

On the Membership in Bergman Spaces of the Derivative of a Blaschke Product With Zeros in a Stolz Domain

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Abstract. It is known that the derivative of a Blaschke product whose zero sequence lies in a Stolz angle belongs to all the Bergman spaces A^p with $0 < p < 3/2$. The question of whether this result is best possible remained open. In this paper, for a large class of Blaschke products B with zeros in a Stolz angle, we obtain a number of conditions which are equivalent to the membership of B' in the space A^p ($p > 1$). As a consequence, we prove that there exists a Blaschke product B with zeros on a radius such that $B' \notin A^{3/2}$.

1 Introduction

We denote by \mathbb{D} the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ and by H^p ($0 < p \leq \infty$) the classical Hardy spaces of analytic functions in \mathbb{D} (see [3]). The Bergman space A^p ($0 < p < \infty$) consists of all functions f analytic in \mathbb{D} which belong to $L^p(\mathbb{D}, dA)$, where $dA(z) = \frac{1}{\pi} dx dy$ denotes the normalized Lebesgue area measure in \mathbb{D} . We mention [4, 6] as general references for the theory of Bergman spaces.

A sequence $\{a_n\}$ of points in \mathbb{D} is said to be a Blaschke sequence if

$$\sum_{n=1}^{\infty} (1 - |a_n|) < \infty.$$

The corresponding Blaschke product B is defined as $B(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \bar{a}_n z}$.

If $\xi \in \partial\mathbb{D}$ and $\sigma \in (1, \infty)$, we set $\Omega_\sigma(\xi) = \{z \in \mathbb{D} : |1 - \bar{\xi}z| \leq \sigma(1 - |z|)\}$. The domains $\Omega_\sigma(\xi)$ ($1 < \sigma < \infty$) are called Stolz angles with vertex at ξ . The domain $\Omega_\sigma(1)$ will be simply denoted by Ω_σ .

If a Blaschke product B has zeros $a_n = r_n e^{it_n}$, we define

$$f_B(t) = \sum_{a_n \neq 0} \frac{1 - |a_n|}{(1 - |a_n|)^2 + (t - t_n)^2}, \quad t \in (-\pi, \pi).$$

Ahern and Clark [2, Lemma 1, p. 121] proved that

$$(1) \quad B' \in H^p \Leftrightarrow f_B \in L^p(-\pi, \pi), \quad 0 < p < \infty.$$

Using this criterion we can deduce:

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- (i) If the zeros of a Blaschke product B all lie in some Stolz angle, then $B' \in \bigcap_{0 < p < 1/2} H^p$.
- (ii) If B is the Blaschke product with zeros $a_n = 1 - 1/(n \log^2 n)$, $n \geq 2$, then $B' \notin H^{1/2}$.

2 The Main Results

Even though we do not have a Bergman space analogue of (1), using [1, Theorem 6.1] (see also [5, Theorem 3]), it follows that if the zeros of a Blaschke product B all lie in some Stolz angle, then $B' \in A^p$ for all $p \in (0, 3/2)$. We shall prove that the exponent $3/2$ is sharp in this result even for Blaschke products with zeros on a radius.

Theorem 1 *The Blaschke product B with zeros $a_n = 1 - 1/(n \log^2 n)$, $n \geq 2$, has the property that $B' \notin A^{3/2}$.*

For a large class of Blaschke products B with zeros in a Stolz angle, we shall obtain a number of conditions which are equivalent to the membership of B' in the space A^p ($1 < p < \infty$). Theorem 1 will follow from these results. We remark that if B is an arbitrary infinite Blaschke product, $B' \notin A^p$ for any $p \geq 2$ (see [7, Theorem 1.1]). Hence, our coming results are really significant only for $3/2 \leq p < 2$.

Following Vinogradov [9], if B is the Blaschke product with zeros $\{a_n\}_{n=1}^\infty$, we define

$$(2) \quad \varphi_B(\theta) = \sum_{a_n \neq 0} \frac{1 - |a_n|}{[\theta + (1 - |a_n|)]^2}, \quad \theta \in (0, \infty).$$

We shall prove the following result.

Theorem 2 *Let B be a Blaschke product whose sequence of zeros lies in a Stolz angle. If there exist a positive constant C and $\theta_0 \in (0, \pi)$ such that*

$$(3) \quad \theta \varphi_B(\theta) \geq C \quad \text{for all } \theta \in (0, \theta_0),$$

then, for any given $p \in (1, \infty)$, we have that $B' \in A^p$ if and only if $\varphi_B \in L^{p-1}(0, 1)$.

Theorem 1 can be deduced from Corollary 10 below, but here we give a direct proof using Theorem 2.

Proof of Theorem 1 If B is the Blaschke product considered in Theorem 1, then

$$\varphi_B(\theta) = \sum_{n=2}^\infty \frac{1 - |a_n|}{[\theta + (1 - |a_n|)]^2} = \sum_{n=2}^\infty \frac{n \log^2 n}{[1 + \theta n \log^2 n]^2}, \quad \theta > 0.$$

For $0 < \theta < 1$, let N_θ be the unique number greater than 1 such that $\theta N_\theta \log^2 N_\theta = 1$. By a standard argument involving summation by parts, we have

$$\varphi_B(\theta) \geq \frac{1}{4} \sum_{2 \leq n \leq N_\theta} n \log^2 n \asymp N_\theta^2 \log^2 N_\theta = \frac{N_\theta}{\theta} = \frac{1}{\theta^2 \log^2 N_\theta}.$$

Now, the definition of N_θ easily implies that $\log N_\theta \sim \log \frac{1}{\theta}$, as $\theta \rightarrow 0$. Then it follows that there exist a positive constant C and $\theta_0 \in (0, 1)$ such that

$$\varphi_B(\theta) \geq C \frac{1}{\theta^2 \log^2 \frac{1}{\theta}}, \quad 0 < \theta < \theta_0.$$

This implies that $\varphi_B \notin L^{1/2}(0, 1)$. Then using Theorem 2 we deduce that $B' \notin A^{3/2}$. ■

Theorem 2 follows immediately from Theorem 3.

Theorem 3 Suppose that $1 \leq p < \infty$ and $\sigma > 1$, and let B be a Blaschke product whose zeros lie in a Stolz angle. Then there exist $C_1 > 0, C_2 > 0, M > 0$ and $\theta_0 \in (0, \pi)$ such that

$$(4) \quad C_1 \int_0^{2\pi} \varphi_B^{p-1}(\theta) d\theta \geq \int_{\mathbb{D}} |B'(z)|^p dA(z) \geq C_2 \int_0^{\theta_0} \varphi_B^{p-1}(\theta) (1 - e^{(-M\theta\varphi_B(\theta))}) d\theta.$$

A number of results will be needed to prove Theorem 3. The pseudo-hyperbolic metric in the unit disc will be denoted by $\varrho: \varrho(z, w) = \left| \frac{z-w}{1-\bar{w}z} \right|, z, w \in \mathbb{D}$. The following result, which is due to Marshall and Sarason, is proved in [8, Proposition 4].

Proposition 4 Let K be a closed convex subset of $\overline{\mathbb{D}}$ with $0 \in K$. Let B be a Blaschke product whose zeros $\{a_n\}$ are all contained in K . If $z \in \mathbb{D} \setminus K$ and $\varepsilon = \varrho(z, K)$, then

$$|B'(z)| \geq \frac{2\varepsilon}{1+\varepsilon^2} \frac{|B(z)|}{1-|z|^2} \sum_{n=1}^{\infty} (1 - \varrho^2(z, a_n))$$

The following lemma can be proved using simple geometric arguments.

Lemma 5 Given $\sigma > 1$ and $0 < \delta < 1$, there exists $\bar{\sigma} > \sigma$ such that $\rho(z, \Omega_\sigma) \geq \delta$ for every $z \in \mathbb{D} \setminus \Omega_{\bar{\sigma}}$.

Lemma 6 Let B be the Blaschke product whose sequence of zeros is $\{a_n\}_{n=1}^{\infty}$ and let $\delta \in (0, 1)$. If $z \in \mathbb{D}$ satisfies that $\varrho(z, a_n) \geq \delta$, for all n , then

$$(5) \quad |B(z)| \geq \exp\left(-\frac{1}{2\delta^2} \sum_{n=1}^{\infty} (1 - \varrho^2(z, a_n))\right).$$

Proof Take $z \in \mathbb{D}$ such that $\varrho(z, a_n) \geq \delta$ for all $n = 1, 2, \dots$; then using the elementary inequality $\log x \leq x - 1$, for $x \geq 1$, we deduce that

$$\log \frac{1}{|B(z)|} = \frac{1}{2} \sum_{n=1}^{\infty} \log \frac{1}{\varrho^2(z, a_n)} \leq \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{\varrho^2(z, a_n)} - 1\right) \leq \frac{1}{2\delta^2} \sum_{n=1}^{\infty} (1 - \varrho^2(z, a_n)),$$

which implies (5). ■

We shall use also the two following elementary lemmas.

Lemma 7 Given $R \in (0, 1)$, there exists $C_R \in (0, 1)$ such that

$$(6) \quad C_R [(1 - r) + (1 - \varrho) + |t|] \leq |1 - \varrho e^{it}| \leq (1 - r) + (1 - \varrho) + |t|, \quad r, \varrho \in [R, 1], t \in [-\pi, \pi].$$

Lemma 8 If $\sigma > 1$ then $\frac{1}{2+\sigma} \leq \frac{|1-\bar{\lambda}z|}{|1-|\lambda|z|} \leq 2 + \sigma$, whenever $z \in \mathbb{D}$ and $\lambda \in \Omega_\sigma$.

Proof of Theorem 3 Take $p \geq 1$ and assume, without loss of generality, that B is a Blaschke product with $B(0) \neq 0$ whose sequence of zeros $\{a_n\}_{n=1}^\infty$ lies in the Stolz angle Ω_σ ($\sigma > 1$). Write φ for φ_B .

There exists $R \in (0, 1)$ such that $|a_n| \geq R$, for all n . Let C_R be the constant associated to R by Lemma 7. Fix a number $\delta \in (0, 1)$. Using Lemma 5, we can take $\bar{\sigma} > \sigma$ such that $\varrho(z, \Omega_\sigma) \geq \delta$, for all $z \in \mathbb{D} \setminus \Omega_{\bar{\sigma}}$. Using Proposition 4 with $K = \overline{\Omega_\sigma}$ and bearing in mind that the function $x \mapsto \frac{2x}{1+x^2}$ is increasing in $(0, 1)$, we obtain that for every $z \in \mathbb{D} \setminus \Omega_{\bar{\sigma}}$,

$$|B'(z)| \geq \frac{2\varrho(z, \Omega_\sigma)}{1 + \varrho(z, \Omega_\sigma)^2} \frac{|B(z)|}{1 - |z|^2} \sum_{n=1}^\infty (1 - \varrho^2(z, a_n)) \geq \frac{2\delta}{1 + \delta^2} |B(z)| \sum_{n=1}^\infty \frac{1 - |a_n|^2}{|1 - \bar{a}_n z|^2}.$$

If $z \in \mathbb{D} \setminus \Omega_{\bar{\sigma}}$, then $\varrho(z, a_n) \geq \delta$ for all n . Lemma 6 and the above inequality yield

$$(7) \quad |B'(z)| \geq \frac{2\delta}{1 + \delta^2} \sum_{n=1}^\infty \frac{1 - |a_n|^2}{|1 - \bar{a}_n z|^2} \exp\left(-\frac{1}{2\delta^2} \sum_{n=1}^\infty \frac{(1 - |z|^2)(1 - |a_n|^2)}{|1 - \bar{a}_n z|^2}\right), \quad z \in \mathbb{D} \setminus \Omega_{\bar{\sigma}}.$$

Using (7), Lemma 8 and Lemma 7, we see that if $z = re^{it} \in \{z \in \mathbb{D} : |z| \geq R\} \setminus \Omega_{\bar{\sigma}}$

$$(8) \quad |B'(z)| \geq \frac{2\delta}{1 + \delta^2} \sum_{n=1}^\infty \frac{1 - |a_n|^2}{|1 - \bar{a}_n z|^2} \exp\left(-\frac{1}{2\delta^2} \sum_{n=1}^\infty \frac{(1 - |z|^2)(1 - |a_n|^2)}{|1 - \bar{a}_n z|^2}\right) \geq \frac{2\delta}{(1 + \delta^2)(2 + \sigma)^2} \sum_{n=1}^\infty \frac{1 - |a_n|^2}{|1 - |a_n||z|^2} \times \exp\left(-\frac{(2 + \sigma)^2}{2\delta^2} \sum_{n=1}^\infty \frac{(1 - |z|^2)(1 - |a_n|^2)}{|1 - |a_n||z|^2}\right) \geq \frac{2\delta}{(1 + \delta^2)(2 + \sigma)^2} \sum_{n=1}^\infty \left(\frac{1 - |a_n|^2}{[(1 - r) + (1 - |a_n|) + |t|]^2}\right) \times \exp\left(-\frac{(2 + \sigma)^2}{2\delta^2} \sum_{n=1}^\infty \frac{(1 - |z|^2)(1 - |a_n|^2)}{C_R^2 [(1 - r) + (1 - |a_n|) + |t|]^2}\right) \geq$$

$$\begin{aligned} &\geq \frac{2\delta}{(1 + \delta^2)(2 + \sigma)^2} \varphi((1 - r) + |t|) \\ &\quad \times \exp\left(-\frac{4(2 + \sigma)^2}{2\delta^2}(1 - r)\varphi((1 - r) + |t|)\right) \\ &= A\varphi((1 - r) + |t|) \exp(-K(1 - r)\varphi((1 - r) + |t|)), \end{aligned}$$

where A and K are two positive constants. Observe that there exists a positive constant β such that

$$(9) \quad |t| \geq \beta(1 - r), \quad \text{for } z = re^{it} \in \{z \in \mathbb{D} : |z| \geq R\} \setminus \Omega_{\overline{\sigma}}.$$

Take $R_0 \geq R$ such that $(\beta + 1)(1 - R_0) \leq \pi$. Using (8), making three consecutive changes of variable: $\theta = \theta(t) = 1 - r + t$, $u = u(r) = 1 - r$, $x = x(u) = u\varphi(\theta)$ and using Fubini’s theorem, we obtain

$$\begin{aligned} (10) \quad \int_{\mathbb{D}} |B'(z)|^p dA(z) &\geq \int_{\{z \in \mathbb{D} : |z| \geq R_0\} \setminus \Omega_{\overline{\sigma}}} |B'(z)|^p dA(z) \\ &\geq 2A^p \int_{R_0}^1 \int_{\beta(1-r)}^{\pi} \varphi^p((1 - r) + t) \exp(-Kp(1 - r)\varphi((1 - r) + t)) dt dr \\ &\geq 2A^p \int_0^{1-R_0} \int_{(\beta+1)u}^{(\beta+1)(1-R_0)} \varphi^p(\theta) \exp(-Kpu\varphi(\theta)) d\theta du \\ &= 2A^p \int_0^{(\beta+1)(1-R_0)} \int_0^{\frac{\theta}{(\beta+1)}} \varphi^p(\theta) \exp(-Kpu\varphi(\theta)) dud\theta \\ &= 2A^p \int_0^{(\beta+1)(1-R_0)} \varphi^{p-1}(\theta) \int_0^{\frac{\theta}{(\beta+1)}} \exp(-Kpx) dx d\theta \\ &= \frac{2A^p}{kp} \int_0^{(\beta+1)(1-R_0)} \varphi^{p-1}(\theta) \left[1 - \exp\left(\frac{-kp\theta\varphi(\theta)}{(\beta + 1)}\right)\right] d\theta. \end{aligned}$$

This proves the second inequality of (4) with $C_2 = \frac{2A^p}{kp}$, $\theta_0 = (\beta + 1)(1 - R_0)$ and $M = kp/(\beta + 1)$.

Now we turn to prove the other inequality. Write $b_n(z) = \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n}z}$ and $B_n(z) = \frac{B(z)}{b_n(z)}$, $n = 1, 2, \dots$. We have,

$$(11) \quad |B'(z)| = \left| \sum_{n=1}^{\infty} b'_n(z) \cdot B_n(z) \right| \leq \sum_{n=1}^{\infty} \frac{1 - |a_n|^2}{|1 - \overline{a_n}z|^2} |B_n(z)|.$$

The elementary inequality $\log(1 - x) \leq -x$, $0 < x < 1$, yields

$$(12) \quad \log |b_n(z)| = \frac{1}{2} \log(1 - (1 - |b_n(z)|^2)) \leq -\frac{1}{2}(1 - |b_n(z)|^2), \quad z \in \mathbb{D}.$$

Summing up over all $j \neq n$ and using the well-known identity $1 - |b_j(z)|^2 = \frac{(1-|z|^2)(1-|a_j|^2)}{|1-\bar{a}_jz|^2}$, we get from (12) that $\log |B_n(z)| \leq -\frac{1}{2} \sum_{j \neq n} \frac{(1-|z|^2)(1-|a_j|^2)}{|1-\bar{a}_jz|^2}$ which, together with (11), Lemma 8 and Lemma 7, implies that whenever $r \in [R, 1)$ and $t \in [-\pi, \pi]$,

$$\begin{aligned} |B'(re^{it})| &\leq \sum_{n=1}^{\infty} \frac{1 - |a_n|^2}{|1 - \bar{a}_n re^{it}|^2} \exp\left(-\frac{1}{2} \sum_{j \neq n} \frac{(1 - r^2)(1 - |a_j|^2)}{|1 - \bar{a}_j re^{it}|^2}\right) \\ &\leq e^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{1 - |a_n|^2}{|1 - \bar{a}_n re^{it}|^2} \cdot \exp\left(-\frac{1}{2} \sum_{n=1}^{\infty} \frac{(1 - r^2)(1 - |a_n|^2)}{|1 - \bar{a}_n re^{it}|^2}\right) \\ &\leq e^{\frac{1}{2}} (2 + \sigma)^2 \sum_{n=1}^{\infty} \frac{1 - |a_n|^2}{|1 - |a_n| re^{it}|^2} \cdot \exp\left(-\frac{1}{2(2 + \sigma)^2} \sum_{n=1}^{\infty} \frac{(1 - r^2)(1 - |a_n|^2)}{|1 - |a_n| re^{it}|^2}\right) \\ &\leq A \sum_{n=1}^{\infty} \frac{1 - |a_n|^2}{[(1 - |a_n|) + (1 - r) + |t|]^2} \cdot \exp\left(-K \sum_{n=1}^{\infty} \frac{(1 - r^2)(1 - |a_n|^2)}{[(1 - |a_n|) + (1 - r) + |t|]^2}\right) \\ &\leq A\varphi((1 - r) + |t|) \exp\left(-K(1 - r)\varphi((1 - r) + |t|)\right), \end{aligned}$$

where, A and K depend only on σ and R . After three changes of variable: $\theta = \theta(t) = 1 - r + t$, $u = u(r) = 1 - r$ and $x = x(u) = u\varphi(\theta)$, some obvious estimates, and using Fubini's theorem, we obtain

$$\begin{aligned} \int_{R \leq |z| < 1} |B'(z)|^p dA(z) &\leq 2A \int_R^1 \int_0^\pi \varphi^p((1 - r) + t) \exp[-Kp(1 - r)\varphi((1 - r) + t)] dt dr \\ &\leq 2A \int_R^1 \int_0^{2\pi} \varphi^p(\theta) \exp[-Kp(1 - r)\varphi(\theta)] d\theta dr \\ &\leq 2A \int_0^{2\pi} \int_0^1 \varphi^p(\theta) \exp[-Kpu\varphi(\theta)] du d\theta \\ &\leq 2A \left(\int_0^{2\pi} \varphi^{p-1}(\theta) d\theta \right) \left(\int_0^{\varphi(\theta)} \exp(-Kp x) dx \right) \leq \frac{2A}{Kp} \int_0^{2\pi} \varphi^{p-1}(\theta) d\theta. \end{aligned}$$

Since $\int_0^{2\pi} |B'(re^{it})|^p dt$ increases with r , this implies the first inequality of (4). ■

Corollary 9 *Suppose that $1 < p < \infty$ and B is a Blaschke product whose zeros lie in a Stolz angle and with the property that there exist $C > 0$ and $\theta_0 \in (0, \pi)$ such that (3) holds. Then the following conditions are equivalent:*

- (i) $B' \in A^p$;
- (ii) $\varphi_B \in L^{p-1}(0, \pi)$;

- (iii) $B' \in H^{p-1}$;
- (iv) $f_B \in L^{p-1}(-\pi, \pi)$.

Proof Theorem 2 shows that (i) \Leftrightarrow (ii). The equivalence (iii) \Leftrightarrow (iv) follows from Lemma 1 and, the implication (iii) \Rightarrow (i) follows from [1, Theorem 6.1].

To prove that (i) \Rightarrow (iii), suppose that B is a Blaschke product with $B(0) \neq 0$, $B' \in A^p$ and such that its zeros $\{a_n\}$ lie in Ω_σ for a certain $\sigma > 1$. Write $a_n = |a_n|e^{i\theta_n}$ with $|\theta_n| \leq \pi$. Since $\{a_n\} \subset \Omega_\sigma$, there exists a positive constant λ such that $|\theta_n| \leq \lambda(1 - |a_n|)$, $n = 1, 2, \dots$. We have $[(1 - |a_n|) + |\theta|]^2 \leq 2[(1 - |a_n|)^2 + \theta^2]$, and $\theta^2 \leq 2((\theta - \theta_n)^2 + \theta_n^2) \leq 2((\theta - \theta_n)^2 + \lambda^2(1 - |a_n|)^2)$ whenever $n \geq 1$ and $\theta \in [-\pi, \pi]$. Then it follows that there exists a constant $C > 0$ such that

$$(13) \quad [(1 - |a_n|) + |\theta|]^2 \leq C \left((1 - |a_n|)^2 + (\theta - \theta_n)^2 \right), \quad n \geq 1, \theta \in [-\pi, \pi].$$

Since (i) \Leftrightarrow (ii), $\varphi_B \in L^{p-1}(0, 1)$. Then (13) gives $f_B \in L^{p-1}(-\pi, \pi)$ and $B' \in H^{p-1}$. ■

Condition (3) is not a simple one. Next we find a simple condition which implies it.

Corollary 10 *If the zeros $\{a_n\}$ of Blaschke product B lie in a Stolz angle and there exist $\lambda > 0$ and $n_0 \geq 1$ such that $1 - |a_{n+1}| \geq \lambda(1 - |a_n|)$, if $n \geq n_0$, then there exist $C > 0$ and $\theta_0 \in (0, \pi)$ such that (3) holds. Hence, $B' \in A^p \Leftrightarrow \varphi_B \in L^{p-1}(0, \pi)$ ($p > 1$).*

Proof Given $\theta \in (0, 1 - |a_{n_0}|)$, take $n \geq n_0$ such that $1 - |a_{n+1}| < \theta \leq 1 - |a_n|$. Then

$$\theta \varphi_B(\theta) \geq \frac{\theta(1 - |a_n|)}{(\theta + (1 - |a_n|))^2} \geq \frac{(1 - |a_{n+1}|)(1 - |a_n|)}{4(1 - |a_n|)^2} = \frac{(1 - |a_{n+1}|)}{4(1 - |a_n|)} \geq \frac{\lambda}{4}.$$

Hence, we have proved (3) with $C = \lambda/4$ and $\theta_0 = 1 - |a_{n_0}|$. ■

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