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On a Conjecture of S. Stahl

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Abstract. S. Stahl conjectured that the zeros of genus polynomials are real. In this note, we disprove this conjecture.

The reader is referred to [3, 5] for the explanation of all terms not defined here.

By the *genus polynomial* of G, we shall mean the polynomial $g_G(x) = \sum_{i=0}^{\infty} g_i x^i$.

Similarly, $f_G(y) = \sum_{j=1}^{\infty} f_j y^j$ is the *crosscap number polynomial* of G.

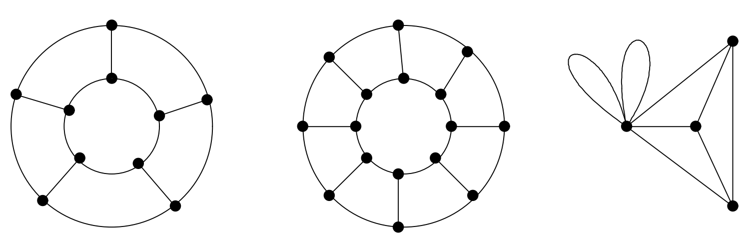
Note that g_i is the number of embeddings of G into the orientable surface with genus i and f_j is the number of embeddings of G into the nonorientable surface N_j $(i = 0, 1, \ldots, j = 1, 2, \ldots)$.

We refer the reader to [1, 3, 5] for the techniques of calculating graph embeddings.

In [5], Stahl posed the following conjecture.

Conjecture The zeros of a genus polynomial $g_G(x)$ are real and negative.

In particular, Stahl considered the H-linear family of graphs obtained by consistently amalgamating additional copies of a graph H, and he verified the conjecture for several infinite families of such graphs. In [4], L. Liu and Y. Wang disproved this conjecture on the basis of [5, Example 6.7]. However, it was pointed out that there is an error in the above example and a new generating matrix and initial vector were provided in [2]. Also see [6]. We also verified the conjecture for graphs with maximum genus 2. However, it is not true for graphs with maximum genus greater than 2. Let G_1 and G_2 be the graphs given in the following figure.



G_1 G_2 G_3

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It is a routine task to compute that the genus polynomials of G_1 and G_2 are

$$g_{G_1}(x) = 2 + 70x + 632x^2 + 320x^3,$$
$$g_{G_2}(x) = 2 + 318x + 5312x^2 + 27520x^3 + 32384x^4.$$

By using Mathematica, approximations of the three zeros of $g_{G_1}(x)$ and the four zeros of $g_{G_2}(x)$ are

$$x_1 = -1.85915,\ x_2 = -0.0579266 - 0.00250235i,$$
$$x_3 = -0.0579266 + 0.00250235i,$$
$$x_1 = -0.605351,\ x_2 = -0.118675 - 0.0168446i,$$
$$x_3 = -0.118675 + 0.0168446i,\ x_4 = -0.00710085.$$

It is obvious that zero is always a root of the crosscap number polynomial $f_G(y)$. Now we give an example of a crosscap number polynomial that has only imaginary roots, except zero. Let G_3 be graph of the above figure. It can be shown that

$$f_{G_3}(y) = 1632y + 9504y^2 + 41424y^3 + 72432y^4 + 53568y^5.$$

By using Mathematica, approximations of the five zeros of $f_{G_3}(y)$ are

$$y_1 = 0,\ y_2 = -0.563995 - 0.367608i,\ y_3 = -0.563995 + 0.367608i,$$
$$y_4 = -0.112081 - 0.233791i,\ y_5 = -0.112081 + 0.233791i.$$

With a computer program, we verified our calculations for $g_{G_1}(x), g_{G_2}(x), f_{G_3}(y)$.

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