1 t_1(x_1, x_2, x_3), & x_1\theta_2 t_1(x_1, x_2, x_3), \\ t_1(x_1, x_2, x_3)\theta_1 x_2, & t_1(x_1, x_2, x_3)\theta_3 x_2. \end{matrix} \tag{2.6$'$}$$

Equating variables as prescribed, we obtain for U_2:

$$x_1 = t_1(x_1, x_1, x_3), \qquad x_1 = t_1(x_1, x_2, x_1), \qquad x_3 = t_1(x_1, x_3, x_3).$$

The following result gives a Malcev condition (in the strict sense of [3]) characterizing $p \le q$ when p is join free.

THEOREM 2.2. *Let p be join free and for each integer $k \geq 2$ let U_k be constructed by Algorithm* 2.1. *For any variety V of algebras of type τ, $p \leq q$ is true in $\Theta(\mathfrak{A})$ for all $\mathfrak{A} \in V$ if and only if there is a $k \geq 2$ and a τ-realization U_k^τ of U_k such that U_k^τ are identities of each algebra of V.*

Proof. First suppose $p \leq q$ is true in $\Theta(\mathfrak{A})$ for all $\mathfrak{A} \in V$. Let $\mathfrak{P}$ be the V-free polynomial algebra freely generated by the set $K = \{x_1, \ldots, x_n\}$ of variables introduced by Algorithm 2.1. With reference to (2.7), for each θ_i occurring in $p \leq q$ define θ_i' on $\mathfrak{P}$ by

$$\theta_i' = \theta_i(x_{j_1}, x_{k_1}) \vee \cdots \vee \theta_i(x_{j_s}, x_{k_s})$$

where, for $m = 1, \ldots, s$, $\theta_i(x_{j_m}, x_{k_m})$ is the least congruence on $\mathfrak{P}$ which identifies x_{j_m} and x_{k_m} as prescribed by the formula $x_{j_m}\theta_i x_{k_m}$ of (2.7). We then have, by our construction of U^1, for the elements x_1, x_2 of $\mathfrak{P}$,

$$x_1 p(\theta_1', \ldots, \theta_r')x_2,$$

and hence, since $p \leq q$,

$$x_1 q(\theta_1', \ldots, \theta_r')x_2.$$

Consequently, by the definition of join, for some $k \geq 2$,

$$x_1 q^k(\theta_1', \ldots, \theta_r')x_2,$$

so that, by our construction of U_k^2, all formulas (2.6) of U_k^2 are true, with θ_i' replacing θ_i and where t_u, t_v are now polynomial symbols of the type τ of V. Forming t_u', t_v' as in Algorithm 2.1, we obtain a τ-realization U_k^τ of U_k. Moreover, in $\mathfrak{P}/\theta_i'$ each $t_u' = t_v'$ in U_k^τ is true. Since $\mathfrak{P}/\theta_i'$ is free we conclude that U_k^τ are identities of each algebra in V.

Conversely, suppose that for some k and τ-realization U_k^τ of U_k, U_k^τ are identities of V. Let $\mathfrak{A} \in V$ and suppose $u_1, u_2 \in \mathfrak{A}$ and $\bar{\theta}_1, \ldots, \bar{\theta}_r \in \Theta(\mathfrak{A})$ are such that

$$u_1 p(\bar{\theta}_1, \ldots, \bar{\theta}_r)u_2.$$

Then for some elements $u_3, \ldots, u_n \in \mathfrak{A}$ all of the formulas (2.4) of U^1 are true. We must show that

$$u_1 q(\bar{\theta}_1, \ldots, \bar{\theta}_r)u_2,$$

for which it suffices to show that

$$u_1 q^k(\bar{\theta}_1, \ldots, \bar{\theta}_r)u_2.$$

This means we must find elements $t_1^*, \ldots, t_{m_k}^* \in \mathfrak{A}$ so that for each formula (2.6) of U_k^2, $t_u^* \bar{\theta}_i t_v^*$. To do this set $t_i^* = t_i(u_1, \ldots, u_n)$ for each polynomial symbol t_i occuring in U_k. We then have, since all of the formulas of U^1 (and in particular (2.7)) are true, by the substitution property for the congruence $\bar{\theta}_i$,

$$t_u^* = t_u(u_1, \ldots, u_n)\bar{\theta}_i t_u'(u_1, \ldots, u_n)$$

and

$$t_v^* = t_v(u_1, \ldots, u_n)\bar{\theta}_i t_v'(u_1, \ldots, u_n).$$

But $t'_u(u_1, \ldots, u_n) = t'_v(u_1, \ldots, u_n)$ by the corresponding identity (2.8) of U_k^τ. Hence we conclude that $t_u^* \bar{\theta}_i t_v^*$ for each formula (2.6) of U_k^2. This completes the proof.

To prove Theorem 1.1 we first state Wille's algorithm in terms of Algorithm 2.1:

ALGORITHM 2.3. *Let p and q be polynomials in the variables $\theta_1, \ldots, \theta_r$ which may involve any of the operations $\vee, \wedge, ;$. For each $n \geq 2$ apply Algorithm* 2.1 *to the formula*

$$p^n \leq q$$

to construct, for each $k \geq 2$, a finite set $U_{n,k}$ of formulas of unspecified type.

Applying Algorithm 2.3 we conclude, by Theorem 2.2, that for any variety V of type τ, $p^n \leq q$ is true in $\Theta(\mathfrak{A})$ for all $\mathfrak{A} \in V$ if and only if there is a $k = k(n) \geq 2$ and a τ-realization $U_{n,k}^\tau$ of $U_{n,k}$ such that $U_{n,k}^\tau$ are identities of V. The proof of Theorem 1.1 is then immediate from the observation that $p \leq q$ in $\Theta(\mathfrak{A})$ if and only if $p^n \leq q$ in $\Theta(\mathfrak{A})$ for all $n \geq 2$.

3. **Local varieties.** The following is our analog, for local varieties, of Theorem 2.2.

THEOREM 3.1. *Let p be join free and for each integer $k \geq 2$ let U_k be constructed by Algorithm* 2.1. *For any algebra $\mathfrak{A}$ of type τ, $p \leq q$ is true in $\Theta(\mathfrak{B})$ for all $\mathfrak{B} \in L(\mathfrak{A})$ if and only if for each finite subset F of $\mathfrak{A}$ there is an integer $k = k(F) \geq 2$ and a τ-realization U_k^τ of U_k such that U_k^τ are identities of F.*

We start with

LEMMA 3.2. *Let F be a finite subset of $\mathfrak{A}$ and let $\mathfrak{F}$ be the subalgebra of $\mathfrak{A}$, with universe $[F]$, generated by F. Let $\mathfrak{P}$ be the $\mathfrak{F}$-free polynomial algebra with n free generators. Then there is a congruence ϕ on $\mathfrak{P}$ such that $\mathfrak{P}/\phi \in L(\mathfrak{A})$ and for $t_1, t_2 \in \mathfrak{P}$, $t_1 \phi t_2$ if and only if t_1 and t_2 agree on F.*

Proof. Let the generating set of $\mathfrak{P}$ be $K = \{x_1, \ldots, x_n\}$. $\mathfrak{P}$ consists of all polynomial symbols of type τ in which any of the variables of K occur, two polynomial symbols t_1, t_2 being considered equal provided $t_1 = t_2$ is an identity of $\mathfrak{F}$.

Set $E = F^K \subseteq [F]^K = \bar{E}$, so that E is finite. $\mathfrak{P}$, which is isomorphic with a subalgebra of $\mathfrak{F}^{\bar{E}}$, is naturally homomorphic to a subalgebra of $\mathfrak{F}^E$ and if we take ϕ to be the kernal of this homomorphism, then for ϕ-congruence classes of $\mathfrak{P}$ (denoted by the subscript ϕ),

$$[t_1(x_1, \ldots, x_n)]_\phi = [t_2(x_1, \ldots, x_n)]_\phi$$

if and only if $t_1(e(x_1), \ldots, e(x_n)) = t_2(e(x_1), \ldots, e(x_n))$ for all $e \in E$, i.e.: if and only if t_1 and t_2 agree on F. Further, $\mathfrak{P}/\phi$ is isomorphic with a subalgebra of $\mathfrak{F}^E$ and thus, by the finiteness of E, locally satisfies the identities of $\mathfrak{F}$(3), and hence locally satisfies the identities of $\mathfrak{A}$, i.e.: $\mathfrak{P}/\phi \in L(\mathfrak{A})$.

(3) This is easily verified directly. See Lemma 4.1 of [9] or Proposition 3.1 of [5].

Proof of Theorem 3.1. First suppose that $p \leq q$ is true in $\Theta(\mathfrak{B})$ for all $\mathfrak{B} \in L(\mathfrak{A})$. Let $K=\{x_1, \ldots, x_n\}$ be the set of variables introduced by Algorithm 2.1 in the construction of U^1, F a finite subset of $\mathfrak{A}$ and $E=F^K$ as in the lemma. By the lemma we conclude that $p \leq q$ is true in $\Theta(\mathfrak{P}/\phi)$.

Next, for each θ_i occurring in $p \leq q$, and referring to (2.7), set

$$E_i = \{e \in E : e(x_{j_1}) = e(x_{k_1}), \ldots, e(x_{j_s}) = e(x_{k_s})\}$$

and define the congruence θ_i' on $\mathfrak{P}$ by

$$t_1(x_1, \ldots, x_n)\theta_i' t_2(x_1, \ldots, x_n)$$

if and only if $t_1(e(x_1), \ldots, e(x_n)) = t_2(e(x_1), \ldots, e(x_n))$ for all $e \in E_i$. Since $E_i \subseteq E$ we have $\theta_i' \geq \phi$. Hence if θ_i'' is the congruence on $\mathfrak{P}/\phi$ induced by θ_i', we have, by our construction of U^1, for the elements $[x_1]_\phi$, $[x_2]_\phi$ of $\mathfrak{P}/\phi$ and the binary relation $p(\theta_1'', \ldots, \theta_r'')$ on $\mathfrak{P}/\phi$,

$$[x_1]_\phi p(\theta_1'', \ldots, \theta_r'')[x_2]_\phi.$$

But since $p \leq q$ is true in $\Theta(\mathfrak{P}/\phi)$, this implies that for some $k \geq 2$,

$$[x_1]_\phi q^k(\theta_1'', \ldots, \theta_r'')[x_2]_\phi.$$

This means, by our construction of U_k^2, that for each formula (2.6) of U_k^2, the formula

$$t_u([x_1]_\phi, \ldots, [x_n]_\phi)\theta_i'' t_v([x_1]_\phi, \ldots, [x_n]_\phi)$$

is true in $\mathfrak{P}/\phi$, where t_u, t_v are now polynomial symbols of type τ. Forming the t_u' and t_v' by equating variables as specified in Algorithm 2.1, it is clear from the definitions of ϕ and θ_i' that $t_u' = t_v'$ for all values of $x_1, \ldots, x_n$ in F. Since this is true for each equation in U_k, we have completed the first half of the proof.

Conversely, suppose that for each finite subset F of $\mathfrak{A}$ there is an integer $k \geq 2$ and a τ-realization U_k^τ of U_k such that U_k^τ are identities of F. Let $\mathfrak{B} \in L(\mathfrak{A})$ and suppose $u_1, u_2 \in \mathfrak{B}$ and $\bar{\theta}_1, \ldots, \bar{\theta}_r \in \Theta(\mathfrak{B})$ are such that

$$u_1 p(\bar{\theta}_1, \ldots, \bar{\theta}_r)u_2.$$

Then for some elements $u_3, \ldots, u_n \in \mathfrak{B}$ all of the formulas (2.4) of U^1 are true. Let $G=\{u_1, \ldots, u_n\}$. Then there is a finite subset F of $\mathfrak{A}$ such that each identity of F is an identity of G. Hence there is a k and a τ-realization U_k^τ of U_k such that the equations U_k^τ are identities of G.

We must show that $u_1 q(\bar{\theta}_1, \ldots, \bar{\theta}_r)u_2$ and hence it suffices to show (as in the proof of Theorem 2.2) that $u_1 q^k(\bar{\theta}_1, \ldots, \bar{\theta}_r)u_2$. We must therefore find elements $t_1^*, \ldots, t_{m_k}^*$ of $\mathfrak{B}$ so that for each formula (2.6) of U_k^2, $t_u^* \bar{\theta}_i t_v^*$. To do this we again set $t_i^* = t_i(u_1, \ldots, u_n)$ and proceed exactly as in the proof of Theorem 2.2. This completes the proof of Theorem 3.1.

REMARK. It is to be emphasized that the integer k of Theorem 3.1 in general depends on the congruence ϕ of Lemma 3.2 and hence on the subset F of $\mathfrak{A}$.

Turning now to the general case where neither p nor q are necessarily join free, Theorem 1.3 now follows directly from Theorem 3.1 and Algorithm 2.3 together with the observation that for all algebras $\mathfrak{A}$ and $\mathfrak{B} \in L(\mathfrak{A})$, $p \leq q$ is true in $\Theta(\mathfrak{B})$ if and only if $p^n \leq q$ is true in $\Theta(\mathfrak{B})$ for all $n \geq 2$.

4. **Extensions.** Applying Theorem 1.3 to each of $p \leq q$ and $q \leq p$, we easily see that Theorem 1.3 remains true if $p \leq q$ is replaced by the equation $p = q$. In the same way we conclude

COROLLARY 4.1. *Theorem 1.3 remains true if $p \leq q$ is replaced by any finite set of inequalities or equalities.*

Finally let E be any equational theory of the type of algebras having operation symbols $\vee$, $\wedge$, and ;. Let

$$p_1 = q_1, \qquad p_2 = q_2, \ldots$$

be a (possibly countable) basis for E. Let $U^i_{n,k}$ be the (finite) set of equations of unspecified type which may be effectively determined from n, k and $p_i = q_i$. We then have

COROLLARY 4.2. *For any algebra $\mathfrak{A}$ of type τ, the equations of E are true in $\Theta(\mathfrak{B})$ for all $\mathfrak{B} \in L(\mathfrak{A})$ if and only if for each $n \geq 2$ and finite subset F of $\mathfrak{A}$, there is a sequence $s = (k_1, k_2, \ldots)$* (depending on n and F) *and a τ-realization $U^\tau_{n,s}$ of*

$$U_{n,s} = U^1_{n,k_1} \cup U^2_{n,k_2} \cup \cdots$$

such that $U^\tau_{n,s}$ are identities of F.

For varieties this specializes to

COROLLARY 4.3. *For every variety V of algebras of type τ, the equations of E are true in $\Theta(\mathfrak{B})$ for all $\mathfrak{B} \in V$ if and only if for each $n \geq 2$ there is a sequence $s = (k_1, k_2, \ldots)$* (depending on n) *and a τ-realization $U^\tau_{n,s}$ of $U_{n,s}$ such that $U^\tau_{n,s}$ are identities of V.*

5. **Remarks.** (a) It is easy to see that the sets of identities $U_{n,k}$ are not independent and, in fact, if $n < m$ then $U_{n,k}$ are deducible from $U_{m,k}$. Thus the last clause of the statement of Theorem 1.3 can be weakened to read "*for each finite subset F of the universe of $\mathfrak{A}$ there are infinitely many integers $n \geq 2$ for which there is an integer $k = k(n, F) \cdots$*" etc.

(b) Let S be a finite set of equalities and inequalities. We might distinguish two special cases from the general characterizations of Theorem 1.3 and Corollary 4.1 by the following definitions:

S has a *finite characterization* if there is a finite set U of equations of unspecified type with the property: *For each algebra $\mathfrak{A}$ of type τ, S are true in $\Theta(\mathfrak{B})$ for each*

$\mathfrak{B} \in L(\mathfrak{A})$ if and only if for each finite subset F of $\mathfrak{A}$ there is a U^τ consisting of identities of F.

S has a *one parameter characterization* if there is a sequence $U_1, U_2, \ldots$ of finite sets of equations of unspecified type with the property: *For each algebra $\mathfrak{A}$ of type τ, S are true in $\Theta(\mathfrak{B})$ for each $\mathfrak{B} \in L(\mathfrak{A})$ if and only if for each finite subset F of $\mathfrak{A}$ there is an integer $k=k(F)$ and a U_k^τ consisting of identities of F.* Characterizations of this type constitute a natural generalization to local varieties of Malcev conditions in the sense of [3].

Analogous definitions can be formulated for the special case of varieties in light of Theorem 2.2.

From the proofs of Theorems 2.2 and 3.1 it is easy to see that if S consists of a finite set of equations involving no occurrences of the join operation, then S has a finite characterization. Malcev's characterization of permutability is typical of this case. On the other hand, Jónsson's and Day's results show that distributivity and modularity each have one parameter characterizations for varieties.

To obtain a one parameter characterization using either Theorem 1.3 or 2.2 it would evidently suffice to show that there is an integer n such that for all $m>n$ and for all $k\geq 2$ there is an r for which $U_{m,r}$ is deducible from $U_{n,k}$. For example Jónsson's and Day's characterizations of distributivity and modularity are each of the form $U_{3,2}, U_{3,3}, U_{3,4}, \ldots$ where the $U_{3,i}$ are produced by Algorithm 2.3. Corresponding to their proofs of the sufficiency of their conditions one can actually find, in each case, for given $m\geq 2$ and some $r=r(m, k)$, a deduction of $U_{m,r}$ from $U_{3,k}$. Either by this method or direct examination of their proofs one sees that their Malcev conditions become "local Malcev conditions" in the more general context of local varieties.

It would be interesting to know of some direct criteria for determining when one parameter characterizations are available, and correspondingly, for determining when Theorems 1.3 and 2.2 are the best one can do.

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