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Periodic and Almost Periodic Functions on Infinite Sierpinski Gaskets

Robert S. Strichartz

Abstract. We define periodic functions on infinite blow-ups of the Sierpinski gasket as lifts of functions defined on certain compact fractafolds via covering maps. This is analogous to defining periodic functions on the line as lifts of functions on the circle via covering maps. In our setting there is only a countable set of covering maps. We give two different characterizations of periodic functions in terms of repeating patterns. However, there is no discrete group action that can be used to characterize periodic functions. We also give a Fourier series type description in terms of periodic eigenfunctions of the Laplacian. We define almost periodic functions as uniform limits of periodic functions.

1 Introduction

The Sierpinski gasket (SG) may be thought of as a fractal analog of the unit interval, while the infinite blow-ups of SG may be thought of as fractal analogs of the real line. On the real line we have periodic and almost periodic functions. What are the analogous functions on the infinite blow-ups of SG? There are two ways to describe periodic functions on the line: (i) as functions invariant under a discrete group of translations, or (ii) as lifts of functions on the circle under covering maps from the line to the circle. On the blow-ups of SG there are no discrete groups acting, so we cannot generalize the first approach. But as we will see, there are covering maps to certain compact fractafolds, analogous to the circle, so we can generalize the second approach. Another approach, starting with a function on *SG* and extending it by rotations, yields a smaller class of functions that we call *extended periodic functions*, in Section 4.

To be specific, we describe SG as the invariant set for the iterated function system (IFS) consisting of three similarities $\{F_1, F_2, F_3\}$ of the plane with fixed-points $\{q_1, q_2, q_3\}$ the vertices of an equilateral triangle, and contraction ratio $1/2$. However, we use the IFS with "twists" as described in [2, 3], so that

$$F_i x = \frac{1}{2} R_i x + \frac{1}{2} q_i,$$

where R_i is the reflection that fixes q_i and permutes the other two vertices of the triangle. Including the reflection does not change the invariant set, but it gives a description of SG that is better suited to our purposes (see Figure 1.1). Note that the three mappings F_i have inverses that may be combined into a single continuous 3-to-1 mapping of *SG* to itself. Let $w = (w_1, w_2, \ldots)$ denote an infinite word where each

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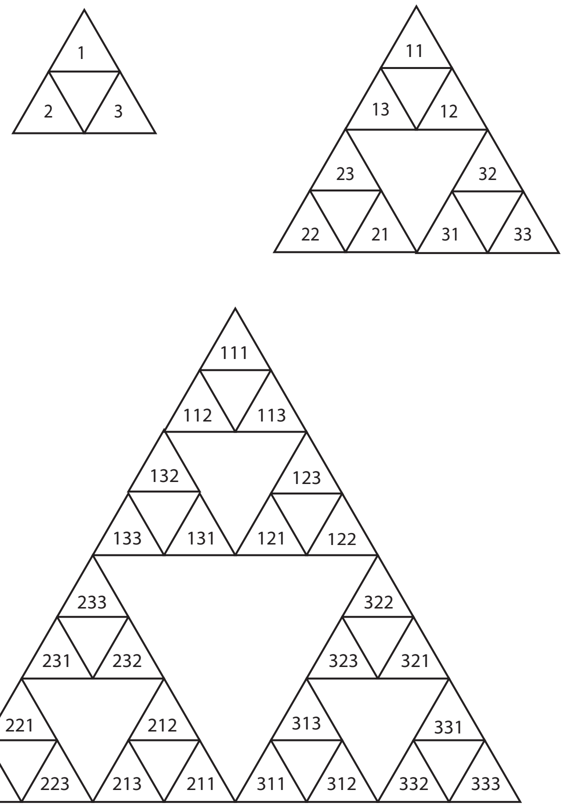


Figure 1.1: SG showing the cell structure on levels $1, 2, 3$, using the IFS with "twists", (label ijk means the cell is $F_i F_j F_k(SG)$).

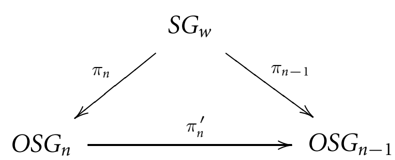
w_i takes on the values $1, 2$, or 3, and let $[w]_m = (w_1, \dots, w_m)$ denote the truncation of w to length m. We define $F^{-1}_{[w]_m} = F^{-1}_{w_1} \circ F^{-1}_{w_2} \circ \cdots \circ F^{-1}_{w_m}$ (note that the order is the reverse of what might be expected), and

$$SG_w = \bigcup_{m=1}^{\infty} F^{-1}_{[w]_m}(SG). \tag{1.1}$$

Note that $SG \subseteq F^{-1}_{[w]_1}(SG) \subseteq F^{-1}_{[w]_2}(SG) \subseteq \cdots$ so (1.1) is an increasing union. We call SG_w an *infinite blow-up* of SG (see [6, 10]).

If $w' = (w'_1, \dots, w'_{m'})$ denotes a finite word of length m', then $F^{-1}_{[w]_m} F_{w'}(SG)$ is called a *cell of order* $m' - m$. Note that the cells of order 0 are isometric to SG, and all cells are similar to SG. Different choices of words lead to different blow-ups, and, in fact, there are uncountably many non-isometric blow-ups (two blow-ups will be isometric only in the case that the words differ in only a finite number of places after a permutation of indices [10, Lemma 2.3]). In general there is no canonical choice of blow-ups. However, we will exclude the case of a word with all but a finite number of w_i being equal. That choice leads to a blow-up with a boundary point, and so is analogous to a half-line rather than a line.

Each blow-up is a *fractafold* [7], defined to be a Hausdorff space in which every point has a neighborhood homeomorphic to a neighborhood of a point in SG. These are noncompact fractafolds without boundary (the neighborhoods avoid the boundary points $\{q_1, q_2, q_3\}$ of SG). We can construct compact fractafolds by gluing together a finite number of copies of SG at boundary points, described by a *cell graph G* whose vertices correspond to the copies of SG and whose edges indicate the gluing. (Because of the symmetry of SG, we do not have to indicate which vertex is glued.) The fractafold $\mathcal{F}$ will be without boundary exactly when G is 3-regular. Two simple examples are the *double cover* $\widetilde{SG}$, whose cell graph has two vertices jointed by three edges, and the *octahedron fractafold* OSG whose cell graph is the complete graph on four vertices. We can visualize OSG as putting a copy of SG in every other face in a regular octahedron. The key observation is that there exists a locally isometric covering map from SG_w to OSG. In fact, we can construct an infinite sequence of fractafolds OSG_n, with $OSG_0 = OSG$, and OSG_n obtained from OSG by enlarging the n-cells of OSG to 0-cells of OSG_n. We will construct locally isometric covering maps $\pi'_n\colon OSG_n \to OSG_{n-1}$ and $\pi_n\colon SG_w \to OSG_n$ so that the following diagram commutes:



We will define a continuous function on SG_w to be *periodic of level n* if it is the lift under π_n of a continuous function on OSG_n. Because of the commuting diagram, a function periodic of level n is also periodic of level n' for any $n' > n$. We define an *almost periodic* function to be a uniform limit of periodic functions. We will give two other characterizations of periodic functions in terms of repeating patterns of restrictions to cells of order $-n$. We will also give a Fourier series type characterization: periodic functions are uniform limits of sums of periodic eigenfunctions of the standard Laplacian on SG_w. We compare and contrast this with the Fourier series type expansions of L^2 functions on SG_w of Teplyaev [10]

In Section 2 we present the construction of the covering maps π_n and π'_n. In Section 3 we prove the two "repeating patterns" characterizations of periodic functions, the second one involving invariance under a family of local isometries. In Section 4 we examine some possible symmetries of periodic functions, arising from the fact that OSG_n has an isometry group isomorphic to the 24-element permutation group S_4. In this section we identify another family of compact fractafolds CSG_n covered by OSG_n and hence by SG_w. As far as we know, these two families are the only compact fractafolds without boundary that have locally isometric covering maps from SG_w. It would be interesting to know if there are any others.

Section 5 is devoted to the Fourier series type expansions of periodic functions. We make use of the results of [7] to identify the eigenfunctions of the Laplacian on

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OSG_n, and lift this to periodic functions. Using the results of [8], we get uniform convergence of eigenfunction expansions. We also discover that there is a family of periodic eigenfunctions with eigenvalues that do not occur in the L^2 spectrum of SG_w. These eigenfunctions, built from the discrete eigenvalue 4, were described in [1]. The value 4 is a fixed point of the polynomial $\lambda(5-\lambda)$ which forms the basis for the spectral decimation method [4] that underlies the explicit description of eigenfunctions. We also briefly discuss the spectrum for almost periodic functions, which is the union of all the spectra for periodic functions. These are the only eigenvalues that are known to have bounded eigenfunctions. In [1] there is some speculation that perhaps there are no others. It would be interesting to resolve this question.

The reader should consult [5, 9] for more detailed description of analysis on SG. The material in Section 5 is based on [7].

2 Covering Maps

First we construct a covering map $\pi\colon SG_w \to OSG$ (all the covering maps in this section will be local isometries). We adopt the following notation. OSG is the union of four 0-cells that are denoted K_0, K_1, K_2, K_3, with the intersection $K_0 \cap K_i$ consisting of the point q_{i0}, while $K_1 \cap K_2 = \{q_{31}\}$, $K_2 \cap K_3 = \{q_{11}\}$, and $K_3 \cap K_1 = \{q_{21}\}$. (See Figure 2.1.) To define the covering map, in which each 0-cell is mapped to one of the K_i, it suffices to describe the image of the boundary points of each 0-cell, namely $\pi(F^{-1}_{[w]_m} F_{w'} q_i)$.

Definition 2.1 The *parity functions* $p_i([w]_m, w')$, $i = 1, 2, 3$, which we write simply as p_i when $[w]_m$ and w' are fixed, are defined to be $p_i = 0$ if i occurs an even number of times among $(w_1, \dots, w_m, w'_1, \dots, w'_m)$, and 1 if it occurs an odd number of times.

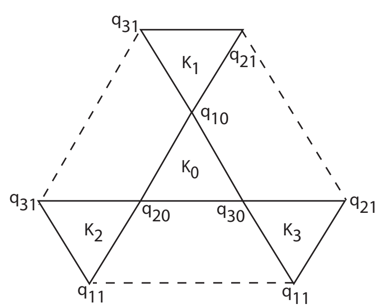

Figure 2.1: OSG, with dotted lines indicating identified points. Each K_i is a copy of SG.

Note that if $|w'| = m$, then $p_1 + p_2 + p_3$ must be even, so if we write $p = (p_1, p_2, p_3)$, then there are just four possibilities for p, namely $(0,0,0)$, $(0,1,1)$, $(1,0,1)$, or $(1,1,0)$. We will define π to map the 0-cell $F^{-1}_{[w]_m} F_{w'}(SG)$ to K_0, K_1, K_2, K_3 in each of these cases. More precisely, we set

(2.1)
$$\pi(F^{-1}_{[w]_m} F_{w'} q_i) = q_{ip_i}.$$

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Lemma 2.2 *This definition is unambiguous.*

Proof First, we note that $F^{-1}_{[w]_m} F_{w'} = F^{-1}_{[w]_{m+1}} F_{w''}$ where $w'' = (w_{m+1}, w'_1, \dots, w'_m)$, but in this case $p_i([w]_m, w') = p_i([w]_{m+1}, w'')$ because we have added two occurences of w_{m+1}. Thus the mapping depends only on the 0-cell, not the way it is represented. Second, we need to check that when cells intersect, the mapping agrees at the intersection point. Here we see the advantage of using the IFS with "twists", because the intersections all occur with $F_{w'} q_i = F_{w''} q_i$, and the index i occurs the same number of times in w' and w'' (in fact w' and w'' agree at all places except one, namely the last k for which $w'_k \neq i$, and there w''_k is the third index). Thus $F_{w'} q_i = F_{w''} q_i$ implies $p_i([w]_m, w') = p_i([w]_m, w'')$ so (2.1) yields the same image point in both cases. ■

In Figure 2.2 we illustrate the mapping restricted to $F^{-1}_{[w]_m}(SG)$ for $m = 3$. Note that each type of cell $(0, 1, 2, 3)$ intersects neighboring cells of the other three types. It is clear where each vertex is mapped because of the types of cells it bounds. For example, a vertex on the boundary of cells of type 2 and 3 is mapped to q_{11}, etc. Of course, the mapping depends on $[w]_m$. However, since OSG has a symmetry group S_4, there are 24 different possibilities, all obtained from each other by permutation of the labels $0, 1, 2, 3$. Note that when m is even, there will be one label that appears one more time than the others, and it appears in the three corner cells. When m is odd, the three corner cells have distinct labels, and the missing label appears one time less than the others.

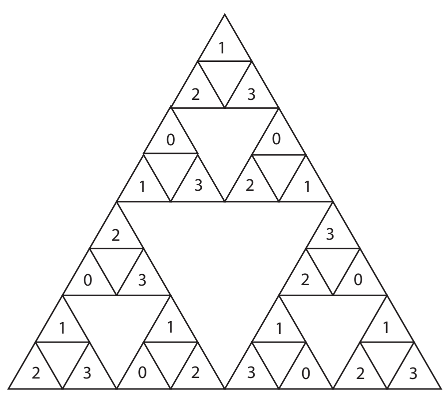

Figure 2.2: The number i inside a triangle indicates that the corresponding cell is mapped to K_i.

Theorem 2.3 $\pi = \pi_0$ *is a locally isometric covering map.*

Proof This follows almost immediately from the definition, since the preimage of $K_i \cup K_j$ for any $i \neq j$ is a disconnected union of pairs of intersecting 0-cells. ■

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Definition 2.4 We define $\pi_n : SG_w \to OSG_n$ as follows. We can identify OSG_n with Figure 2.1 if we simply take each K_i to be a cell of level $-n$ (containing 3^n 0-cells). We can identify OSG_n with Figure 2.1 if we simply take each K_i to be a cell of level $-n$ (containing 3^n 0-calls). To define π_n we need to map $(-n)$-cells of SG_w to one of the K_i. All $(-n)$-cells have the form $F^{-1}_{[w]_m} F_{w'}(SG)$ where now $|w'| = m - n$. When n is even we use (2.1) to define π_n on the boundary of the $(-n)$-cell, and continuing isometrically to the interior of the cell. The description of which K_i is the image of $F^{-1}_{[w]_m} F_{w'}(SG)$ is the same as before. When n is odd, then $p_1 + p_2 + p_3$ is also odd, so we need to replace p_i by $1 - p_i$ in (2.1).

We would also like to have a description of π_n on the level 0 cells. To do this, we need to introduce some more notation for the $4 \cdot 3^n$ 0-cells in OSG_n. When $n = 1$, we split each of the four (-1)-cells K_j into three 0-cells K_{jk} for $k \neq j$, and we make K_{jk} intersect K_{kj} (as well as the other 0-cells $K_{j\ell}$ in K_j). In general OSG_n will have 0-cells labeled $K_{j_1,\dots,j_{n+1}}$ with consecutive indices distinct. The 0-cell $K_{j_1,\dots,j_n}$ in OSG_{n-1} will split into three intersecting 0-cells $K_{j_1,\dots,j_n,k}$ in OSG_n with $k \neq j_n$. Moreover, if $K_{j_1,\dots,j_n}$ intersects $K_{j'_1,\dots,j'_{n-1},k}$ in OSG_{n-1}, then $K_{j_1,\dots,j_n,k}$ will also intersect $K_{j'_1,\dots,j'_{n-1},k,j_n}$ in OSG_n. Figure 2.3 illustrates the cases $n = 1, 2$.

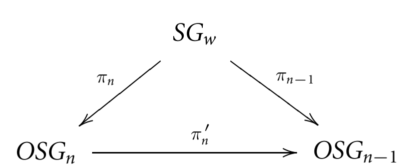
Definition 2.5 In terms of the above description it is easy to define the covering maps $\pi'_n : OSG_n \to OSG_{n-1}$ by deleting the first index: $\pi'_n(K_{j_1,\dots,j_{n+1}}) = K_{j_2,\dots,j_{n+1}}$.

For this to make sense we need to verify that if $K_{j'_1,\dots,j'_{n+1}}$ and $K_{j_1,\dots,j_{n+1}}$ intersect in OSG_n, then $K_{j'_2,\dots,j'_{n+1}}$ and $K_{j_2,\dots,j_{n+1}}$ intersect in OSG_n, so

$$\pi'_n(K_{j_1,\dots,j_{n+1}} \cap K_{j'_1,\dots,j'_{n+1}}) = K_{j_2,\dots,j_{n+1}} \cap K_{j'_2,\dots,j'_{n+1}}.$$

But this intersection property follows from the inductive construction above.

Theorem 2.6 *The following diagram of covering maps commutes.*



Proof Consider the map $\pi_1 : SG_w \to OSG_1$. A 0-cell has the form

$$F^{-1}_{[w]_m} F_{w'}(SG) \quad \text{for } w' = (w'_1, \dots, w'_m),$$

and it lies in the (-1)-cell $F^{-1}_{[w]_m} F_{(w'_1,\dots,w'_{m-1})}(SG)$. (We can always take $m \geq 1$.) We know the K_i cells of level -1 in OSG_1 to which this gets mapped, depending on $p([w_m], w'_1, \dots, w'_{m-1})$. Let us define the index function $I(p)$ by the following rules:

$$I(0,0,0) = I(1,1,1) = 0, \tag{2.2}$$
$$I(0,1,1) = I(1,0,0) = 1,$$
$$I(1,0,1) = I(0,1,0) = 2,$$
$$I(1,1,0) = I(0,0,1) = 3.$$

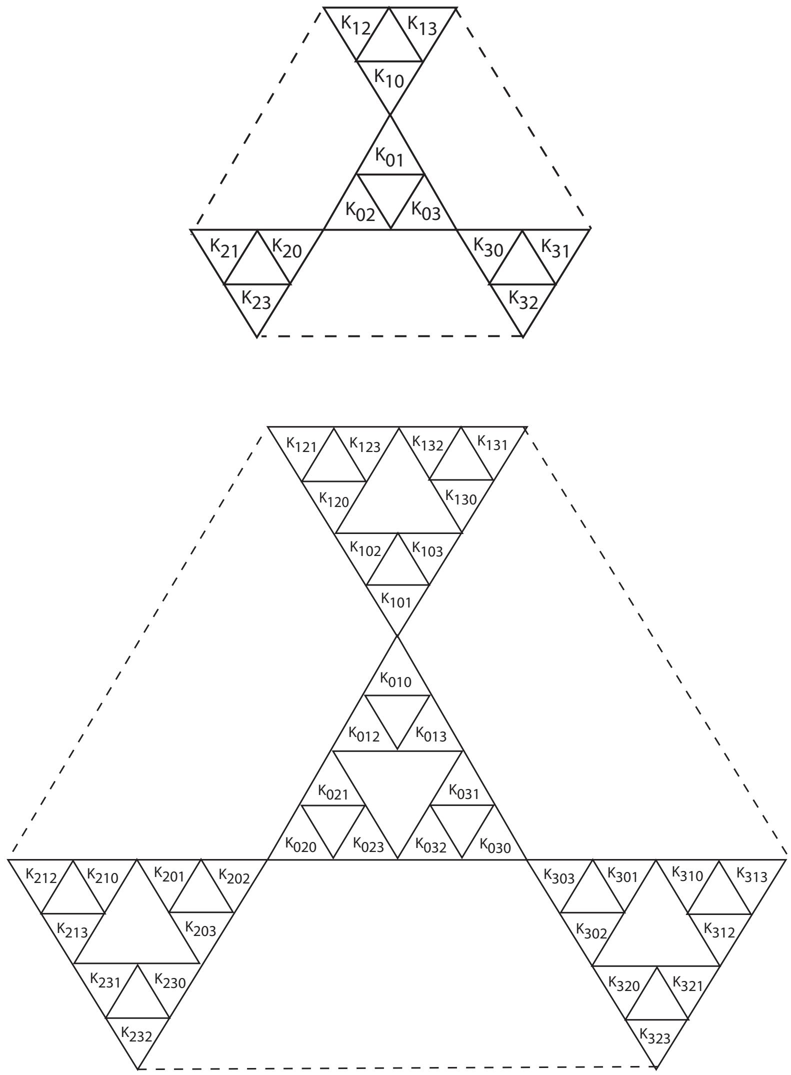


Figure 2.3: OSG_n for $n = 1, 2$.

Then $\pi_1(F^{-1}_{[w]_m} F_{(w'_1,\dots,w'_{m-1})}(SG)) = K_{j_1}$ for $j_1 = I(p([w]_m, (w'_1, \dots, w'_{m-1}))$, and $\pi_1(F^{-1}_{[w]_m} F_{w'}(SG)) = K_{j_1 j_2}$ for $j_2 = I(p([w]_m, (w'_1, \dots, w'_m))$. Note that

$$p_i([w]_m, \big(w'_1, \dots, w'_{m-1})\big) \quad \text{and} \quad p_i\big([w]_m, (w'_1, \dots, w'_m)\big)$$

differ only for $i = w'_m$, so it follows from (2.2) that $j_2 \neq j_1$. For this to make sense we need to verify that if $F_{w'}(SG)$ and $F_{w''}(SG)$ intersect ($w' = (w'_1, \dots, w'_m)$

and $w'' = (w_1'', \dots, w_m'')$), then so do $K_{j_1', j_2'} = \pi_1(F_{[w]_m}^{-1} F_{w'}(SG))$ and $K_{j_1'', j_2''} = \pi_1(F_{[w]_m}^{-1} F_{w''}(SG))$. There are two ways this can happen. The first is if $F_{w'}(SG)$ and $F_{w''}(SG)$ belong to the same $(m-1)$-cell, in which case $w_1' = w_1'', \dots, w_{m-1}' = w_{m-1}''$ and $w_m' \neq w_m''$. In this case clearly $j_1' = j_1''$ and $j_2' \neq j_2''$, so $K_{j_1', j_2'}$ and $K_{j_1'', j_2''}$ are different 0-cells in the same (-1)-cell $K_{j_1'}$, and so intersect. Or, we could have $F_{w'}(SG)$ and $F_{w''}(SG)$ belong to intersecting $(m-1)$-cells, in which case $w_m' = w_m'' = k$, and moreover $w_1' = w_1'', \dots, w_\ell' = w_\ell''$ (for some $\ell \leq m-2$), $w_{\ell+1}'$, $w_{\ell+1}''$, and k are distinct, and $w_{\ell+2}' = \cdots = w_m' = w_{\ell+2}'' = \cdots = w_m'' = k$. For example, $w' = (w_1', \dots, w_\ell', 1, 2, \dots, 2)$, $w'' = (w_1', \dots, w_\ell', 3, 2, \dots, 2)$. In this case, if $p([w]_m, w') = (p_1, p_2, p_3)$, then $p([w])_m, w'') = (1-p_1, p_2, 1-p_3)$. But when we delete that last $w_m' = w_m'' = 2$, we flip p_2 and leave the others alone, so $p([w]_m, (w_1', \dots, w_{m-1}')) = (p_1, 1-p_2, p_3)$ and $p([w]_m, (w_1'', \dots, w_{m-1}'')) = (1-p_1, 1-p_2, 1-p_3)$. Thus $I(p([w]_m, (w_1', \dots, w_{m-1}')) = I(p([w]_m, (w_1'', \dots, w_m''))$ and $I(p([w]_m, (w_1'', \dots, w_{m-1}'')) = I(p([w]_m, (w_1', \dots, w_m'))$. This means $j_1' = j_2''$ and $j_1'' = j_2'$, so $K_{j_1', j_2'}$ and$K_{j_1'', j_2''}$ intersect in OSG_1. Figure 2.4 shows π_1 restricted to $F_{[w]_3}^{-1}(SG)$.

The description of π_n is similar. Without loss of generality we may take $m \geq n$. Define $j_k = I(p([w]_m, (w_1', \dots, w_{m-n-1+k}')$ for $k = 1, \dots, n+1$. Then

$$\pi_n(F_{[w]_m}^{-1} F_{(w_1', \dots, w_m')}(SG)) = K_{j_1, \dots, j_{m-1}}.$$

Essentially the same reasoning as in the case $n = 1$ shows that this makes sense and agrees with our previous definition in terms of $(-n)$-cells.

It is clear from the descriptions that

$$\pi_n' \circ \pi_n = \pi_{n-1}. \tag{2.3}$$

This can be seen in Figures 2.2 and 2.4: if you delete all the first labels in Figure 2.4 (the action of π_1'), you obtain Figure 2.2. ■

3 Characterization of Periodic Functions

Let Per_n denote the periodic functions of level n on SG_w. Recall that $F \in \mathrm{Per}_n$ if and only if F is continuous, and there exists a continuous function f on OSG_n such that $F = f \circ \pi_n$ (so F is the lift of f). (Functions here may be real-valued, complex-valued, or vector-valued.) Note that Per_n is an increasing family of function spaces because of (2.3). We will give two characterizations of periodic functions based on repeating patterns. The first is a simple consequence of the definition of π_n.

Theorem 3.1 *A continuous function F is in* Per_n *if and only if*

$$F(F_{[w]_m}^{-1} F_{w'} x) = F(F_{[w]_m}^{-1} F_{w''} x) \text{ for } x \in SG, \tag{3.1}$$

whenever $|w'| = |w''| = m - n$ and

$$p([w]_m, w') = p([w]_m, w''). \tag{3.2}$$

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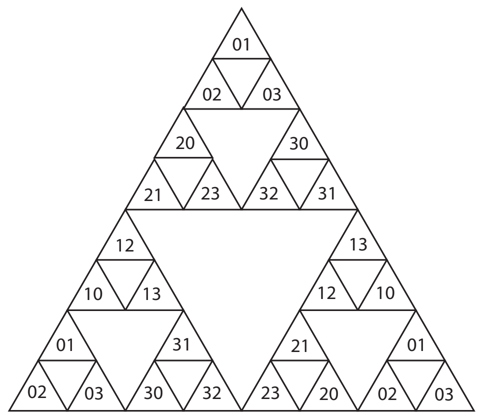

Figure 2.4: The map π_1 restricted to $F^{-1}_{[w]_3}(SG)$. Each 0-cell is labeled jk if it is mapped to K_{jk}.

Proof If $F = f \circ \pi_n$ then (3.1) holds because $\pi_n \circ F^{-1}_{[w]_m} F_{w'} = \pi_n \circ F^{-1}_{[w]_n} F_{w'}$ on SG if (3.2) holds by Definition 2.4. Conversely, if (3.1) holds, then define f on OSG_n by $f(x) = F(\pi_n^{-1}(x))$, where the value of F at any pre-image is the same because of (3.1). ■

Although there are no global isometries of SG_w, there are local isometries, and these can be used to characterize periodicity. These can be imagined as "conveyer belt" rotations around cycles. A *cycle* of $(-n)$-c ells of order k ($k = 0, 1, 2, \dots$) is a union of $3 \cdot 2^k$ $(-n)$-cells surrounding the inner deleted triangle in a $(-n-k-1)$-cell. Every $(-n)$-cell belongs to exactly three cycles, one being of order 0 and the other two being of order greater than 0. The cells in the cycle are cyclicly ordered, with consecutive cells intersecting. The rotation ρ is defined to move each cell to its neighbor, counterclockwise, translating along each of the three sides of the cycle, and rotating around the three corners, conveyer-belt style. A cycle of 0-cells of order 2 is shown in Figure 3.1.

Periodicity is essentially characterized by invariance under ρ^3. Note that for cycles of order 0, we have ρ^3 equal to the identity, so we need only consider $k \geq 1$.

Theorem 3.2 *A continuous function F on SG_w is in* Per_n *if and only if its restriction to every cycle of $(-n)$-cells is invariant under ρ^3.*

Proof For clarity of exposition we just treat the case $n = 0$. First we prove that functions in Per_0 have the ρ^3 invariance. We want to give an induction argument, not directly on the order k, but essentially on the size of a cell containing the cycle. Observe that the three outer edges of a cell are part of a larger cycle. So we assume the following induction hypothesis: for any cell C of order $-k$, F is invariant under ρ^3 for any cycle in C, and also F is invariant under translation by three 0-cells along each of the three outer edges of C. For $k = 1$ these statements are vacuously true. We need to show that the induction hypothesis implies the same statement for $k+1$.

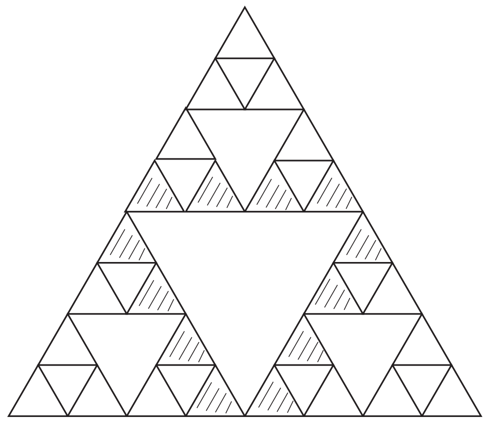

Figure 3.1: A cycle of 0-cells of order 2.

Now each $(-k-1)$-cell C splits into three $(-k)$-cells C_1, C_2 and C_3, and the induction hypothesis covers all cycles of order less than k, since they lie in one of the $(-k)$-cells. So it suffices to prove the invariance for the central cycle of order k, and for the three outer edges of C. Note that we already have the invariance for the portions of these cycles lying in C_1, C_2 and C_3, so we only have to check that the invariance persists near the three corners. By symmetry it suffices to see what happens at one such corner. In Figure 3.2 we show a neighborhood of the corner where C_1 and C_3 intersect when $k = 2$ and 3 (the labels in the 0-cells are the last $k + 1$ indices in w'). Along the inner cycle we verify $p(131) = p(322) = (0, 0, 1)$, $p(121) = p(323) = (0, 1, 0)$, and $p(122) = p(313) = (1, 0, 0)$ when $k = 2$, and $p(1213) = p(3222) = (0, 1, 1)$, $p(1223) = p(3221) = (1, 0, 1)$, and $p(1222) = p(3231) = (1, 1, 0)$ when $k = 3$. Along the outer edge we verify $p(113) = p(322) = (0, 0, 1)$, $p(123) = p(321) = (1, 1, 1)$, and $p(122) = p(331) = (1, 0, 0)$ when $k = 2$, and $p(1231) = p(3222) = (0, 1, 1)$, $p(1221) = p(3223) = (0, 0, 0)$, and $p(1222) = p(3213) = (1, 1, 0)$ when $k = 3$. In general, the even k's are like $k = 2$ and the odd k's are like $k = 3$, the only difference being an even number of indices 2 inserted after the first index, which does not change the parities. This completes the induction argument.

For the converse, assume the invariance holds for all cycles. We want to prove (3.1) holds if (3.2) holds. We do this again by induction on k, under the additional hypothesis that $F^{-1}_{[w]_m} F_{w'}(SG)$ and $F^{-1}_{[w]_m} F_{w''}(SG)$ belong to the same $(-k)$-cell. This is trivially true when $k = 1$ because (3.2) never holds for distinct 0-cells. For the induction argument, assume it is true for k. Let C be any $(-k-1)$-cell, and split it into $(-k)$-cells C_1, C_2, C_3 as before. By the induction hypothesis, (3.1) holds for any pair of 0-cells satisfying (3.2) and both lying in the same C_j. So we just need to verify that the pairs match up across different C_j's. But this follows from our previous analysis of what happens in a neighborhood of the corner points, running the argument backward. We also need to observe that all four indices $I(p)$ occur. ■

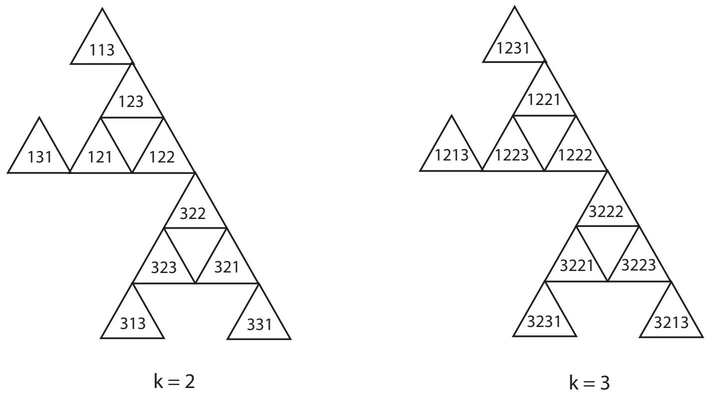

Figure 3.2: A neighborhood of the corner where C_1 and C_3 intersect.

We define AP, the space of *almost periodic functions* on S_w to be the closure (in the uniform norm) of $\bigcup_{n=0}^{\infty}\operatorname{Per}_n$. In other words, almost periodic functions are uniform limits of periodic functions. We have the following characterization analogous to Theorem 3.1.

Theorem 3.3 *A continuous function F on SG_w is in AP if and only if for all $\varepsilon > 0$ there exists n such that*

$$|F(F_{[w]_m}^{-1}F_{w'}x) - F(F_{[w]-m}^{-1}F_{w''}x)| \le \varepsilon \text{ for } x \in SG$$

whenever $|w'| = |w''| = m - n$ *and* (3.2) *holds.*

We omit the straightforward proof.

It is also possible to characterize AP functions in terms of the analog of a Bohr compactification of SG_w. For this we consider the inverse (sometimes called "projective") limit of the sequence of covering maps

$$OSG_0 \xleftarrow{\pi_1'} OSG_1 \xleftarrow{\pi_2'} OSG_2 \longleftarrow \cdots,$$

defined to be the compact subset of the infinite product $\Pi_{n=0}^{\infty} OSG_n$ of sequences $(x_0, x_1, \ldots)$ satisfying $x_{n-1} = \pi_n'(x_n)$ for all n. We denote this space OSG_∞. We define a map $\pi_\infty \colon SG_w \to OSG_\infty$ by $\pi_\infty(x) = (\pi_0(x), \pi_1(x), \ldots)$. This map is not a covering map. It is easy to see that it is injective, since if $x \neq y$ in SG_w, then $\pi_n(x) \neq \pi_n(y)$ for n sufficiently large, but it is not onto, and the dense image $\pi_\infty(SG_w)$ will depend on w. We use π_∞ to identify SG_w with a subset of OSG_∞. Any function $F = f \circ \pi_n$ in Per_n extends to a continuous function $\widetilde{F}$ on OSG_∞ via $\widetilde{F}(x_1, x_2, \ldots) = f(x_n)$. It follows that every AP function on SG_w extends to a continuous function on OSG_∞. The converse is also true. It is not clear, however, that this point of view leads to any nontrivial conclusions.

4 Symmetries of Periodic Functions

Periodic functions may have additional symmetries. These can be understood best by considering the isometry group of OSG (or OSG_n), which is isomorphic to the 24-element permutation group S_4. For example, the vertical reflection in Figure 2.1 preserves K_0 and K_1 and permutes K_2 and K_3. We denote the symmetry $S((2,3))$, and identify it with the permutation $(2,3)$. Similarly, there are symmetries $S((j,k))$ for $0 \le j < k \le 3$, and the permutations (j,k) generate S_4.

For any subgroup $\mathcal{G}$ of S_4, we can consider the subspace of Per_n of functions that are lifts of functions on OSG_n invariant under isometries in $\mathcal{G}$. (More generally, we could also consider lifts of functions transforming according to different irreducible representations of $\mathcal{G}$.)

We will look closely at two examples. The first is the 4-element group $\mathcal{G}_4$ consisting of all double pair permutations:

$$(0,1)(2,3),(0,2)(1,3),(0,3)(1,2)$$

as well as the identity. Note that $S((0,1))S((2,3))$ permutes K_0 and K_1 by fixing q_{10} and rotating, so q_{20} and q_{21} are permuted, as are q_{30} and q_{31}, with a similar story for K_2 and K_3. Thus any continuous function on OSG_n invariant under $\mathcal{G}_4$ is uniquely determined by its restriction to K_0, and then extended to K_1, K_2, K_3 by rotation. In particular, there is no restriction on the values on the boundary points q_{10}, q_{20}, q_{30}. The extended function satisfies $f(q_{10}) = f(q_{11})$, $f(q_{20}) = f(q_{21})$, $f(q_{30}) = f(q_{31})$.

Lifts of such functions will be called *extended periodic functions*, with the class of functions denoted EP_n. The characterization analogous to Theorem 3.1 is that (3.1) holds whenever $|w'| = |w''| = m - n$, without imposing condition (3.2). There is also a characterization analogous to Theorem 3.2, but, instead of the conveyer belt rotation ρ, we must rotate the cells by $1/3$, pivoting on the intersection point of neighboring cells. If we call this map $\widetilde{\rho}$, we have $\widetilde{\rho}^3 = \rho^3$. A function belongs to EP_n if and only if it is invariant under $\widetilde{\rho}$ on every cycle of $(-n)$-cells. Figure 4.1 shows the boundary values on 0-cells for a function in EP_0 restricted to a cell of order -2. Again the containments $EP_0 \subseteq EP_1 \subseteq EP_2 \subseteq \cdots$ hold.

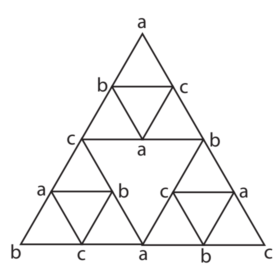

Figure 4.1: Values of a function in EP_0.

Because the isometries in $\mathcal{G}_4$ have fixed points, we cannot factor out the fractafolds OSG_n under the action of $\mathcal{G}_4$ within the category of fractafolds. (The quotients will

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be fractal analogs of orbifolds, but there does not appear to be a strong imperative to develop a theory of such objects at this time.) In other words, we cannot characterize extended periodic functions as lifts of covering maps to specific fractafolds.

On the other hand, there are subgroups $\mathcal{G}$ of S_4 whose action on OSG_n is fixed point free. There are four such subgroups, all conjugate, so we describe just one, which we denote $\mathcal{G}_3$. It is a three element group corresponding to the permutations that fix 0 and cyclicly permute 1, 2, 3. The action in Figure 2.1 just consists of rotations by $1/3$ and $2/3$. Regarding Figure 2.3 ($n = 1$) as a diagram of $(1-n)$-cells in OSG_n, it is clear that the union of the cells $K_{10}, K_{12}, K_{13}, K_{01}$ is a fundamental domain for the action of $\mathcal{G}_3$. Because of identification of boundary points we end up with the fractafold CSG_n shown in Figure 4.2, with the four $(1-n)$-cells labeled J_0, J_1, J_2, J_3. The covering map $\pi_n'' \colon OSG_n \to CSG_n$ is shown n Figure 4.3. We call the fractafolds CSG_n *conical* since they can be wrapped around a double cone. We then have a covering map $\pi_n'' \circ \pi_n \colon SG_w \to CSG_n$, and a periodic function in Per_n is $\mathcal{G}_3$ invariant if and only if it is a lift of a continuous function on CSG_n. A portion of this covering map is shown in Figure 4.4.

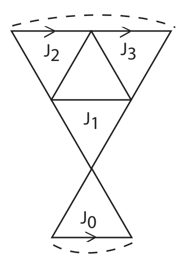

Figure 4.2: The conical fractafold CSG_n, with each J_k being a $(1-n)$-cell, and dotted lines indicating identified points.

We note that there are also covering maps $\pi_n''' \colon CSG_n \to CSG_{n-1}$, as illustrated in Figure 4.5. This leads to the commutative diagram of covering maps

$$\begin{array}{ccccc} & & SG_w & & \\ & \swarrow^{\pi_n} & & \searrow^{\pi_{n-1}} & \\ OSG_n & & \xrightarrow{\pi_n'} & & OSG_{n-1} \\ \downarrow{\scriptstyle \pi_n''} & & & & \downarrow{\scriptstyle \pi_{n-1}''} \\ CSG_n & & \xrightarrow{\pi_n'''} & & CSG_{n-1} \end{array}$$

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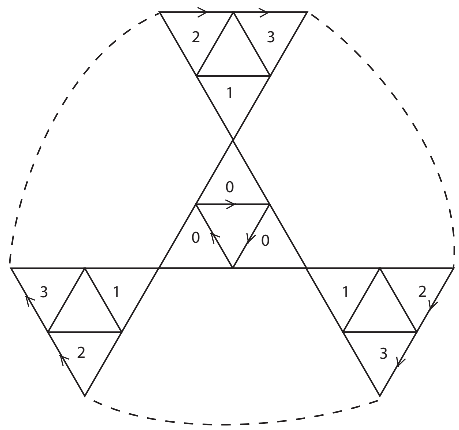

Figure 4.3: The covering map $\pi_n''\colon OSG_n \to CSG_n$, each index j indicating that the cell is mapped to J_j, with arrows lining up.

5 Eigenfunction Expansions

Let Δ denote the standard Laplacian on any of our spaces [5,9]. We can expand functions on OSG_n in an infinite series of eigenfunctions of Δ on OSG_n, and then lift the eigenfunctions to eigenfunctions of Δ on SG_w in Per_n, and so obtain an eigenfunction expansion for functions in Per_n. The exact spectrum of Δ on OSG_n is described in [7] generalizing the spectral decimation method on SG of [4]. It is interesting to compare it with the L^2 spectrum of Δ on SG_w as described in [10]. As usual we write the eigenfunction equation as $-\Delta u = \lambda u$. Since OSG_n is compact, it has a complete set of eigenfunctions with nonnegative eigenvalues tending to infinity. It is remarkable that SG_w, although not compact, also has a complete set of eigenfunctions (in fact compactly supported), but each eigenvalue has infinite multiplicity.

Let Λ_n denote the set of distinct eigenvalues of Δ on OSG_n, with the multiplicity of $\lambda \in \Lambda_n$ denoted $m_n(\lambda)$. It is convenient to subdivide Λ_n into four disjoint series,

$$\Lambda_n = \Lambda_n^{(0)} \cup \Lambda_n^{(4)} \cup \Lambda_n^{(5)} \cup \Lambda_n^{(6)}.$$

Here $\Lambda_n^{(0)}$ consists of the single eigenvalue $\lambda = 0$ with multiplicity one, and the corresponding eigenfunction is constant. Eigenvalues in the other series have a "generation of birth" m_0, and are determined by a sequence $\{\lambda_k\}$ for $k \geq m_0$, with $\lambda_{m_0} = 4, 5$ or 6 depending on the series. The values λ_k are interpreted as eigenvalues of the graph Laplacian Δ_k on the graph of vertices V_k (the boundary points of k-cells) in OSG_n, with the restriction of the eigenfunction u on OSG_n to V_k giving the associated eigenfunction of Δ_k. Since OSG_n is built of $(-n)$-cells, we have $k \geq -n$, with V_{-n} consisting of the six vertices $\{q_{10}, q_{20}, q_{30}, q_{11}, q_{21}, q_{22}\}$ in Figure 2.1, and $\#V_{-n} = 2 \cdot 3^{k+n+1}$.

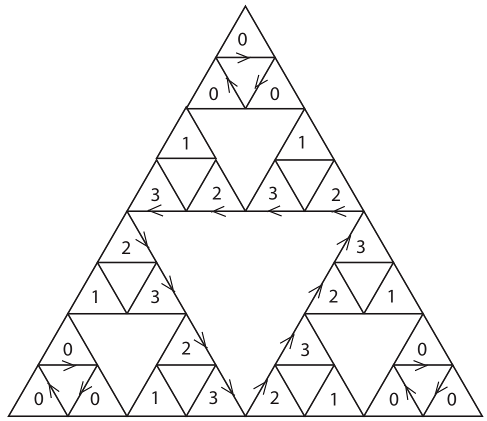

Figure 4.4: A portion of the covering map $\pi_n'' \circ \pi_n \colon SG_w \to CSG_n$, with index j in a $(1-n)$-cell of SG_w indicating it is mapped to J_j, with the arrows lining up.

The eigenvalues λ_k and λ_{k+1} are related by the quadratic equation

$$\lambda_k = \lambda_{k+1}(5 - \lambda_{k+1}),$$

which we can solve

$$\lambda_{k+1} = \frac{1}{2}\big(5 + \varepsilon_k\sqrt{25 - 4\lambda_k}\big)$$

for $\varepsilon_k = \pm 1$. The value of λ is then given by $\lambda = \lim_{k\to\infty} 5^k \lambda_k$. For the limit to exist we must have all but a finite number of $\varepsilon_k = -1$. We also have a few specific rules.

(i) If $\lambda \in \Lambda_n^{(6)}$, then $\varepsilon_{m_0} = +1$ so $\lambda_{m_0+1} = 3$.
(ii) If $\lambda \in \Lambda_n^{(5)}$, then $m_0 \geq -n + 1$.
(iii) If $\lambda \in \Lambda_n^{(4)}$, then $m_0 = -n$.

$$m_n(\lambda) = \begin{cases} 3 & \text{if } \lambda \in \Lambda_n^{(4)}, \\ 2 \cdot 3^{m_0+n} & \text{if } \lambda \in \Lambda_n^{(6)}, \\ 2 \cdot 3^{m_0+n-1} + 1 & \text{if } \lambda \in \Lambda_n^{(5)}. \end{cases}$$

The spectra are nested: $\Lambda_n \subseteq \Lambda_{n+1}, m_n(\lambda) \leq m_{n+1}(\lambda)$ (this is not obvious for the 4-series, but it follows from the observation that if $\lambda_k = 4$ and we take $\varepsilon_k = +1$, we obtain $\lambda_{k+1} = 4$). We also have the relationship $\Lambda_{n+1} = \frac{1}{5}\Lambda_n$. Figure 5.1 illustrates the eigenfunctions with $m_0 = -n$, and Figure 5.2 illustrates the eigenfunctions with $m_0 = 1 - n$. For $m_0 > 1 - n$ the eigenfunctions are miniaturized versions of the $m_0 = 1 - n$ case. For $\lambda_{m_0} = 6$ there is one for every vertex in V_{m_0-1}. For $\lambda_{m_0} = 5$ there is one for every independent cycle of length $3 \cdot 2^k$ for $1 \leq k \leq m_0 + n$ (there is one linear relation). The actual eigenfunctions on OSG_n are then determined by the choices of ε_k (the discrete eigenvalue equation determines the extension from V_k to V_{k+1}, and $V_* = \bigcup_k V_k$ is dense).

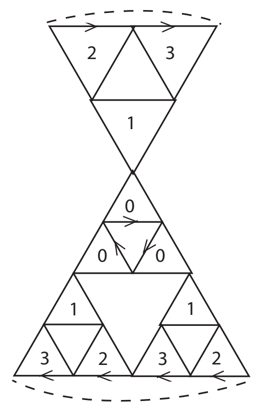

Figure 4.5: The covering map $\pi_n''': CSG_n \to CSG_{n-1}$. The index j in a $(2-n)$-cell of CSG_n indicates that it is mapped to the $(2-n)$-cell J_j in CSG_{n-1}, with the arrows lining up.

Note that in the cases of high multiplicity we have bases for the eigenspaces that are not orthogonal.

For each eigenspace there is a projection operator $P_\lambda^{(n)}$ from functions on OSG_n to eigenfunctions, and there is a corresponding Fourier series expansion

$$f = \sum_{\lambda \in \Lambda_n} P_\lambda^{(n)} f. \tag{5.1}$$

As shown in [8], this series actually converges uniformly for continuous functions f if we take appropriate partial sums (for example, up to the first $\#V_k$ eigenvalues, counting multiplicity). All this lifts to Per_n functions on SG_w (lifts of eigenfunctions are again eigenfunctions). Thus, without changing notation, we can view (5.1) as a Fourier expansion for Per_n functions on SG_w.

The eigenfunctions on SG_w corresponding to the 4-series were first described in [1], where it was observed that they are bounded. The eigenvalues in $\Lambda_n^{(5)}$ and $\Lambda_n^{(6)}$ are also in the L^2 spectrum of SG_w, but, of course, the Per_n eigenfunctions are never in L^2. In fact, the entire discrete L^2 spectrum is just the union of $\Lambda_n^{(5)}$ and $\Lambda_n^{(6)}$ for all n. An interesting observation about the L^2 spectrum is that the eigenfunctions all have total integral zero. This implies that the L^2 eigenfunction expansion does not converge in L^1 no matter how well behaved the function is. (The analogous statements are true for Haar series and related wavelet series.) This appears to rule out any analog of the Poisson summation formula. Indeed, it is easy to define a periodization map from $L'(SG_w)$ to $L^1(OSG_n)$ by summing over pre-images

$$\Pi_n f(x) = \sum f(\pi_n^{-1}(x)).$$

However, there is no relationship between the spectral projections of f and $\Pi_n f$, at least for the $\Lambda_n^{(0)}$ and $\Lambda_n^{(4)}$ eigenvalues.

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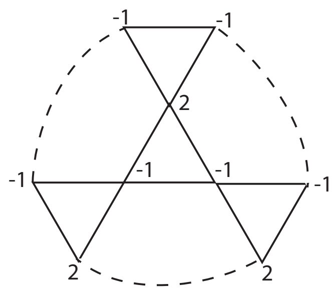

Figure 5.1: Examples of discrete eigenfunctions on OSG_n with $m_0 = -n$, $\lambda_{-n} = 6$ and $\lambda_{-n} = 4$. (Values shown on V_{-n}.) Under the symmetry group $\mathcal{G}_3$ they generate eigenspaces of multiplicity 2 and 3.

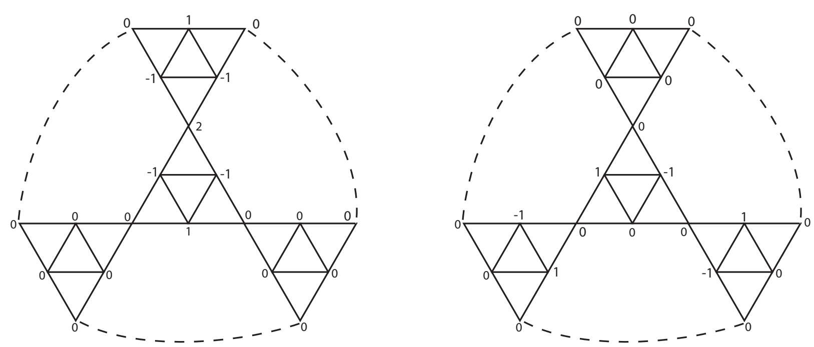

Figure 5.2: Examples of discrete eigenfunctions on OSG_n with $m_0 = 1 - n$, $\lambda_{1-n} = 6$ and $\lambda_{1-n} = 5$. (Values shown on V_{1-n}.) Under the symmetry group S_4 they generate eigenspaces of multiplicity 6 and 3.

We can also describe explicitly the spectrum for periodic functions with symmetry. For the extended periodic functions there are no 4-series eigenfunctions. It is clear that the discrete eigenfunction in Figure 5.1 with $\lambda_n = 4$ is orthogonal to all $\mathcal{G}_4$ invariant functions because it is skew-symmetric with respect to one of the $\mathcal{G}_4$ symmetries, and this orthogonality persists on OSG_n. On the other hand, the eigenfunction with $\lambda_{-n} = 6$ in Figure 5.1 is clearly $\mathcal{G}_4$ invariant. Altogether, the spectrum of EP_0 functions is identical (including multiplicities) to the Neumann spectrum of SG, as the Neumann boundary conditions are exactly what is needed to obtain an eigenfunction in the extension from K_0 to OSG. There is a similar description of the spectrum for $\mathcal{G}_3$ invariant periodic functions. The cell graph of CSG_1 has spectrum $\{0, 1, 4, 5\}$, and this can be used with the results of [7] to obtain the explicit spectrum of CSG_n. We omit the details.

Finally, we discuss briefly the spectral theory of AP functions. First we note the existence of the analog of the Bohr mean,

$$M(F) = \lim_{m\to\infty} 3^{-m} \int_{F^{-1}_{[w]_m}(SG)} F\, d\mu,$$

where μ is the standard measure on SG_w. If $F = f \circ \pi_n$ is in Per_n then $M(F)$ is just the mean value of f on OSG_n. It follows easily that $M(F)$ exists for AP functions.

The projection operators $P^{(n)}_\lambda$ on Per_n functions may be written as integral operators on OSG_n. If $u_1, \dots, u_{m(\lambda)}$ is an orthonormal basis for the λ-eigenspace on OSG_n, then

$$P^{(n)}_\lambda f(x) = \int_{OSG_n} P^{(n)}_\lambda(x,y) f(y)\, d\mu(y)$$

for $P^{(n)}_\lambda(x,y) = \sum_{j=1}^{m(\lambda)} u_j(x)u_j(y)$. We may lift this to SG_w ($F = f \circ \pi_n$, etc.) to obtain

$$P^{(n)}_\lambda F(x) = M(P^{(n)}_\lambda(x,\cdot\,)F(\,\cdot\,)). \tag{5.2}$$

Now if $F \in AP$, then $F = \lim_{n\to\infty} F_n$ with $F_n \in \mathrm{Per}_n$. It follows easily that F is the uniform limit of sums of periodic eigenfunctions with eigenvalues in $\bigcup_{n=0}^\infty \Lambda_n$. However, we would like to say more than this. Note that (5.2) makes sense for $F \in AP$. We would like to take the limit as $n \to \infty$ to define

$$P_\lambda F(x) = \lim P^{(n)}_\lambda F(x) = M(P_\lambda(x,\cdot\,)F(\,\cdot\,))$$

for $P_\lambda(x,y) = \lim_{n\to\infty} P^{(n)}_\lambda(x,y)$. It is not hard to see that this holds in the L^2 mean norm $\|F\|_2^2 = M(|F|^2)$. However, this does not imply that $P_\lambda F$ is in AP, or even that it is bounded. It is possible that the limits exist uniformly; one way to prove this would be to prove supnorm estimates on $P^{(n)}_\lambda$ that are independent of n. If that were the case, one could try to prove that

$$F = \lim_{n\to\infty} \sum_{\varepsilon_n \le \lambda \le N_n} P_\lambda F \tag{5.3}$$

uniformly for some sequences $\varepsilon_n \to 0$ and $N_n \to \infty$. Again it is true that (5.3) holds in the L^2 mean norm (for any such sequences). We leave this to the future.

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Mathematics Department, Malott Hall, Cornell University, Ithaca, NY 14853, USA
e-mail: str@math.cornell.edu

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