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ON GENERALIZED NÖRLUND METHODS OF SUMMABILITY II

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The object of this paper is to establish some inclusion relations between two generalized Nörlund methods of summability. Our results are generalizations of Das's Theorem and one of them is also a generalization of a theorem of the author. They include the well-known fact that the Cesàro method (C, α) is weaker than (C, β) for $\beta > \alpha > 0$.

1. Introduction

The object of this paper is to prove Theorems 1 and 2 below which are generalizations of Das's Theorem ([1], Theorem 1), in particular Theorem 1 is also a generalization of a theorem of the author ([4], Theorem 1). They are proved in §3, and in §2 we state some preliminary lemmas.

Let $p = \{p_n\}$, $\alpha = \{\alpha_n\}$ be given sequences of real numbers such that

$$(p * \alpha)_n = \sum_{\nu=0}^{n} p_{n-\nu}\alpha_\nu \neq 0 \text{ for all } n .$$

Given a series $\sum a_n$ with its partial sum s_n , if $t_n^{p,\alpha} \to s$ as $n \to \infty$, where

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$$t_n^{p,\alpha} = \left(1/(p * \alpha)_n\right) \sum_{\nu=0}^{n} p_{n-\nu}\alpha_\nu s_\nu ,$$

then the series $\sum a_n$ is said to be summable (N, p, α) to s and we write $\sum a_n = s(\mathrm{N}, p, \alpha)$. Necessary and sufficient conditions for the regularity of the (N, p, α) method are, for each $\rho \geq 0$, $p_{n-\rho} = o\left((p * \alpha)_n\right)$, and $\sum_{\nu=0}^{n} |p_{n-\nu}\alpha_\nu| = O\left((p * \alpha)_n\right)$. The method (N, p, α) reduces to the Nörlund method (N, p) when $\alpha_n = 1$, to the method $(\overline{\mathrm{N}}, \alpha)$ when $p_n = 1$.

Let A and B be two summability methods. If every series summable (A) to a finite sum is also summable (B) to the same sum, we write $\mathrm{A} \subseteqq \mathrm{B}$, and in addition if $\sum a_n = \pm\infty$ (A) implies $\sum a_n = \pm\infty$ (B) , we shall write B t.s. A (see [4]). We define recursively the two difference operators Δ^k and ∇^k on a sequence $\{a_n\}$ by $\Delta^0 a_n = \nabla^0 a_n = a_n$, $\Delta^1 a_n = \Delta a_n = a_n - a_{n+1}$, $\nabla^1 a_n = \nabla a_n = a_n - a_{n-1}$, $\Delta^k a_n = \Delta\left\{\Delta^{k-1} a_n\right\}$ and $\nabla^k a_n = \nabla\left\{\nabla^{k-1} a_n\right\}$ for $k = 1, 2, \ldots$.

In the following theorems we suppose that $p_n > 0$, $q_n > 0$, $\alpha_n > 0$ and $\beta_n > 0$ for all n .

THEOREM 1. *Let* $\{p_n\}$, $\{q_n\}$, $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be such that for some integer* $k \geq 0$,

(i) $\nabla^s p_n > 0$ $(s = 0, 1, \ldots, k)$, $\nabla^{k+1} p_n \leq 0$ $(n \geq 1)$,
$\nabla^k p_{n+2}/\nabla^k p_{n+1} \geq \nabla^k p_{n+1}/\nabla^k p_n$;

(ii) $\nabla^s q_n > 0$ $(s = 0, 1, \ldots, k)$, $\nabla^k q_{n+1}/\nabla^k q_n \geq \nabla^k p_{n+1}/\nabla^k p_n$;

(iii) $\Delta^t(\beta_n/\alpha_n) \geq 0$ $(t = 1, 2, \ldots, k+1)$;

(iv) (N, q, β) *is regular.*

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Then (N, q, β) t.s. (N, p, α) . *If* $\alpha_n = \beta_n$ *for all* n , $\nabla^s p_n > 0$ *and* $\nabla^s q_n > 0$ ($s = 0, 1, \ldots, k$) *in (i) and (ii) may be* $\nabla^k p_n > 0$ *and* $\nabla^k q_n > 0$ *only*.

The case in which $k = 0$ is the author's result ([4], Theorem 1) and so involves Das ([1], Theorem 1, Case A). In our theorem we can obtain the following well-known result, if we take $p_n = A_n^{\gamma-1}$, $q_n = A_n^{\delta-1}$ and $\alpha_n = \beta_n = 1$.

COROLLARY. (C, δ) t.s. (C, γ) *for* $\delta > \gamma > 0$.

In the following result the case in which $k = 0$ is also due to Das ([1], Theorem 1, Case B).

THEOREM 2. *Let* $\{p_n\}$, $\{q_n\}$, $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be such that for some integer* $k \geq 0$, *the conditions (i) and (ii) of Theorem* 1 *hold and*

(iii) $\Delta^t(\beta_n/\alpha_n) \leq 0$ ($t = 1, 2, \ldots, k+1$) ,

$$\beta_n(q * \alpha)_n/\alpha_n(q * \beta)_n = O(1) ;$$

(iv) (N, q, α) *is regular*.

Then (N, p, α) $\subseteqq$ (N, q, β) . *If* $\alpha_n = \beta_n$ *for all* n , $\nabla^s p_n > 0$ *and* $\nabla^s q_n > 0$ ($s = 0, 1, \ldots, k$) *in (i) and (ii) may be* $\nabla^k p_n > 0$ *and* $\nabla^k q_n > 0$ *only*.

REMARK. In our theorems when $k \geq 1$ we can not obtain an inclusion relation for (N, p, α) $\subseteqq$ ($\overline{\mathrm{N}}$, β) directly since $\{q_n\}$ depends on $\{p_n\}$ with the integer k . In fact the above relation does not hold when $\alpha_n = 1$ and $p_n = A_n^{\delta-1}$ ($\delta > 1$) which satisfies the condition *(i)* with $k = [\delta]$.

2. Preliminary lemmas

We introduce several definitions. For given sequences $\{p_n\}$ and

$\{q_n\}$, let

$$(2.1)\qquad \{c_n\} : (c * p)_n = 1\ (n = 0),\ = 0\ (n \geq 1)\ ,$$

$$(2.2)\qquad \{k_n\} : (k * p)_n = q_n \quad (n \geq 0)\ .$$

It follows from these definitions that

$$(2.3)\qquad k_n = (c * q)_n \quad \text{for all} \quad n\ .$$

LEMMA 1. *Let* $\alpha_n \neq 0$ *for all* n. *Then the necessary and sufficient conditions that* $(\mathrm{N}, p, \alpha) \subseteqq (\mathrm{N}, q, \beta)$ *holds are*

$$(2.4)\qquad \sum_{\rho=0}^{n} \left|(p * \alpha)_\rho K_\rho^n\right| = O\big((q * \beta)_n\big)\ ,$$

$$(2.5)\qquad K_\rho^n = o\big((q * \beta)_n\big) \quad \textit{as} \quad n \to \infty\ , \textit{ for each} \quad \rho \geq 0\ ,$$

where

$$(2.6)\qquad K_\rho^n = \sum_{\nu=\rho}^{n} q_{n-\nu} c_{\nu-\rho} \beta_\nu / \alpha_\nu \quad \textit{for} \quad n \geq \rho \geq 0\ .$$

This lemma is due to Das ([*1*], Lemma 1). It easily follows from (2.1) and (2.6) that

$$(2.7)\qquad \sum_{\rho=0}^{n} (p * \alpha)_\rho K_\rho^n = (q * \beta)_n \quad \text{for all} \quad n\ .$$

LEMMA 2 ([*4*], Lemma 3). *Let* $p_n > 0$, $q_n > 0$, $\alpha_n > 0$ *and* $\beta_n > 0$ *for all* n. *Then necessary and sufficient conditions that* (N, q, β) t.s. (N, p, α) *are* (2.5) *and for some integer* $N \geq 0$,

$$(2.8)\qquad K_\rho^n \geq 0 \quad \textit{for} \quad n \geq \rho \geq N\ .$$

Now given any sequence $\{a_n\}$, let $a_n^{(m)} = \sum_{\nu=0}^{n} a_\nu^{(m-1)}$ for integer $m \geq 1$.

LEMMA 3. *Let* $\{p_n\}$ *be such that for some integer* $k \geq 0$,

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$\nabla^k p_n > 0$, $\nabla^k p_{n+2} \nabla^k p_n \ge \left\{\nabla^k p_{n+1}\right\}^2$ *hold. Then* $c_n^{(k)} \le 0$ *for* $n \ge 1$, *and* $c_0^{(k)} > 0$. *Furthermore if* $\nabla^{k+1} p_n \le 0$, *then* $c_n^{(k+1)} \ge 0$ *for all* n .

This lemma is a generalization of Kulza ([2], Theorem 22) which is our case $k = 0$, and is proved in a similar way, replacing p_n by $\nabla^k p_n$.

LEMMA 4. *Let* $\{p_n\}$ *and* $\{q_n\}$ *be such that for some integer* $k \ge 0$, *the condition of Lemma* 3 *holds and* $\nabla^k q_n > 0$,

$$\nabla^k q_{n+1} \nabla^k p_n \ge \nabla^k q_n \nabla^k p_{n+1} .$$

Then

$$0 \le k_n \le \sum_{\nu=0}^{m} \left\{\nabla^k_n q_{n-\nu}\right\} c_\nu^{(k)} \quad \textit{for} \quad m = 0, 1, \ldots, n .$$

Proof. By (2.3), using Abel's transformation, we have

$$\begin{aligned} k_n &= \sum_{\nu=0}^{n} \left\{\nabla^k_n q_{n-\nu}\right\} c_\nu^{(k)} \\ &= \left\{\nabla^k q_n\right\} \sum_{\nu=0}^{n} \left\{\nabla^k q_{n-\nu} / \nabla^k q_n\right\} c_\nu^{(k)} \\ &\ge \left\{\nabla^k q_n / \nabla^k p_n\right\} \sum_{\nu=0}^{n} \nabla^k p_{n-\nu} c_\nu^{(k)} \\ &= 0 \quad \text{for all} \quad n . \end{aligned}$$

Since $c_\nu^{(k)} \le 0$ for $n \ge 1$, it follows that for $m = 0, 1, \ldots, n$,

$$k_n \le \sum_{\nu=0}^{m} \left\{\nabla^k q_{n-\nu}\right\} c_\nu^{(k)} ,$$

and so our lemma is proved.

It is worth noting that from this result we see that an inclusion relation $(N, p) \subseteq (N, q)$ holds when $\{p_n\}$ and $\{q_n\}$ satisfy the conditions that (N, q) is regular and $\nabla^{k+1} p_n \le 0$ with the conditions of Lemma 4. The same kind of inclusion relation for the absolute Nörlund

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summability is discussed by Kishore ([3], Theorem 2).

3. Proofs of Theorems 1 and 2

3.1. Proof of Theorem 1. By Lemma 2 we may show that the conditions (2.5) and (2.8) hold. To show (2.8) holds, using repeated application of Abel's transformation to (2.6), we will have

$$
\begin{aligned}
(3.1)\quad K_\rho^n &= \sum_{\nu=0}^{n-\rho} \left(\beta_{\nu+\rho}/\alpha_{\nu+\rho}\right) q_{n-\rho-\nu} c_\nu \\
&= \sum_{\nu=0}^{n-\rho} \left\{\Delta\left(\beta_{\nu+\rho}/\alpha_{\nu+\rho}\right)\right\} q_{n-\rho-\nu} c_\nu^{(1)} \\
&\qquad + \sum_{\nu=0}^{n-\rho} \left(\beta_{\nu+\rho+1}/\alpha_{\nu+\rho+1}\right)\left(\nabla_n q_{n-\rho-\nu}\right) c_\nu^{(1)} \\
&= \dots \\
&= \sum_{r_1=1}^{2} \sum_{r_2=1}^{2} \cdots \sum_{r_k=1}^{2} I_\rho^n\left(r_1, r_2, \dots, r_k\right)
\end{aligned}
$$

where

$$
I_\rho^n\left(r_1, r_2, \dots, r_k\right) = \sum_{\nu=0}^{n-\rho} \left\{\Delta^{s_1}\left(\beta_{\nu+\rho+s_2}/\alpha_{\nu+\rho+s_2}\right)\right\}\left\{\nabla_n^{s_2} q_{n-\rho-\nu}\right\} c_\nu^{(k)} \;;
$$

here let s_1 be the number of occurrences of the digit 1 in the set $\{r_n : 1 \le n \le k\}$ and $s_2 = k - s_1$. Now if $s_2 \ne k$, then $\nabla^{s_2+1} q_n > 0$. Hence by *(ii)* and *(iii)* it follows from Lemma 3 that

$$
\begin{aligned}
I_\rho^n\left(r_1, r_2, \dots, r_k\right) &\ge \left\{\Delta^{s_1}\left(\beta_{\rho+s_2}/\alpha_{\rho+s_2}\right)\right\}\left\{\nabla^{s_2} q_{n-\rho}\right\} c_{n-\rho}^{(k+1)} \\
&\ge 0 .
\end{aligned}
$$

On the other hand when $s_2 = k$, $s_1 = 0$. We also have

$$
\begin{aligned}
I_\rho^n(2, 2, \dots, 2) &= \sum_{\nu=0}^{n-\rho} \left(\beta_{\nu+\rho+k}/\alpha_{\nu+\rho+k}\right)\left\{\nabla^k q_{n-\rho-\nu}\right\} c_\nu^{(k)} \\
&\ge \left(\beta_{\rho+k}/\alpha_{\rho+k}\right) k_{n-\rho} \\
&\ge 0 \text{ , by Lemma 3.}
\end{aligned}
$$

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Hence it follows from (3.1) that $K_\rho^n \ge 0$ for $n \ge \rho \ge 0$.

Next to show that (2.5) holds we use the above notation. Since $c_n^{(k)} \le 0$ for $n \ge 1$, we have

$$I_\rho^n(r_1, r_2, \ldots, r_k) \le \left\{\Delta^{s_1}\left(\beta_{\rho+s_2}/\alpha_{\rho+s_2}\right)\right\}\left\{\nabla^{s_2} q_{n-\rho}\right\} c_0 \; .$$

Hence we have by *(iv)*, for each fixed $\rho \ge 0$,

$$I_\rho^n(r_1, r_2, \ldots, r_k) = o\big((q * \beta)_n\big) \quad \text{as} \quad n \to \infty \; ,$$

and so it follows from (3.1) that for each fixed $\rho \ge 0$,

$$K_\rho^n = o\big((q * \beta)_n\big) \quad \text{as} \quad n \to \infty \; .$$

Finally, if $\alpha_n = \beta_n$ for all n , then $K_\rho^n = k_{n-\rho}$, and so the required result follows from Lemma 4.

This completes the proof of our theorem.

3.2. Proof of Theorem 2 . We first prove (2.4) in Lemma 1. By repreated application of Abel's transformation to (2.6), we will have

$$\begin{aligned} K_\rho^n &= \sum_{\nu=0}^{n-\rho} \left(\beta_{\nu+\rho}/\alpha_{\nu+\rho}\right) q_{n-\rho-\nu} c_\nu \\ &= \sum_{\nu=0}^{n-\rho} \left(\beta_{\nu+\rho}/\alpha_{\nu+\rho}\right)\left(\nabla_n q_{n-\rho-\nu}\right) c_\nu^{(1)} + \sum_{\nu=0}^{n-\rho-1} \left(\Delta\left(\beta_{\nu+\rho}/\alpha_{\nu+\rho}\right)\right) q_{n-\rho-\nu-1} c_\nu^{(1)} \\ &= \ldots \\ &= \sum_{r_1=1}^{2} \sum_{r_2=1}^{2} \cdots \sum_{r_k=1}^{2} J_\rho^n(r_1, r_2, \ldots, r_k) \; , \end{aligned}$$

where

$$J_\rho^n(r_1, r_2, \ldots, r_k) = \sum_{\nu=0}^{n-\rho-s_1} \left\{\Delta^{s_1}\left(\beta_{\nu+\rho}/\alpha_{\nu+\rho}\right)\right\}\left\{\nabla^{s_2} q_{n-\rho-\nu-s_1}\right\} c_\nu^{(k)} \; ;$$

here s_1 and s_2 are denoted in the same way of the proof of Theorem 1.

Furthermore let us divide the factor $J_\rho^n(2, 2, \ldots, 2)$ in the following

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$$\mathcal{J}^n_\rho(2,\ 2,\ \ldots,\ 2) = \sum_{\nu=0}^{n-\rho} \left(\beta_{\nu+\rho}/\alpha_{\nu+\rho}\right)\left\{\nabla^k q_{n-\rho-\nu}\right\}c_\nu^{(k)}$$
$$= \sum_{\nu=0}^{n-\rho-1} \left(\Delta\left(\beta_{\nu+\rho}/\alpha_{\nu+\rho}\right)\right) \sum_{i=0}^{\nu} \left\{\nabla^k q_{n-\rho-i}\right\}c_i^{(k)} + \left(\beta_n/\alpha_n\right)k_{n-\rho}$$
$$= \mathcal{J}^n_\rho(2,\ 2,\ \ldots,\ 2;\ 1) + \mathcal{J}^n_\rho(2,\ 2,\ \ldots,\ 2;\ 2)\ .$$

Now since $c_n^{(k+1)} \geq 0$ for all n and from *(iii)* it follows that $\mathcal{J}^n_\rho(2,\ 2,\ \ldots,\ 2;\ 2) \geq 0$ and others are negative. So by (2.7),

$$\sum_{\rho=0}^{n} \left|(p * \alpha)_\rho K^n_\rho\right| \leq 2 \sum_{\rho=0}^{n} (p * \alpha)_\rho \mathcal{J}^n_\rho(2,\ \ldots,\ 2;\ 2) - (q * \beta)_n$$
$$\leq 2\ \left(\beta_n/\alpha_n\right)\left\{\sum_{\rho=0}^{n} (p * \alpha)_\rho k_{n-\rho}\right\}$$
$$= 2\left(\beta_n/\alpha_n\right)(q * \alpha)_n\ .$$

Therefore (2.4) follows from *(iii)*.

To prove (2.5) we also use the above notations:

$$(3.2) \qquad \sum \mathcal{J}^n_\rho\left(r_1,\ r_2,\ \ldots,\ r_k\right) \leq K^n_\rho \leq \mathcal{J}^n_\rho(2,\ 2,\ \ldots,\ 2;\ 2)\ ;$$

$\left(r_1,\ r_2,\ \ldots,\ r_k\right) \neq (2,\ 2,\ \ldots,\ 2;\ 2)$.

Now since $c_n^{(k)} \leq 0$ for $n \geq 1$, we have

$$\mathcal{J}^n_\rho(2,\ \ldots,\ 2;\ 2) \leq \left(\beta_n/\alpha_n\right)\left\{\nabla^k q_{n-\rho}\right\}c_0$$
$$= \left(\beta_n(q * \alpha)_n/\alpha_n(q * \beta)_n\right)\left\{\nabla^k q_{n-\rho}/(q * \alpha)_n\right\}c_0(q * \beta)_n\ .$$

Hence it follows from *(iii)* and *(iv)* that $\mathcal{J}^n_\rho(2,\ \ldots,\ 2;\ 2) = o\left((q * \beta)_n\right)$ as $n \to \infty$, for each $\rho \geq 0$. Similarly we get

$$\mathcal{J}^n_\rho(2,\ \ldots,\ 2;\ 1) \geq \sum_{\nu=0}^{n-\rho-1} \left(\Delta\left(\beta_{\nu+\rho}/\alpha_{\nu+\rho}\right)\right)\left\{\nabla^k q_{n-\rho}\right\}c_0$$
$$\geq \left(-\left(\beta_n/\alpha_n\right)\right)\left\{\nabla^k q_{n-\rho}\right\}c_0\ ,$$

and so $\mathcal{J}^n_\rho(2,\ \ldots,\ 2;\ 1) = o\left((q * \beta)_n\right)$ as $n \to \infty$, for each $\rho \geq 0$.

Finally when $(r_1, r_2, \ldots, r_k) \neq (2, 2, \ldots, 2)$, we have by Lemma 3,

$$\begin{aligned} J_\rho^n(r_1, r_2, \ldots, r_k) &= \sum_{\nu=0}^{n-\rho-s_1} \left\{\Delta^{s_1}(\beta_{\nu+\rho}/\alpha_{\nu+\rho})\right\}\left\{\nabla_n^{s_2} q_{n-\nu-s_1}\right\} c_\nu^{(k+1)} \\ &\geq \left\{\nabla^{s_2} q_{n-s_1}\right\} c_0 \sum_{\nu=0}^{n-\rho-s_1} \left\{\Delta^{s_1}(\beta_{\nu+\rho}/\alpha_{\nu+\rho})\right\} \\ &\geq c_0 \left\{\nabla^{s_2} q_{n-s_1}\right\}\left\{\Delta^{s_1-1}(\beta_\rho/\alpha_\rho)\right\} , \end{aligned}$$

and hence by *(iv)*, $J_\rho^n(r_1, r_2, \ldots, r_k) = o\big((q * \beta)_n\big)$ as $n \to \infty$, for each $\rho \geq 0$. Therefore (2.5) follows from (3.2).

Thus the proof of Theorem 2 is completed.

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