IDEALS IN PSEUDO-RINGS

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(Received 5th February 1970)

1. Introduction

The concept of a pseudo-ring was introduced by Patterson (1). Briefly, a pseudo-ring is an algebraic system consisting of an additive abelian group A, a distinguished subgroup A^*, and a multiplication operation $A^* \times A \to A$ under which A^* is a ring and A a left A^*-module. For convenience, we denote the pseudo-ring by $\mathfrak{A} = (A^*, A)$. For the definitions of the various types of ideal, we refer the reader to (1).

We know that if $\mathfrak{M} = (M^*, M)$ is a maximal modular normal right ideal of $\mathfrak{A}$, then M^* is a maximal modular right ideal of the ring A^*. However, given a maximal modular right ideal M^* of A^*, there need not exist a subgroup M of A such that (M^*, M) is a maximal modular normal right ideal of $\mathfrak{A}$. For example, the ring Z of integers has maximal modular right ideals of the form pZ, where p is a prime integer; the pseudo-ring (Z, Q), however, has no proper normal right ideals (Patterson, (2)).

In Section 2 we shall give a necessary and sufficient condition that a maximal modular right ideal M^* of A^* extend to a maximal modular normal right ideal (M^*, M) of $\mathfrak{A} = (A^*, A)$. We shall also show that, if this condition holds, there exists a distinguished subgroup M of A such that (M^*, M) is a maximal modular normal right ideal of $\mathfrak{A}$, and, if (M^*, M') is another such right ideal, then $M' \supseteq M$.

In Section 3 we shall discuss the existence of primitive normal ideals in a pseudo-ring, using the results of Section 2.

In Section 4 we shall give conditions that an element a_* of A may be considered a right multiplier in the pseudo-ring, and we shall show that certain ideals defined in Sections 2 and 3 remain invariant under right multiplication of this type.

Our notation will follow that of (1), except as follows.

A right ideal $\mathfrak{B} = (B^*, B)$ of $\mathfrak{A}$ will be called accessible if $B = B^* + B^*A$, following the notation of (3).

We shall denote the set $\{a - e^*a \mid a \in S\}$ by $(1 - e^*)S$, for any subset S of A, whether A^* possesses a unit element or not.

We shall denote by A_0 the set $(A \setminus (A^* + A^*A)) \cup \{0\}$. It is clear that $(A^* + A^*A) \cap A_0 = \{0\}$, and $A^* + A^*A + A_0 = (A^* + A^*A) \cup A_0 = A$.

2. Maximal Modular Normal Right Ideals

Let $\mathfrak{A} = (A^*, A)$ be a pseudo-ring. We make the following definitions.

Definition. A right ideal B^* of A^* is $\mathfrak{A}$-*extensible* if $A^* \cap (B^*A) \subseteq B^*$.

Definition. A normal right ideal $\mathfrak{M} = (M^*, M)$ of $\mathfrak{A}$ is *quasi-accessible* if there exists an $e^* \in A^*$ and $M = M^* + M^*A + (1-e^*)A$.

We may now state the following result on general right ideals of A^*.

Theorem 2.1. *Let $\mathfrak{A} = (A^*, A)$ be a pseudo-ring and B^* a right ideal of A^*. Then B^* is $\mathfrak{A}$-extensible if and only if there exists a subgroup B of A such that $\mathfrak{B} = (B^*, B)$ is a normal right ideal of $\mathfrak{A}$.*

Proof. Suppose B^* is $\mathfrak{A}$-extensible; consider $B = B^* + B^*A$. Then $\mathfrak{B} = (B^*, B)$ is clearly a right ideal of $\mathfrak{A}$. Also, $B \cap A^* = B^* + ((B^*A) \cap A^*) = B^*$; thus $\mathfrak{B}$ is normal in $\mathfrak{A}$. Conversely, suppose there exists a subgroup B of A such that $\mathfrak{B} = (B^*, B)$ is a normal right ideal of $\mathfrak{A}$. Then $B \supseteq B^*A$; but $B^* = B \cap A^* \supseteq (B^*A) \cap A^*$. Therefore B^* is $\mathfrak{A}$-extensible.

We now consider the case where M^* is a modular right ideal of A^*. We require the following result of ring theory, stated explicitly as a lemma.

Lemma 2.2. *Let A^* be a ring and M^* a maximal right ideal modular with respect to e^*. Then M^* is also modular with respect to f^* if and only if*

$$e^* - f^* \in M^*.$$

Proof. Suppose $e^* - f^* = m^* \in M^*$. Then for all $a^* \in A^*$,

$$a^* - f^*a^* = (a^* - e^*a^*) + m^*a^* \in M^*.$$

Thus M^* is modular with respect to f^*.

Conversely, suppose M^* is modular with respect to f^*, and suppose $(e^* - f^*) \notin M^*$. Then, since $a^* - f^*a^* \in M^*$, and $a^* - e^*a^* \in M^*$, we have $(e^* - f^*)a^* \in M^*$ for all $a^* \in A^*$. Then, if G^* is the additive group generated by $e^* - f^*$, $G^*A^* \subseteq M^*$ and hence $M^* + G^*$ is a right ideal of A^*. As $e^* - f^* \notin M^*$, $M^* \subset M^* + G^* \subseteq A^*$. By the maximality of M^*, $M^* + G^* = A^*$. In particular, $e^* = m^* + g^*$ for some $m^* \in M^*$ and $g^* \in G^*$. Thus $e^*A^* \subseteq m^*A^* + g^*A^* \subseteq M^*$; but $(1 - e^*)A^* \subseteq M^*$. Therefore $M^* = A^*$ which contradicts the maximality of M^*. Hence $e^* - f^* \in M^*$; the proof is now complete.

We may now state the following results concerning quasi-accessible normal right ideals.

Theorem 2.3. *Let $\mathfrak{A} = (A^*, A)$ be a pseudo-ring. Then*

(i) *Every quasi-accessible normal right ideal $\mathfrak{M}$ of $\mathfrak{A}$ is modular;*

(ii) *If $\mathfrak{M}' = (M^*, M')$ is a modular normal right ideal of $\mathfrak{A}$, there exists a subgroup M of M', such that $\mathfrak{M} = (M^*, M)$ is a quasi-accessible normal right ideal of $\mathfrak{A}$;*

https://doi.org/10.1017/S0013091500026924 Published online by Cambridge University Press

(iii) *If* $\mathfrak{M} = (M^*, M)$ *is a quasi-accessible normal right ideal of* $\mathfrak{A}$ *such that* M^* *is a maximal right ideal of* A^*, *then* M *is unique*;

(iv) *If* $\mathfrak{M} = (M^*, M)$ *is a quasi-accessible normal right ideal of* $\mathfrak{A}$, *then* $M = M^*+M^*A+(1-e^*)A_0$.

Proof. (i) If $\mathfrak{M}$ is a quasi-accessible normal right ideal of $\mathfrak{A}$, then

$$M = M^*+M^*A+(1-e^*)A \supseteq (1-e^*)A,$$

i.e. $\mathfrak{M}$ is modular.

(ii) Let $\mathfrak{M}' = (M^*, M')$ be a normal right ideal of $\mathfrak{A}$ modular with respect to $e^* \in A^*$. Then, consider $M = M^*+M^*A+(1-e^*)A$. M is a subgroup of M', and $\mathfrak{M} = (M^*, M)$ is a right ideal of $\mathfrak{A}$.
Also $M^* = M' \cap A^* \supseteq M \cap A^* \supseteq M^*$, i.e. $M^* = M \cap A^*$ and thus $\mathfrak{M}$ is normal. Thus $\mathfrak{M} = (M^*, M)$ is a quasi-accessible normal right ideal of $\mathfrak{A}$.

(iii) Let $\mathfrak{M} = (M^*, M)$ be a quasi-accessible normal right ideal of $\mathfrak{A}$, and let M^* be a maximal right ideal of A^*. Then $M = M^*+M^*A+(1-e^*)A$ for some $e^* \in A^*$. Suppose $M' = M^*+M^*A+(1-f^*)A$ for some $f^* \in A^*$, and $\mathfrak{M}' = (M^*, M')$ is also a quasi-accessible normal right ideal of $\mathfrak{A}$. Then $(1-e^*)A^* \subseteq M \cap A^* = M^*$. Hence M^* is modular with respect to e^*. Similarly, M^* is modular with respect to f^*; by Lemma 2.2, $e^*-f^* = m^* \in M^*$. Therefore $M \subseteq M^*+M^*A+(1-f^*)A+m^*A \subseteq M^*+M^*A+(1-f^*)A = M'$. Similarly $M' \subseteq M$ and hence $M' = M$. Thus M is unique.

(iv) Let $\mathfrak{M} = (M^*, M)$ be a quasi-accessible normal right ideal of $\mathfrak{A}$. $M = M^*+M^*A+(1-e^*)A$ for some $e^* \in A^*$. As in (iii) above, M^* is modular with respect to e^*, i.e. $(1-e^*)A^* \subseteq M^*$. Now $A = A^*+A^*A+A_0$ and thus

$$(1-e^*)A \subseteq M^*+M^*A+(1-e^*)A_0.$$

Therefore, $M = M^*+M^*A+(1-e^*)A_0$.

It is an immediate consequence of Theorem 2.3 (iv) that, if $\mathfrak{A}$ is accessible so that $A^*+A^*A = A$, and therefore $A_0 = \{0\}$, then every quasi-accessible normal right ideal of $\mathfrak{A}$ is accessible. Thus, if M^* is a modular right ideal of A^*, it extends to a quasi-accessible normal right ideal of $\mathfrak{A}$ if and only if (M^*, M^*+M^*A) is normal. Theorem 2.1 shows that this will occur if and only if M^* is $\mathfrak{A}$-extensible. We extend this result to the general case in the following theorem.

Theorem 2.4. *Let* $\mathfrak{A} = (A^*, A)$ *be a pseudo-ring, and let* M^* *be a modular right ideal of* A^*. *Then the following three conditions are equivalent.*

(i) M^* *is* $\mathfrak{A}$-*extensible.*

(ii) *There exists a subgroup* M *of* A *such that* $\mathfrak{M} = (M^*, M)$ *is a quasi-accessible normal right ideal of* $\mathfrak{A}$.

(iii) *There exists a subgroup* M' *of* A *such that* $\mathfrak{M}' = (M^*, M')$ *is a modular normal right ideal of* $\mathfrak{A}$.

Proof. Suppose condition (i) holds, i.e. $A^*\cap(M^*A) \subseteq M^*$. Suppose M^* is modular with respect to $e^* \in A^*$; consider

$$M = M^*+M^*A+(1-e^*)A = M^*+M^*A+(1-e^*)A_0$$

as in the proof of Theorem 2.3 (iv). Then $\mathfrak{M} = (M^*, M)$ is a right ideal of $\mathfrak{A}$. Suppose $m \in M\cap A^*$; then $m = m^*+m'+(1-e^*)a$ for some $m' \in M^*A$, and some $a \in A_0$. Then $a = (m-m^*)+(e^*a-m') \in A^*+A^*A$ as $m \in A^*$. Thus $a \in (A^*+A^*A)\cap A_0 = \{0\}$, i.e. $a = 0$. Therefore $m = m^*+m'$, i.e.

$$m' \in A^*\cap(M^*A) \subseteq M^*.$$

Hence $m \in M^*$ and so $\mathfrak{M}$ is normal in $\mathfrak{A}$. Thus $\mathfrak{M}$ is a quasi-accessible normal right ideal of $\mathfrak{A}$, i.e. condition (ii) holds.

It is an immediate consequence of Theorem 2.3 (i) that condition (ii) implies condition (iii).

Suppose condition (iii) holds. Then, since $\mathfrak{M}'$ is a normal right ideal of $\mathfrak{A}$, $A^*\cap(M^*A) \subseteq M^*$, i.e. M^* is $\mathfrak{A}$-extensible. Thus condition (i) holds. This completes the proof.

Finally, we consider the case where M^* is not only a modular right ideal of A^*, but also maximal in A^*.

Theorem 2.5. *A modular normal right ideal $\mathfrak{M} = (M^*, M)$ of the pseudo-ring $\mathfrak{A}$ is maximal if and only if M^* is maximal in A^*.*

Proof. Suppose M^* is maximal in A^*, and suppose $\mathfrak{N} = (N^*, N)$ is a right ideal of $\mathfrak{A}$ such that $\mathfrak{M} \subseteq \mathfrak{N} \subseteq \mathfrak{A}$ and $M^* \neq N^*$. Then, since N^* is a right ideal of A^*, $N^* = A^*$. In particular $e^* \in N^*$, and thus $e^*A \subseteq N$. But $(1-e^*)A \subseteq M \subseteq N$ and hence $A = N$. Thus $\mathfrak{N} = \mathfrak{A}$ i.e. $\mathfrak{M}$ is maximal in $\mathfrak{A}$. Conversely, suppose M^* is not maximal in A^*; then there exists a right ideal N^* such that $M^*\subset N^*\subset A^*$. Then the right ideal (N^*, A) of $\mathfrak{A}$ contradicts the maximality of $\mathfrak{M}$.

Theorems 2.4 and 2.5 together with Theorem 2.3 (iii) give the following result for maximal modular right ideals of A^*.

Theorem 2.6. *Let $\mathfrak{A} = (A^*, A)$ be a pseudo-ring, and M^* a maximal modular right ideal of A^*. Then the following conditions are equivalent.*

(i) *M^* is $\mathfrak{A}$-extensible.*

(ii) *There exists a unique subgroup M of A such that $\mathfrak{M} = (M^*, M)$ is a maximal quasi-accessible normal right ideal of $\mathfrak{A}$.*

(iii) *There exists a subgroup M' of A such that $\mathfrak{M}' = (M^*, M')$ is a maximal modular normal right ideal of $\mathfrak{A}$.*

It is now straightforward to prove the next theorem.

Theorem 2.7. *The Jacobson radical of $\mathfrak{A}$ is the intersection of the maximal quasi-accessible normal right ideals of $\mathfrak{A}$.*

We now consider the embedding of $\mathfrak{A}$ in a pseudo-ring with identity, as in the case of ring theory.

Let A_1 be the group $A \oplus Z$; let $A_1^* = A^* \oplus Z$. Define multiplication $A_1^* \times A_1 \to A_1$ by

$$(a^* + z_1)(a + z_2) = (a^*a + z_1 a + z_2 a^*) + z_1 z_2$$

for all $a^* \in A^*$, $a \in A$ and all z_1 and z_2 in Z. Then under this multiplication operation $\mathfrak{A}_1 = (A_1^*, A_1)$ is clearly a pseudo-ring and $\mathfrak{A}$ a normal ideal of $\mathfrak{A}_1$. We shall require the following lemma in Section **4**.

Lemma 2.8. *Let $\mathfrak{M}$ be a quasi-accessible normal right ideal of $\mathfrak{A}$. Then $\mathfrak{M} = \mathfrak{A} \cap \mathfrak{M}_1$, where $\mathfrak{M}_1$ is an accessible normal right ideal of $\mathfrak{A}_1$.*

Proof. Let $\mathfrak{M}$ be modular with respect to e^*; let G^* be the additive subgroup of A_1^* generated by $1 - e^*$. Let $M_1^* = M^* + G^*$ and $M_1 = M + G^*$. Then $\mathfrak{M}_1 = (M_1^*, M_1)$ is a normal right ideal of $\mathfrak{A}_1$, and $\mathfrak{M} = \mathfrak{A} \cap \mathfrak{M}_1$. It remains to show that $\mathfrak{M}_1$ is accessible. Let m_1 be any element of M_1; then $m_1 = m + k(1 - e^*)$ where $m \in M$ and $k \in Z$. Now

$$m = m_0^* + \sum_{i=1}^{n} m_i^* a_i + (1 - e^*) a_0,$$

where $m_j^* \in M^*$ and $a_j \in A$ for $j = 0, 1, 2, \ldots, n$. Then

$$m_1 = m_0^* + \sum_{i=1}^{n} m_i^* a_i + (1 - e^*)(a_0 + k) \in M_1^* + M_1^* A_1.$$

Therefore $\mathfrak{M}_1$ is accessible, as required.

3. Primitive Normal Ideals

Let $\mathfrak{A} = (A^*, A)$ be a pseudo-ring. Patterson (**1**) showed that a normal ideal of $\mathfrak{A}$ is primitive if and only if it is of the form $(\mathfrak{M}: A^*)$ where $\mathfrak{M}$ is a maximal modular normal right ideal of $\mathfrak{A}$. We make the following definition.

Definition. A primitive normal ideal $\mathfrak{P}$ of $\mathfrak{A}$ is *semi-accessible* if $\mathfrak{P} = (\mathfrak{M}: A^*)$ where $\mathfrak{M}$ is a maximal quasi-accessible normal right ideal of $\mathfrak{A}$.

Theorem 3.1. *If $\mathfrak{P}' = (P^*, P')$ is a primitive normal ideal of $\mathfrak{A}$, there exists a subgroup P of P' such that $\mathfrak{P} = (P^*, P)$ is a semi-accessible primitive normal ideal of $\mathfrak{A}$.*

Proof. Let $\mathfrak{P}' = (\mathfrak{M}': A)$, where $\mathfrak{M}' = (M^*, M')$ is a maximal modular normal right ideal of $\mathfrak{A}$. Then by Theorem 2.3, there exists a subgroup M of M' such that $\mathfrak{M} = (M^*, M)$ is a quasi-accessible normal right ideal of $\mathfrak{A}$. By Theorem 2.5, $\mathfrak{M}$ is maximal in $\mathfrak{A}$.

Let $P = \{a \in A: A^*a \subseteq M\}$; since M is normal, $P \cap A^* = P^*$ and

$$\mathfrak{P} = (P^*, P) = (\mathfrak{M}: A^*).$$

Thus $\mathfrak{P}$ is a semi-accessible primitive normal ideal of $\mathfrak{A}$. Also, P is a subgroup of P', for $\mathfrak{P} = (\mathfrak{M}: A^*) \subseteq (\mathfrak{M}': A^*) = \mathfrak{P}'$.

The following result is an immediate consequence of Theorem 3.1.

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Theorem 3.2. *The Jacobson radical of $\mathfrak{A}$ is the intersection of the semi-accessible primitive normal ideals of $\mathfrak{A}$.*

Using the results of Section 2, we may now give a condition that a primitive ideal P^* of A^* extend to a primitive normal ideal $\mathfrak{P} = (P^*, P)$ of $\mathfrak{A}$. We use the fact that $P^* = (M^*: A^*) = \{a^* \in A^*: A^*a^* \subseteq P^*\}$, where M^* is a maximal modular right ideal of A^*.

Theorem 3.3. *Let $\mathfrak{A} = (A^*, A)$ be a pseudo-ring, and P^* a primitive ideal of A^*. Then the following conditions are equivalent.*

(i) *$P^* = (M^*: A^*)$ where M^* is an $\mathfrak{A}$-extensible maximal modular right ideal of A^*.*

(ii) *There exists a subgroup P of A such that $\mathfrak{P} = (P^*, P)$ is a semi-accessible primitive normal ideal of $\mathfrak{A}$.*

(iii) *There exists a subgroup P' of A such that $\mathfrak{P}' = (P^*, P')$ is a primitive normal ideal of $\mathfrak{A}$.*

Proof. Suppose condition (i) holds; then, by Theorem 2.6, there exists a subgroup M of A such that $\mathfrak{M} = (M^*, M)$ is a maximal quasi-accessible normal right ideal of $\mathfrak{A}$. Then $(\mathfrak{M}: A^*)$ is a semi-accessible primitive normal ideal of $\mathfrak{A}$. Let $P = \{a \in A: A^*a \subseteq M\}$; then, since $\mathfrak{M}$ is normal in $\mathfrak{A}$, $P \cap A^* = P^*$. Thus $\mathfrak{P} = (P^*, P) = (\mathfrak{M}: A^*)$, as required.

Clearly condition (ii) implies condition (iii).

Suppose condition (iii) holds; then by a result of Patterson (**1**), $\mathfrak{P}' = (\mathfrak{M}': A^*)$ where $\mathfrak{M}' = (M^*, M')$ is a maximal modular normal right ideal of $\mathfrak{A}$. By Theorem 2.6, M^* is an $\mathfrak{A}$-extensible maximal modular right ideal of A^*. Also, $P^* = (M^*: A^*)$; thus condition (i) holds.

4. Right Multipliers

In (**1**), Patterson remarked that it would be possible to study a two-sided pseudo-ring, consisting of a bimodule containing both its underlying rings. One advantage of this would be that we may multiply elements of the bimodule on the right. It is possible, however, to consider multiplication on the right in the usual (left) pseudo-rings.

In this section we shall give conditions for an element of a pseudo-ring to be regarded as a right multiplier, and we shall show that both quasi-accessible normal right ideals and semi-accessible primitive normal ideals are invariant under right multiplication as defined.

Let $\mathfrak{A} = (A^*, A)$ be a pseudo-ring. An element $a_* \in A$ will be called a *right multiplier* in $\mathfrak{A}$ if a product $a \circ a_*$ is defined and belongs to A for every $a \in A$, and the following conditions are satisfied.

(i) $a^* \circ a_* = a^*a_*$ for all $a^* \in A^*$.

(ii) $a^*(a \circ a_*) = (a^*a) \circ a_*$ for all $a^* \in A^*$ and all $a \in A$.

(iii) $(a_1 + a_2) \circ a_* = (a_1 \circ a_*) + (a_2 \circ a_*)$ for all a_1 and a_2 in A.

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It should be noted that these conditions are not sufficient to ensure that multiplication on the right by a single element extends to multiplication on the right by elements of a ring, to form a two-sided pseudo-ring; the following example illustrates this.

Example 1. Let Z^* be a copy of the ring of integers, with multiplicative identity 1^*; and let Z_1, Z_2 be copies of the additive group of integers, generated by 1_1 and 1_2 respectively. Let Z be the group $Z^* \oplus Z_1 \oplus Z_2$. Then $\mathfrak{Z} = (Z^*, Z)$ is a pseudo-ring under multiplication defined by $1^* z = z$ for all $z \in Z$.

Define $1^* \circ 1_1 = 1_1$; $1_1 \circ 1_1 = 1_1$; $1_2 \circ 1_1 = 2_2$. Then, under this law of composition, 1_1 is a right multiplier. Suppose that this law of composition may be extended to form the multiplication operation of a ring Z_* containing 1_1. Then Z is not a right Z_*-module, because

$$1_2 \circ (1_1 \circ 1_1) = 2_2, \qquad (1_2 \circ 1_1) \circ 1_1 = 4_2.$$

For the purpose of this paper, however, the given conditions are sufficient. Also, the next example shows that, even where right multiplication may be extended to form a bimodule, primitive normal ideals and modular normal right ideals in general are not invariant under right multiplication.

Example 2. Let $\mathfrak{Z}$ be the pseudo-ring defined in Example 1. Define $1^* \circ 1_1 = 1_1$; $1_1 \circ 1_1 = 1_1$; $1_2 \circ 1_1 = 1_1$. Then, under this law of composition 1_1 is a right multiplier in $\mathfrak{Z}$. Also since $1_1 \circ 1_1 = 1_1$, the law of composition extends to define a multiplication operation on Z_1, under which Z_1 is a ring isomorphic to the ring of integers. Then Z is a bimodule over Z^* and Z_1.

Let p be any prime integer; then $\mathfrak{P} = (pZ^*, pZ^* \oplus pZ_1 \oplus Z_2)$ is a maximal normal right ideal of $\mathfrak{Z}$, modular with respect to 1^*. $\mathfrak{P}$ is also a primitive normal ideal of $\mathfrak{Z}$, being $(\mathfrak{P}: Z^*)$. However, $\mathfrak{P}$ is not invariant under right multiplication, as $Z_2 \circ 1_1 = Z_1 \nsubseteq pZ^* \oplus pZ_1 \oplus Z_2$.

We now prove that every quasi-accessible normal right ideal and every semi-accessible primitive normal ideal is invariant under right multiplication.

Theorem 4.1. *Let $\mathfrak{A} = (A^*, A)$ be a pseudo-ring, $\mathfrak{B} = (B^*, B)$ an accessible right ideal of $\mathfrak{A}$, and $a_* \in A$ a right multiplier in $\mathfrak{A}$. Then $B \circ a_* \subseteq B$.*

Proof. Let $b \in B = B^* + B^* A$. Then $b = b_0^* + \sum_{i=1}^{n} b_i^* a_i$ where $b_j^* \in B^*$, $j = 0, 1, 2, \ldots, n$ and $a_j \in A$, $j = 1, 2, \ldots, n$. Using the rules for right multipliers,

$$\begin{aligned} b \circ a_* &= (b_0^* \circ a_*) + \sum_{i=1}^{n} ((b_i^* a_i) \circ a_*) \\ &= b_0^* a_* + \sum_{i=1}^{n} (b_i^* (a_i \circ a_*)) \in B^* A \subseteq B. \end{aligned}$$

Therefore $B \circ a_* \subseteq B$.

Theorem 4.2. *Let* $\mathfrak{A} = (A^*, A)$ *be a pseudo-ring,* $\mathfrak{M} = (M^*, M)$ *a quasi-accessible normal right ideal of* $\mathfrak{A}$, *and* $a_* \in A$ *a right multiplier in* $\mathfrak{A}$. *Then* $M \circ a_* \subseteq M$.

Proof. Embed $\mathfrak{A}$ in the pseudo-ring $\mathfrak{A}_1$ with identity as in Section **2**. Lemma 2.8 shows that $\mathfrak{M} = \mathfrak{A} \cap \mathfrak{M}_1$ where $\mathfrak{M}_1$ is an accessible normal right ideal of $\mathfrak{A}_1$. Define $(a+z) \circ a_* = (a \circ a_*) + za_*$ for all $a \in A$ and all $z \in Z$. Then it is easy to show that, under this law of composition, a_* is a right multiplier in $\mathfrak{A}_1$. By Theorem 4.1, $M_1 \circ a_* \subseteq M_1$; hence $M \circ a_* \subseteq M_1$. But $M \circ a_* \subseteq A \circ a_* \subseteq A$. Therefore $M \circ a_* \subseteq M_1 \cap A = M$, as required.

Theorem 4.3. *Let* $\mathfrak{A} = (A^*, A)$ *be a pseudo-ring,* $\mathfrak{P} = (P^*, P)$ *a semi-accessible primitive normal ideal of* $\mathfrak{A}$, *and* $a_* \in A$ *a right multiplier in* $\mathfrak{A}$. *Then* $P \circ a_* \subseteq P$.

Proof. $\mathfrak{P} = (\mathfrak{M}\colon A^*)$ where $\mathfrak{M}$ is a maximal quasi-accessible normal right ideal of $\mathfrak{A}$. Let $p \in P$, and $a^* \in A^*$. Then $a^*(p \circ a_*) = (a^*p) \circ a_* \in M \circ a_*$. Hence, using Theorem 4.2,

$$A^*(p \circ a_*) \subseteq M \circ a_* \subseteq M.$$

Thus $p \circ a_* \in \{a \in A \mid A^*a \subseteq M\} = P$. Therefore $P \circ a_* \subseteq P$, as required.

Thus, primitive normal ideals and modular normal right ideals in general need not be invariant under right multiplication; but semi-accessible primitive normal ideals and quasi-accessible normal right ideals, as defined in this paper, are invariant under right multiplication.

Acknowledgement

The major portion of this research work was carried out at the University of Dundee, and was financed by a Research Scholarship from that University. I wish to thank my supervisor, Dr. A. D. Sands, for his advice and encouragement during the preparation of this paper.

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