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ARE NON-COMMUTATIVE L_p SPACES REALLY NON-COMMUTATIVE?

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1. The central objects in integration theory can be considered to be an abelian Von Neumann algebra, L_∞, of the measure space, together with a (not necessarily finite-valued) positive linear functional on it, the integral (see [**10**]). It is natural, therefore, to attempt to construct a "non-commutative" integration theory starting with a non-abelian Von Neumann algebra. Segal [**9**] and Dixmier [**2**] have developed such a theory, and constructed the Non-Commutative L_p spaces associated with a Von Neumann algebra M and a normal, faithful, semifinite trace (i.e. a unitarily invariant weight) t on M. They show that there exists a unique ultra-weakly dense *-ideal J of M such that t (extends to) a positive linear form on J. A generalisation of the Hölder inequality then shows that, for $1 \leqq p < \infty$, the function

$$x \mapsto (t(|x|^p))^{1/p}$$

is a norm on J, denoted by $\|\cdot\|_p$. $L_p(M, t)$ denotes the completion of $(J, \| \ \|_p)$ and its elements can be identified with (unbounded) closed "measurable" operators affiliated to M (see [**8**], [**5**]); $L_\infty(M, t)$ is identified with M itself. Of course, if M is abelian, this construction yields what it ought to, namely the classical L_p spaces. The restriction that t be unitarily invariant is necessary for the Hölder inequality to be valid, and hence for the construction of the L_p spaces to be possible. In the absence of unitary invariance, one has to resort to Tomita–Takesaki theory [**12**].

The problem now presenting itself is whether these non-commutative L_p spaces form a class of Banach spaces distinguishable from classical ones. More precisely, if

$$T: L_p(M, t) \to L_p(X, m)$$

(where (X, m) is a measure space) is an onto isometry, then does it follow that M is isomorphic, as a Von Neumann algebra, to $L_\infty(X, m)$? The purpose of this note is to show that the answer is yes in the finite case ($t(1) < \infty$) when $p > 2$. The answer is obviously no in case $p = 2$, since $L_2(M, t)$ is a Hilbert space. But if the condition that T be positivity preserving is added to the hypotheses, then the answer is yes, even in the semifinite case (see [**1**]).

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2. In order to solve the problem posed above, one needs to study isometries of non-commutative L_p spaces. For $p = \infty$, these were classified by Kadison [**3**]: Isometries of Von Neumann algebras are Jordan *-homomorphisms (i.e. linear mappings preserving squares and the involution) composed with left multiplication by a fixed unitary. In a previous paper [**4**] this result was partially extended to the case $2 < p < \infty$.

THEOREM 1. *Let (M_i, t_i) be two Von Neumann algebras equipped with faithful normal traces t_i such that $t_i(1) = 1$, and let $p > 2$. If*

$$T: L_p(M_1, t_1) \to L_p(M_2, t_2)$$

*is a *-linear into isometry such that $T(1) = 1$, then:*
(i) *T is a Jordan *-homomorphism*
(ii) *T is positivity preserving*
(iii) *T preserves $\|\cdot\|_{2n}$, for $n = 1, 2, \ldots \infty$*
(iv) *T is ultraweakly continuous*
(v) *there exists a projection $e \in T(M_1) \cap T(M_1)'$ such that*

$$x \mapsto T(x)e \text{ is a homomorphism}$$

$$x \mapsto T(x)\ (1 - e) \text{ is an anti-homomorphism.}$$

Here we shall need the following variant of this result:

THEOREM 2. *Let (M_i, t_i) be as in Theorem 1. Suppose that, for some $n \in \mathbf{N}$, $n \geqq 2$,*

$$T: L_{2n}(M_1, t_1) \to L_{2n}(M_2, t_2)$$

is an into linear mapping such that $T(1) = 1$ and T maps normal elements to normal elements. If T is isometric on normal elements, then all the conclusions of Theorem 1 follow.

Proof. Let $z \in \mathbf{C}$ and $x \in M_1$ be normal. Then $T(1 + zx) = 1 + T(zx)$ is normal, and hence

$$\|1 + zTx\|_{2n} = \|1 + zx\|_{2n}.$$

Now

$$\|1 + zx\|_{2n}^{2n} = t_1(((1 + zx)^*(1 + zx))^n)$$
$$= \sum_{k=0}^{n} \sum_{j=0}^{n} \binom{n}{k}\binom{n}{j} \bar{z}^k z^j t_1(x^{*k} x^j)$$

and similarly

$$\|1 + zTx\|_{2n}^{2n} = \sum_{k=0}^{n} \sum_{j=0}^{n} \binom{n}{k}\binom{n}{j} \bar{z}^k z^j t_2((Tx)^{*k}(Tx)^j).$$

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Hence, comparing coefficients of $\bar{z}^k z^j$, we find

(1) $t_1(x^{*k}x^j) = t_2((Tx)^{*k}(Tx)^j) \quad j, k = 1, 2, \ldots n.$

Putting $j = k = 1$ in (1), we get

(2) $\|x\|_2 = \|Tx\|_2 \quad \forall x \in M_1$ normal.

Replacing x by $y + zx$, with x, y commuting self-adjoint elements of M_1 and $z \in \mathbf{C}$, we find, since $y + zx$ (and hence also $T(y + zx)$) is normal,

$$t_1((y + zx)^*(y + zx)) = t_2((Ty + zTx)^*(Ty + zTx)).$$

Expanding and comparing coefficients of z, this yields

(3) $t_1(y^*x) = t_2((Ty)^*(Tx)).$

Now (1) with $k = 1, j = 2$ gives

$$t_1(x^*x^2) = t_2((Tx)^*(Tx)^2).$$

Again replacing x by $y + zx$ as above, expanding, and comparing coefficients of z^2, we find

(4) $t_1(y^*x^2) = t_2((Ty)^*(Tx)^2).$

Replace x by x^2 in (3) and compare the result with (4) to get

$$t_2((Ty)^*T(x^2)) = t_2((Ty)^*(Tx)^2)$$

or, with $y = x^2$,

(5) $t_2(T(x^2)^*T(x^2)) = t_2(T(x^2)^*(Tx)^2).$

On the other hand, (1) with $j = k = 2$ gives

$$t_1(x^{*2}x^2) = t_2((Tx)^{*2}(Tx)^2)$$

while (4) with $y = x^2$ gives

$$t_1(x^{*2}x^2) = t_2(T(x^2)^*(Tx)^2) = \overline{t_2(T(x^2)^*(Tx)^2)}$$

(since the left hand side is real)

$$= t_2((Tx)^{*2}T(x^2)).$$

Comparing the last two equalities, we find

(6) $t_2((Tx)^{*2}(Tx)^2) = t_2((Tx)^{*2}T(x^2)).$

Using (5) and (6) we can now conclude that

$$\|T(x^2) - (Tx)^2\|_2{}^2 = t_2(T(x^2)^*T(x^2) - T(x^{2*}(Tx)^2 - (Tx)^{*2}T(x^2) + (Tx)^*{}_2(Tx)^2 = 0$$

and hence

$$T(x^2) = (Tx)^2 \quad \text{for each self-adjoint } x \in M_1.$$

Therefore

$$T(xy + yx) = (TxTy + TyTx) \quad \forall x, y \in M_1 \text{ self-adjoint.}$$

Writing an arbitrary $x \in M_1$ in the form $x = x_1 + ix_2$ with x_j self-adjoint, and using the previous equality, we have

$$T(x^2) = (Tx)^2 \quad \forall x \in M_1$$

and hence

$$T(xy + yx) = TxTy + TyTx \quad \forall x, y \in M_1.$$

Using this and induction, it follows that

(7) $\quad T(x^n) = (Tx)^n \quad \forall x \in M_1, \quad \forall n \in \mathbf{N}.$

Now let $x \in M_1$ be normal, and $n \in \mathbf{N}$. We have

$$\|Tx\|_{2n}{}^{2n} = t_2(((Tx)^*(Tx))^n) = t_2((Tx)^{*n}(Tx)^n)$$

since Tx is normal

$$= t_2(T(x^n)^*T(x_n)) \quad \text{by (7)}$$
$$= \|T(x)^n\|_2{}^2 = \|x_n\|_2{}^2 \quad \text{by (2)}$$
$$= t_1(x^*{}_nx_n) = \|x\|_{2n}{}^{2n}.$$

Since it is known that $\|x\|_\infty = \sup_n \|x\|_n$ (see e.g. [**5**]) it follows that

$$\|Tx\|_\infty = \|x\|_\infty \quad \forall x \in M_1 \text{ normal.}$$

For arbitrary $x = x_1 + ix_2 \in M_1$ with x_j self-adjoint

$$\begin{aligned}\|Tx\|_\infty &= \|Tx_1 + iTx_2\|_\infty \leqq \|Tx_1\|_\infty + \|Tx_2\|_\infty \\ &= \|x_1\|_\infty + \|x_2\|_\infty < \infty\end{aligned}$$

($\|\cdot\|_\infty$ denoting the operator norm). It follows that $T(M_1) \subset M_2$.

We can now, adapting a proof of Kadison [**3**], show that T must be *-linear.

Suppose, to the contrary, that $\exists\, x \in M_1$ self-adjoint with $\|x\|_\infty = 1$ such that $Tx = y + iz$ with $y, z \in M_1$ self-adjoint. We may assume that the spectrum of z contains a $k \in \mathbf{R}$ with $k > 0$ (otherwise consider $-x$). Let $n \in \mathbf{N}$ be such that $(1 + n^{-2})^{1/2} < 1 + k/n$. Now $x + in \in M_1$ is normal, so that

$$\|T(x + in)\|_\infty = \|x + in\|_\infty.$$

We have:

$$\begin{aligned}\|x + in\|_\infty &= (1 + n^2)^{1/2} < n + k \leqq \|n + z\|_\infty \\ &\leqq \|y + i(z + n)\|_\infty = \|T(x + in)\|_\infty = \|x + in\|_\infty\end{aligned}$$

a contradiction.

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Let $y \in M_1$ be such that $y^2 = x^*x$ and $y = y^*$. Then

$$T(x^*x) = T(y^2) = (Ty)^2 \geqq 0$$

since Ty is self-adjoint. This shows that T is positivity preserving.

Now let $x = x_1 + ix_2 \in M_1$ with x_j self-adjoint. We have

$$\begin{aligned}\|Tx\|_2^2 &= t_2(Tx_1^2 + Tx_2^2 + iTx_1Tx_2 - iTx_2Tx_1)\\ &= t_2(Tx_1^2 + Tx_2^2) = \|Tx_1\|_2^2 + \|Tx_2\|_2^2 = \|x_1\|_2^2\\ &\qquad + \|x_2\|_2^2 \quad \text{by (2)}\\ &= \|x\|_2^2.\end{aligned}$$

Thus

$$\|Tx\|_2 = \|x\|_2 \quad \forall x \in M_1.$$

Further, $\forall n \in \mathbf{N}$,

$$\|Tx\|_{2n}^{2n} = t_2(((Tx)^*(Tx))^n) = t_2((Tx)((Tx)^*(Tx))^{n-1}(Tx)^*)$$

which by centrality equals

$$t_2((Tx)((Tx^*)(Tx))^{n-1}(Tx^*)) = t_2(TxT((x^*x)^{n-1}x^*))$$

which since T is a Jordan *-homomorphism (see [**3**]) equals

$$\begin{aligned}&\langle Tx^*, T((x^*x)^{n-1}x^*)\rangle \quad \text{where} \quad \langle a, b\rangle = t_2(a^*b)\\ &= \langle x^*, (x^*x)^{n-1}x^*\rangle\end{aligned}$$

since T is an L_2-isometry

$$= \|x\|_{2n}^{2n}.$$

Thus, $\forall x \in M_1$,

$$\|Tx\|_\infty = \sup_n \|Tx\|_{2n} = \sup_n \|x\|_{2n} = \|x\|_\infty$$

which shows (iii).

The remaining conclusions of the Theorems now follow from results of [**3**] and [**11**] on the structure of Jordan *-homomorphisms between Von Neumann algebras. This concludes the proof of Theorem 3.

3. The results of the previous paragraph do not answer the question raised in the Introduction, because of the restrictions that $T(1) = 1$, that T is *-linear (Theorem 1) and that $p = 2n$ (Theorem 2). However, based on Theorem 2, we can show

THEOREM 3. *Let M be a Von Neumann algebra, t a faithful normal trace on M such that $t(1) = 1$, and (X, m) a finite measure space. For $p > 2$, suppose that*

$$T: L_p(M, t) \to L_p(X, m)$$

is an into linear mapping, isometric on normal elements. Then M is w^*-

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isomorphic to a Von Neumann subalgebra of $L_\infty(X, m)$, and hence is abelian.

Proof. (i) Let $x \in M$ be a (self-adjoint) projection, $y = 1 - x$. We have

$$\begin{aligned}&\|x + y\|_p{}^p + \|x - y\|_p{}^p - 2\|x\|_p{}^p - 2\|y\|_p{}^p \\ &= t(|x + y|^p) + t(|x - y|^p) - 2t(|x|^p) - 2t(|y|^p) \\ &= t(x + y) + t(x + y) - 2t(x) - 2t(y) = 0\end{aligned}$$

since $|x \pm y|^2 = (x \pm y)(x \pm y) = x + y$, since $xy = 0$, so that $|x \pm y| = x + y$ and so, since T is isometric on normal elements,

$$\|Tx + Ty\|_p{}^p + \|Tx - Ty\|_p{}^p - 2\|Tx\|_p{}^p - 2\|Ty\|_p{}^p = 0.$$

Now Tx, $Ty \in L_p(X, m)$. Since $p > 2$, the quantity on the left hand side of the previous equality is well known to be always non-negative, and to vanish if and only if $(Tx)(\omega).\ (Ty)(\omega) = 0$ for almost all $\omega \in X$ (see, e.g. [**6**]). Now for $\omega \in X$,

$$(T1)(\omega) = (Tx)(\omega) + (Ty)(\omega)$$

and thus, if $(Tx)(\omega) \neq 0$, then $(Ty)(\omega) = 0$, so that $(T1)(\omega) = (Tx)(\omega) \neq 0$ and therefore ω belongs to the support of $T1$. Thus if $x \in M$ is a projection, the support of Tx is contained in the support of $T1$. But an arbitrary $x \in M$ may be approximated, in the L_p norm, by a finite linear combination of projections (this is a consequence of the spectral theorem and the fact that the L_p norm is smaller than the operator norm). Hence for each $x \in M$, the support of Tx is contained in the support on $T1$. Restricting attention to that support, we see that there is no loss of generality in assuming that $T1$ is non-zero almost everywhere.

Since $T1$ is an almost everywhere non-zero measurable function on (X, m), we may define the mapping S by

$$Sx = Tx/T1$$

and the measure

$$\mu(A) = ((T1)^*(T1))^{p/2}m(A)$$

defined on measurable subsets of X, and μ will be equivalent to m. Thus $L_\infty(X, \mu) = L_\infty(X, m)$. We have

$$\int|Sx|^p d\mu = \int|Tx|^p dm = \|x\|_p{}^p \quad \text{for normal } x \in M.$$

Thus

$$S: L_p(M, t) \to L_p(X, \mu)$$

is isometric on normal elements, and $S1 = 1$.

Now let $x \in M$ be normal. Then $Sx \in L_p(X, \mu)$ and hence may be regarded as a normal operator affiliated to $L_\infty(X, \mu) = L_\infty(X, m)$. In case $p = 2n$, with $n \in \mathbf{N}$, $n > 2$, Theorem 2 is applicable. If not, we use the following result ([**4**]):

PROPOSITION 4. *Let (M_j, t_j) be two Von Neumann algebras equipped with normal faithful traces t_j such that $t_j(1) = 1$, let $p > 2$, and let $x_j \in L_p(M_j, t_j)$ be normal. If, for small $z \in \mathbf{C}$, we have*

$$\|1 + zx_1\|_p = \|1 + zx_2\|_p$$

then

$$\text{(a)} \quad \|x_1\|_2 = \|x_2\|_2$$

and

$$\text{(b)} \quad \|x_1\|_4 = \|x_2\|_4.$$

Returning to the proof of Theorem 3, if $x \in M$ is normal, then so is $1 + zx$, $\forall z \in \mathbf{C}$. Hence

$$\|1 + zx\|_p = \|1 + zSx\|_p$$

and thus by Proposition 4,

$$\|x\|_4 = \|Sx\|_4.$$

Thus Theorem 2 is again applicable, and shows that S is a Jordan *-homomorphism, and isometric in the operator norm, therefore injective. Since it is also ultra-weakly continuous, it is a w^* isomorphism onto its range, a w^*-subalgebra of $L_\infty(X, m)$. Finally, let $x, y \in M$ be self-adjoint. Then $i(xy - yx) \in M$ is self-adjoint, and we have

$$S((ixy - iyx)^2) = (iSxSy - iSySx)^2$$

which by a well-known property of Jordan homomorphisms equals 0 since $L_\infty(X, m)$ is abelian. Therefore $(ixy - iyx)^2 = 0$, so that $ixy - iyx = 0$, and hence M is abelian. This concludes the proof of Theorem 3.

PROPOSITION 5. *Let (M, t) and (X, m) be as in Theorem 3, let $p \in (1, \infty]$, $p \neq 2$ and let*

$$T: L_p(M, t) \rightarrow L_p(X, m)$$

be an onto linear isometry. Then M is w^*-isomorphic to $L_\infty(X, m)$, and hence is abelian.

Proof. If $p > 2$, this is just Theorem 3, since the mapping S constructed in the proof of that theorem maps M onto $L_\infty(X, m)$, if T is onto. If $p \in [1, 2)$, we use duality to reduce the problem to the case $p > 2$.

It is well-known (see, e.g. [**2**]) that the dual of $L_p(M, t)$ is $L_q(M, t)$, where $1/p + 1/q = 1$, so that $q > 2$. Since T is an onto isometry, the dual map:

$$T^*: L_q(X, m) \to L_q(M, t)$$

is also an onto isometry, and hence so is its inverse

$$(T^*)^{-1}: L_q(M, t) \to L_q(X, m) \quad q > 2.$$

We may thus apply Theorem 3 to $(T^*)^{-1}$.

4. Concluding remarks. Theorem 3 and Proposition 5 show that Non-Commutative L_p spaces do form a class of Banach spaces distinct from classical ones, in the sense that they can never be isometric to subspaces of classical L_p spaces. It is of interest to study the structure of the isometries of these spaces. More precisely, do these isometries preserve the structure of the underlying Von Neumann algebras? The problem here is to relax the restrictions of preservation of the involution and the identity in Theorem 1. This problem will be investigated in a future paper.

C. A. McCarthy [**7**] has proved that in case M is the algebra of all bounded operators on an infinite dimensional Hilbert space, and t is the usual trace, then $L_p(M, t)$ can never be linearly and bicontinuously embedded in an ordinary L_p space. His method is completely different from ours, and does not seem to be readily extendible to the general case, as it is based on explicit calculations of various L_p norms using orthonormal bases of the underlying Hilbert space.

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