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ABSOLUTELY FREE ALGEBRAS IN A TOPOS CONTAINING AN INFINITE OBJECT

BY
D. SCHUMACHER

0. **Introduction.** This note confirms that the existence proof for absolutely free algebras originated by Dedekind in [2] and completely developed for instance in [4] can still be carried out in a topos containing an infinite object i.e. an object N for which $N \simeq N+1$ if the type of the algebras considered is finite, pointed and internally projective i.e. is a finite sequence of objects, $(I_j)_{1\le j\le k}$ for which the functors $(\ )^{I_j}$ preserve epimorphisms and each of which has a global section.

However restrictive these requirements in the case of non-finitary operations might be, the omission of nullary operations is not serious: if m of the I_j are zero the absolutely free algebra over an object X can be obtained as an algebra with no nullary operations which is absolutely free over the coproduct of X with the m-fold coproduct of the terminal object 1 of the given topos.

Henceforth all types are understood to be finite, pointed and internally projective.

The present paper represents a vast improvement over [7] which came out at roughly the same time B. Lesaffre [5] had obtained the very same result. The existence proof for free finitary algebras in a topos containing a natural number object in [5] is an adaption of the existence proof for free finitary algebras over sets as it may be found in [1].

As in the classical case, we draw from, the existence of absolutely free algebras will be established in two steps:

PROPOSITION 1. *For every type and every object X of the given topos* **E** *there is an algebra* **A** *of this type in* **E** *containing X the operations of which are monomorphic, mutually disjoint and disjoint from X.*

PROPOSITION 2. *For every subobject X of (the underlying object of) an algebra* **A** *there can be defined a subalgebra $[X]$ of* **A** *containing X in such a way that two homomorphisms from $[X]$ coinciding over X are equal and moreover any morphism from X into an algebra* **B** *has a unique extension to a homomorphism from $[X]$ into* **B** *if* **A** *and X are as in Proposition* 1.

1. **The proof of Proposition 1.** This proof requires only that **E** contains an infinite object N and is a finitely complete and cocomplete cartesian closed

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category whose initial objects are strict initial and whose coproducts are disjoint, i.e. have monomorphic and mutually disjoint injections.

Given a type $(I_j)_{1\le j\le k}$ it is sufficient to show that for any object X there is an object A such that for all $1\le j\le k$ $A^{I_j}\simeq A$ and X and $A+A$ are retracts of A. For then

$$\begin{array}{l} X \longrightarrow A \xrightarrow{i_0} \coprod_{i=0}^{k} A \longrightarrow A \\ A^{I_1}\simeq A \xrightarrow{i_1} \coprod_{i=0}^{k} A \\ \quad\vdots \\ A^{I_k}\simeq A \xrightarrow{i_k} \coprod_{i=0}^{k} A \end{array}$$

is an algebra of type $(I_j)_{1\le j\le k}$ containing X the operations of which are monomorphic, mutually disjoint and disjoint from X.

Now, $A=X^J\times 2^{N\times J}$ with $J=I_1^N\times\cdots\times I_k^N$ has the required properties:

J has a global section, say $1\xrightarrow{\gamma}J$. Hence $X^\gamma X^{!_J}=X$ with $!_J$ the unique morphism from J into 1 and therefore X is a retract of X^J. But X^J in turn is a retract of $X^J\times 2^{N\times J}$ since $2^{N\times J}$ has a global section and there is hence a morphism f from X^J to $2^{N\times J}$: $X^J\xrightarrow{\langle X^J,f\rangle}X^J\times 2^{N\times J}$.

$A+A\simeq A\times 2\simeq X^J\times 2^{(N\times J)+1}$. Since J has a global section, $(N\times J)+1$ is a retract of $(N\times J)+J\simeq(N+1)\times J\simeq N\times J$. Therefore $2^{(N\times J)+1}$ is a retract of $2^{N\times J}$ which finally gives that $A+A$ is a retract of A.

For every $1\le j\le k$ $J\times I_j\simeq I_1^N\times\cdots\times I_j^{N+1}\times\cdots\times I_k^N\simeq J$ and hence $A^{I_j}\simeq(X^J)^{I_j}\times(2^{N\times J})^{I_j}\simeq X^{J\times I_j}\times 2^{N\times J\times I_j}\simeq A$.

2. **The proof of Proposition 2.** The following proof is based on the theorem of Mikkelsen's [6] that for every endomap Φ of the class $\mathrm{Sub}(A)$ of all subobjects of an object A, which is "induced" by an order preserving endomorphism ϕ of Ω^A, there is a monomorphism m into A which is smallest among all $\mu\in\mathrm{Sub}(A)$ with $\Phi(\mu)\le\mu$. Calling a monomorphism m into a product $B\times Y$ and a morphism $Y\xrightarrow{g}\Omega^B$ transpose of each other iff g is the exponential adjoint of the characteristic function of m, ϕ induces Φ means that for all $s\in\mathrm{Sub}(A)$ $\Phi(s)$ is a transpose of $\phi\bar{s}$ where $1\xrightarrow{\bar{s}}\Omega^A$ is the transpose of s. The order on Ω^A, which ϕ preserves, is the canonical one i.e. the equalizer of $(\Omega\times\Omega)^A\xrightarrow{\wedge^A}\Omega^A$ and $(\Omega\times\Omega)^A\xrightarrow{p_1^A}\Omega^A$. Note that for morphisms $Y\xrightarrow{g}\Omega^A$ and $Y\xrightarrow{h}\Omega^A$ $\langle h,g\rangle$ factors through the order on Ω^A iff the transpose of g factors through the transpose of h. From this it follows easily that an endomorphism ϕ of Ω^A is order preserving if and only if for all morphisms g and h the transpose of ϕg factors through the transpose of ϕh if the transpose of g factors through the transpose of h.

With the help of his theorem and Freyd's Proposition 2.21 (unique existentiation) [3] Mikkelsen had succeeded in translating into a topos Dedekind's proof of the existence of natural numbers from the existence of infinite sets. Since the passage from infinite objects to natural number objects is a special case of passing from an algebra in Proposition 1 to the corresponding absolutely free algebra, it is rather obvious that a slight generalization of Mikkelsen's proof should provide a proof for Proposition 2.

For every morphism $A \xrightarrow{f} B$ let $\exists_f$ be the transpose of an image of $(f \times \Omega^A)\epsilon_A$, where ϵ_A is a subobject of the evaluation $A \times \Omega^A \xrightarrow{ev} \Omega$. This assignment is functorial i.e. we have a functor $\exists_{(\ )}$ usually called the direct image functor.

For any objects A and I of the topos we define the morphism $\Pi_1 : \Omega^A \to \Omega^{A^1}$ (called "raising to the I-th power") as follows: the map $(\epsilon_A)^I$ into $(A \times \Omega^A)^I$ is a monomorphism, which we regard as having codomain $A^I \times (\Omega^A)^I$, thus obtaining as its transpose a morphism $(\Omega^A)^I \to \Omega^{A^I}$. Composing this morphism with the evident morphism $(\Omega^A)^{!_I} : \Omega^A \to (\Omega^A)^I$ yields Π_I.

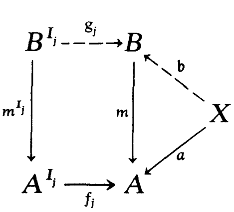
Now, let $X \xrightarrow{a} A$ be a monomorphism into the underlying object of an algebra $\mathbf{A} = (A, (f_j)_{1 \le j \le k})$. Then the morphism

$$\phi : \Omega^A \xrightarrow{\langle \overline{a}!,\, \exists_{f_1}\Pi_{I_1}, \dots, \exists_{f_k}\Pi_{I_k} \rangle} \Omega^{(k+1)\cdot A} \xrightarrow{\exists_\nabla} \Omega$$

induces an endomap Φ of Sub(A) which sends any monomorphism μ into A to a union of a and the monomorphisms $f_1[\mu^{I_1}], \dots, f_k[\mu^{I_k}]$, where $f[\iota]$ denotes the image of $f\iota$. For (i) the transpose of the composition $Y \xrightarrow{h} \Omega^B \xrightarrow{\exists_g} \Omega^C$ is the image of $(g \times Y)\eta$ with η transpose of h; (ii) the transpose of the "pointwise union" $Y \xrightarrow{\langle h_1, \dots, h_n \rangle} \Omega^{n.C} \xrightarrow{\exists_\nabla} \Omega^C$ of a family $(Y \xrightarrow{h_j} \Omega^C)_{1 \le j \le n}$ is a union of transposes of the morphisms h_j; and (iii) the transpose of $Y \xrightarrow{g} \Omega^A \xrightarrow{\Pi_I} \Omega^{A^I}$ is the pullback of γ^I along $A^I \times Y^! : A^I \times Y \to (A \times Y)^I$ with γ transpose of g, which gives in particular that for any monomorphism μ into A the transpose of $\Pi_I \overline{\mu}$ is μ^I.

ϕ is order preserving since (i), (ii) and (iii) imply that for all morphisms $Y \xrightarrow{g} \Omega^A$ and $Y \xrightarrow{h} \Omega^A$ the transpose of ϕg factors through the transpose of ϕh if the transpose of g factors through the transpose of h.

Thus for every algebra $\mathbf{A} = (A, (f_j)_{1 \le j \le k})$ and every monomorphism $X \xrightarrow{a} A$ there is a monomorphism $B \xrightarrow{m} A$ which is smallest among all subobjects μ of A, through which a and all $f_j[\mu^{I_j}]$ factor. For the morphisms b and g_j $(1 \le j \le k)$ for which the diagrams



commute, $\mathbf{B}=(B,(g_j)_{1\le j\le k})$ is an algebra of the same type as $\mathbf{A}$, m is a homomorphism from $\mathbf{B}$ into $\mathbf{A}$ and finally b generates $\mathbf{B}$ i.e. every monomorphism m' into B through which b and all $g_j[m'^{I_j}]$ factor, is an isomorphism. $\mathbf{B}$ is reasonably called the subalgebra of $\mathbf{A}$ generated by a. In particular, the image of $X+B^{I_1}+\cdots+B^{I_k}\xrightarrow{[b,\,g_1,\,\ldots,\,g_k]}B$ is isomorphic and thus $[b,g_1,\ldots,g_k]$ epimorphic. If the f_j are monomorphic, mutually disjoint and disjoint from a then also the g_j are monomorphic, mutually disjoint and disjoint from b and hence $[b,g_1,\ldots,g_k]$ is an isomorphism.

An algebra $\mathbf{B}=(B,(g_j)_{1\le j\le k})$ the operations of which are monomorphic, mutually disjoint and disjoint from a monomorphic $X\xrightarrow{b}B$ generating B is however in any topos an absolutely free algebra over X: Given an algebra $\mathbf{S}=(S,(\sigma_j)_{1\le j\le k})$ and a morphism x from X into S, a monomorphism $C\xrightarrow{\lambda}B\times S$, which is smallest among all monomorphisms μ into $B\times S$ through which $\langle b,x\rangle$ and all $(g_j\times\sigma_j)[\mu^{I_j}]$ factor, is up to an isomorphism the graph of a homomorphism h from $\mathbf{B}$ into $\mathbf{S}$ for which $x=hb$ (such a homomorphism is apparently unique since $\mathbf{B}$ is generated by b). For $p_1\lambda$ turns out to be both an epimorphism and a monomorphism (i.e., an isomorphism) and hence, if we note that λ is a homomorphism from the subalgebra $\mathbf{C}=(C,(h_j)_{1\le j\le k})$ of $\mathbf{B}\times\mathbf{S}=(B\times S,(g_j\times\sigma_j)_{1\le j\le k})$ generated by $\langle b,x\rangle$, it follows easily that $(p_2\lambda)(p_1\lambda)^{-1}$ is the homomorphism from $\mathbf{B}$ into $\mathbf{S}$ requested.

Obviously b and all $g_j(p_1\lambda)^{I_j}$ factor through $p_1\lambda$. It is in order to conclude from here, that besides b all the $g_j[\iota^{I_j}]$ factor through the image ι of $p_1\lambda$ and thus ι is an isomorphism (i.e., $p_1\lambda$ is an epimorphism), that we required the arities to be internally projective.

The much harder problem of showing that $p_1\lambda$ is monomorphic can be surprisingly smoothly settled following Mikkelsen's advice to test for monomorphy by Freyd's Proposition of Unique Existentiation:

LEMMA (compare with [1; Lemma 5.431]). *Let q be a homomorphism from an algebra $\mathbf{C}=(C,(h_j)_{1\le j\le k})$ generated by a monomorphism $X\xrightarrow{c}C$, into an algebra $\mathbf{B}=(B,(g_j)_{1\le j\le k})$ the operations of which are monomorphic, mutually disjoint and disjoint from a monomorphism $X\xrightarrow{b}B$ for which $b=qc$. Then q is a monomorphism.*

The proof of the Lemma is, with Mikkelsen's suggestion in mind, straightforward: If

$$\begin{array}{ccc} Q & = & Q \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle m} \\ C & \xrightarrow[q]{} & B \end{array}$$

is the pullback of unique existentiation, then in order to show that w is isomorphic and hence q is monomorphic, it is sufficient to prove that c and all $h_j w^{I_j}$ factor through w; which in turn is true if

$$(1)\qquad \begin{array}{ccc} X & = & X \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle b} \\ C & \xrightarrow[q]{} & B \end{array} \qquad \text{and (2) the} \qquad \begin{array}{ccc} Q^{I_j} & = & Q^{I_j} \\ {\scriptstyle h_j w^{I_j}}\downarrow & & \downarrow{\scriptstyle q h_j w^{I_j}} \\ C & \xrightarrow[q]{} & B \end{array}$$

are pullbacks. For the latter (as for Freyd's Lemma 5.431) the fact that $[c, h_1, \ldots, h_k]$ is epimorphic turns out to be rather essential: The pullback $b \cap q[c, h_1, \ldots, h_n]$ of b and $q[c, h_1, \ldots, h_n]$ is $b \cap [b, g_1 q^{I_1}, \ldots, g_k q^{I_k}] = [b \cap b, 0, \ldots, 0] = b$ and hence

$$\begin{array}{ccccc} X & = & = & = & X \\ {\scriptstyle i_0}\downarrow & & & & \downarrow{\scriptstyle b} \\ X + C^{I_1} + \cdots + C^{I_k} & \xrightarrow[{[c, h_1, \ldots, h_k]}]{} & C & \xrightarrow[q]{} & B \end{array}$$

is a pullback. But in general if

$$\begin{array}{ccccc} D & = & = & = & D \\ {\scriptstyle e}\downarrow & & & & \downarrow \\ E & \xrightarrow[\varepsilon]{} & G & \longrightarrow & F \end{array}$$

is a pullback and ε is an epimorphism then also

$$\begin{array}{ccc} D & = & D \\ {\scriptstyle \varepsilon e}\downarrow & & \downarrow \\ G & \longrightarrow & F \end{array}$$

is a pullback because, fitting in a pullback of GF and DF

$$\begin{array}{ccccc} D & = & = & = & D \\ & \searrow & & \nearrow & \\ \downarrow & & H & & \downarrow \\ & & \downarrow & & \\ E & \longrightarrow & G & \longrightarrow & F \end{array}$$

also DHGE with the induced morphism from D into H is a pullback. Whence DH is epimorphic and leftinvertible and thus an isomorphism.

For (2) note that because

$$\begin{array}{ccc} Q & \xrightarrow{w} & C \\ \big\downarrow & & \big\downarrow{\scriptstyle q} \\ Q & \xrightarrow[qw=m]{} & B \end{array} \quad \text{is a pullback then all} \quad \begin{array}{ccc} Q^{I_j} & \xrightarrow{w^{I_j}} & C^{I_j} \\ \big\downarrow & & \big\downarrow{\scriptstyle g_j q^{I_j}} \\ Q^{I_j} & \xrightarrow[g_j(qw)^{I_j}]{} & B \end{array}$$

are pullbacks and that hence the pullback $qh_jw^{I_j} \cap q[c, h_1, \ldots, h_k]$ of $qh_jw^{I_j}$ and $q[c, h_1, \ldots, h_k]$ is $qg_j(qw)^{I_j} \cap [b, g_1q^{I_1}, \ldots, g_kq^{I_k}] = [0, \ldots, g_j(qw)^{I_j}, 0, \ldots, 0] = g_j(qw)^{I_j}$. This implies that the diagram

$$\begin{array}{ccccc} Q^{I_j} & = & = & = & Q^{I_j} \\ {\scriptstyle i_jw^{I_j}}\big\downarrow & & & & \big\downarrow{\scriptstyle qh_jw^{I_j} = g_j(qw)^{I_j}} \\ X + C^{I_1} + \cdots + C^{I_k} & \xrightarrow[[c, h_1, \ldots, h_k]]{} & C & \xrightarrow[q]{} & B \end{array}$$

is a pullback to which the general remark above on pullbacks of this form applies.

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DEPARTMENT OF MATHEMATICS,
ACADIA UNIVERSITY,
WOLFVILLE, NOVA SCOTIA, CANADA