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A MINIMAX EQUALITY RELATED TO THE LONGEST DIRECTED PATH IN AN ACYCLIC GRAPH

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1. Introduction. As an analog of a recently established minimax equality for directed graphs [**1**], I. Simon has suggested that the following be investigated.

1.1. *For a finite acyclic directed graph G, a minimum collection of directed coboundaries whose union is the edge set of G has cardinality equal to that of a maximum strong matching of G.*

This minimax equality is here proved, using a characterization of a maximum strong matching of an acyclic graph as the set of edges of a longest directed path in the graph.

The terms employed in the above theorem are defined as follows. Let G be a finite *directed graph* with *vertex set* VG and *edge set* eG. For each edge α, one of its ends is specified the *positive end* $p\alpha$, the other the *negative end* $n\alpha$. For a subset X of VG, *coboundary* $\delta_G X$, or simply δX, is the set of edges each with one end in X and the other in $VG - X$. If each edge has its positive end in X, then δX is *outdirected*; if each edge has its negative end in X, then δX is *indirected*; in either case, δX is a *directed coboundary*. A *strong matching* is a set of edges no two of which lie in the same directed coboundary. A set is *minimum* with a given property if it has that property but no set with smaller cardinality has that property; on the other hand, a set with a given property is *minimal* if no proper subset has the property. "Maximum" and "maximal" are defined analogously.

2. Maximum strong matching and a longest directed path. A *directed path from* a *to* b in directed graph G is a finite sequence $(v_0, \alpha_1, v_1, \alpha_2, \ldots, \alpha_n, v_n)$, whose terms are alternately vertices v_i and edges α_j, such that each edge α_j has positive end v_{j-1} and negative end v_j. The *origin* v_0 is equal to a, the *terminus* v_n is equal to b. Let $a \to b$ denote the existence in G of a directed path from a to b. The set of edges in path π is denoted $e\pi$. The *length* of path π is equal to n, the subscript on the terminus v_n; for an acyclic graph and a directed path, $|e\pi| = n$. A longest directed path is one of maximum length.

2.1. *For any two distinct edges of an acyclic graph, there is a directed path that contains them both if and only if there is no directed coboundary that contains them both.*

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Proof. For α and β distinct edges in acyclic graph G, let X_α equal $\{x \in VG: x \to p\alpha\}$ and let X_β equal $\{x \in VG: x \to p\beta\}$.

Consider first the case in which no directed path contains both α and β. Then $n\alpha$ and $n\beta$ each lie in $VG - X_\alpha \cup X_\beta$, whence α and β each lie in $\delta(X_\alpha \cup X_\beta)$, which is an outdirected coboundary, since each of δX_α and δX_β is outdirected.

Consider next the case in which α and β lie in the same directed path. Then either $n\alpha \to p\beta$ or $n\beta \to p\alpha$; adjust notation if necessary so that $n\alpha \to p\beta$. Suppose that some directed coboundary δX contains both α and β; choose X so that δX is outdirected. Since $n\alpha \in VG - X$, $p\beta \in X$ and $n\alpha \to p\beta$, there is some edge with positive end in $VG - X$ and negative end in X, in contradiction to δX outdirected. So no directed coboundary δX contains both α and β. The proof is complete.

2.2. *In an acyclic graph G, a set m of edges is a strong matching if and only if there is a directed path in G that contains all the elements of m.*

Proof. Any subset of the edge set of a directed path is a strong matching by 2.1. To prove the converse, assume that m is a strong matching. The proof proceeds by induction on $|m|$, the cardinality of m. For $|m|$ equal to 0 or 1, the assertion holds trivially. For $|m|$ equal to 2, it follows from 2.1. For $|m|$ greater than 2, assume as induction hypothesis that the assertion holds for each proper subset of m. For α in m, let m' equal $m - \{\alpha\}$. There is a directed path π' such that $m' \subseteq e\pi'$. Let $\alpha_0, \alpha_1, \ldots, \alpha_k$ be the edges of m', arranged in order of their occurrence as edge-terms of π'; then $n\alpha_{i-1} \to p\alpha_j$ for each i between 1 to k. Since α does not lie in the same directed coboundary as any of the edges of m', thus by 2.1 either $n\alpha_i \to p\alpha$ or $n\alpha \to p\alpha_i$ for each i between 0 and k. Say that α_i *precedes* α if $n\alpha_i \to p\alpha$ and that α_i *succeeds* α if $n\alpha \to p\alpha_i$. If α_0 succeeds α or if α_k precedes α, then the assertion follows directly. Assume that α_0 precedes α and α_k succeeds α. There must then exist a subscript i such that α_{i-1} precedes α and α_i succeeds α. That is, there is a directed path from $n\alpha_{i-1}$ to $p\alpha_i$ that includes α; call that path π_α. Let path π be obtained from π' by replacing the segment of π' from $n\alpha_{i-1}$ to $p\alpha_i$ by π_α. Then π is a directed path such that $m \subseteq e\pi$; this is the asserted path.

Proposition 2.2 has the following corollary.

2.3. *A set m of edges in acyclic graph G is a maximum strong matching if and only if m is the edge set of a longest directed path in G.*

3. A minimax equality.

3.1. *For a finite acyclic graph G, a minimum collection of directed coboundaries whose union is the edge set of G has cardinality equal to the length of a longest directed path in G.*

Proof. For any collection D of directed coboundaries whose union is eG and any directed path π, $|D| \geqq |e\pi|$. Hence it suffices to show that there is a pair D and π such that $|D| = |e\pi|$. Such a path π must be a longest directed path in G and so its origin must be a *source*, i.e. a vertex not the negative end of any edge in G.

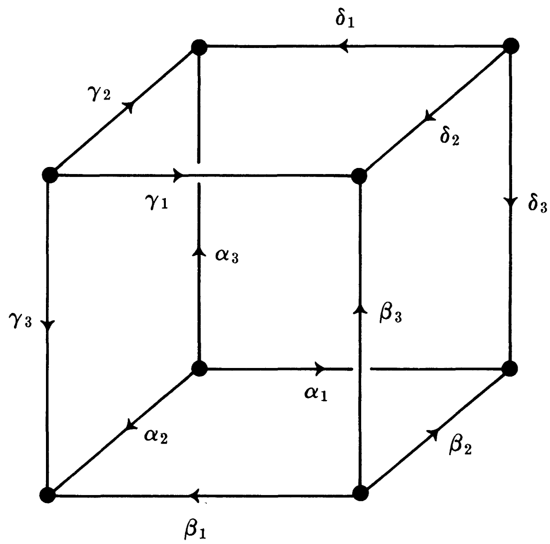
The proof proceeds by induction on the length of a longest directed path π. If $|e\pi| = 0$, then the null set is the required collection D. Assume that $|e\pi| = k > 0$ and, as induction hypothesis, that the assertion holds for all acyclic graphs in which the length of a longest directed path is at most $k - 1$. Let S be the set of source vertices in G; then δS is an outdirected coboundary in G. Let G' be the subgraph of G obtained by deleting each vertex of S and its incident edges.

Of course, G' is acyclic. Moreover, a longest directed path in G' has length one less than that of a longest directed path in G, since the origin of the latter must lie in S. By the induction hypothesis, there is a collection D' of directed coboundaries in G' whose union is eG', such that $|D'|$ is equal to the length of a longest directed path π' in G'. Expressing each element of D' as an outdirected coboundary $\delta_{G'}X$, let D equal

$$\{\delta_G S\} \cup \{\delta_G(S \cup X): \delta_{G'}X \in D'\}.$$

Each coboundary in D is outdirected. Moreover, since $\delta_{G'}X \subseteq \delta_G(S \cup X)$, thus $\bigcup D = \delta S \cup eG' = eG$. Finally, $|D| = |D'| + 1 = |e\pi'| + 1 = |e\pi|$, for π a longest directed path in G. Consequently the assertion holds for G. By induction, 3.1 holds generally.

Implicit in the proof just given is an algorithm for finding a minimum collection D and a longest directed path π. The number of execution steps in



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that algorithm is bounded above by some fairly small power of the number of edges and vertices in the graph.

Theorem 1.1 follows directly from 3.1 and 2.3.

It is natural to ask whether replacement of "directed coboundary" by "minimal nonnull directed coboundary" in the statement of Theorem 1.1, and in the definition of strong matching, yields another valid proposition. That this is not the case is shown by the counterexample pictured in Figure 1.

In the graph shown there, $\{\alpha_1, \beta_1\}$ is a maximum set of edges no two of which lie in the same minimal nonnull directed coboundary. On the other hand, a minimum collection of minimal nonnull directed coboundaries whose union contains every edge of G has cardinality 3. The observation that this variant is invalid was first made by G. N. Robertson.

Reference

1. C. L. Lucchesi and D. H. Younger, *A minimax theorem for directed graphs* (to appear).

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