On the Relation between Pincherle's Polynomials and the Hypergeometric Function.

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1. The Pincherle polynomials are defined * as the coefficients in the expansion of $\{1 - 3tx + t^3\}^{-\frac{1}{2}}$ in ascending powers of t. If the coefficient of t^n be denoted by $P_n(x)$, then the polynomials satisfy the difference equation

$$2(n+1)P_{n+1}(x) - 3(2n+1)xP_n(x) + (2n-1)P_{n-2}(x) = 0 \quad \ldots\ldots(1)$$

and $P_n(x)$ satisfies the differential equation

$$4(4x^3-1)\frac{d^3y}{dx^3} + 96x^2\frac{d^2y}{dx^2} - (12n^2+24n-91)x\frac{dy}{dx} - n(2n+3)(2n+9)y = 0 \quad \ldots\ldots(2)$$

Dr Humbert † has defined a more general type of polynomial from the relation

$$\{1 - 3tx + t^3\}^{-\nu} = \sum_{n=0}^{\infty} t^n P_n^\nu(x),$$

and has shown that these polynomials satisfy the difference equation

$$(n+1)P_{n+1}^\nu(x) - 3x(n+\nu)P_n^\nu(x) + (n+3\nu-2)P_{n-2}^\nu(x) = 0 \ldots\ldots(3)$$

and that $P_n^\nu(x)$ satisfies the differential equation

$$(4x^3-1)\frac{d^3y}{dx^3} + 6x^2(2\nu+3)\frac{d^2y}{dx^2} - x\{3n^2+3n(2\nu+1) - (3\nu+2)(3\nu+5)\}\frac{dy}{dx} - n(n+3\nu)(n+3\nu+3)y = 0 \ldots\ldots(4)$$

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* S. Pincherle, *Mem. della R. Accad. di Bologna*, S. 5, T. I. (1890), p. 337

† Humbert, *Proc. Edin. Math. Soc.*, (1921).

As is to be expected from their definitions, these polynomials have properties which are closely analagous to those of Legendre's and Gegenbauer's polynomials. The object of this present note is to exhibit this similarity by obtaining expressions for Pincherle's and Humbert's polynomials in terms of the hypergeometric function.

2. The hypergeometric function of the second order, defined as

$$F(\alpha, \beta, \gamma\,;\ \delta, \epsilon\,;\ x) = 1 + \frac{\alpha\,.\,\beta\,.\,\gamma\,.}{1\,!\ \delta\,.\,\epsilon}\,x + \frac{\alpha(\alpha+1)\,.\,\beta(\beta+1)\,.\,\gamma(\gamma+1)}{2\,!\ \delta(\delta+1)\,.\,\epsilon(\epsilon+1)}\,x^2$$
$$+\frac{\alpha(\alpha+1)(\alpha+2)\,.\,\beta(\beta+1)(\beta+2)\,.\,\gamma(\gamma+1)(\gamma+2)}{3\,!\ \delta(\delta+1)(\delta+2)\,.\,\epsilon(\epsilon+1)(\epsilon+2)}\,x^3 + \ldots,$$

satisfies the differential equation

$$(1-x)\,x^2\frac{d^3y}{dx^3} + \{(\delta+\epsilon+1)-(\alpha+\beta+\gamma+3)\,x\}\,x\frac{d^2y}{dx^2}$$
$$+\{\delta\epsilon - x(\alpha\beta+\beta\gamma+\gamma\alpha+\alpha+\beta+\gamma+1)\}\frac{dy}{dx} - \alpha\beta\gamma\,.\,y = 0 \ldots\ldots (5)$$

The complete solution of this equation may be written in the form

$$y = A\,F(\alpha, \beta, \gamma\,;\ \delta, \epsilon\,;\ x)$$
$$+ B\,.\,x^{1-\delta}\,F(\alpha-\delta+1, \beta-\delta+1, \gamma-\delta+1\,;\ 2-\delta, \epsilon-\delta+1\,;\ x)$$
$$+ C\,x^{1-\epsilon}\,F(\alpha-\epsilon+1, \beta-\epsilon+1, \gamma-\epsilon+1\,;\ \delta-\epsilon+1, 2-\epsilon\,;\ x),$$

A, B, C being arbitrary constants.

If in equation (5) we make the substitution $x = 4\,t^3$, it reduces to

$$4\,(4\,t^3-1)\frac{d^3y}{dt^3} + \left\{48\,(\alpha+\beta+\gamma+1)\,t^2 - 12\,(\delta+\epsilon+1)\,\frac{1}{t}\right\}\frac{d^2y}{dt^2}$$
$$+\Big[16\{1-6\,(\alpha+\beta+\gamma)+9(\alpha\beta+\beta\gamma+\gamma\alpha+\alpha+\beta+\gamma)\}\,t$$
$$-4\,(4-6\,\delta-6\,\epsilon+9\,\delta\epsilon)\,\frac{1}{t}\Big]\frac{dy}{dt} + 432\,\alpha\beta\gamma\,.\,y = 0 \ldots\ldots\ldots (6)$$

Identifying this equation with equation (4), satisfied by Humbert's polynomials, we obtain

$$\left.\begin{aligned} 2\,(\alpha+\beta+\gamma+1) &= 2\,\nu+3 \\ \delta+\epsilon-1 &= 0 \\ -4\,\{1-6\,(\alpha+\beta+\gamma)+9\,(\alpha\beta+\beta\gamma+\gamma\alpha+\alpha+\beta+\gamma)\}& \\ = 3n^2+3n\,(2\nu+1)-(3\nu+2)&(3\nu+5) \\ 4-6\delta-6\epsilon+9\delta\epsilon &= 0 \\ 108\,\alpha\beta\gamma &= -n\,(n+3\,\nu)(n+3\,\nu+3) \end{aligned}\right\}\quad (7)$$

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Equations (7) give the results

$$\alpha = -\frac{n}{3},\ \beta = \frac{n}{6}+\frac{\nu}{2},\ \gamma = \frac{n}{6}+\frac{\nu+1}{2},\ \delta = \frac{1}{3},\ \epsilon = \frac{2}{3}.$$

The complete solution of Humbert's equation is therefore

$$y = A\,.\,F\left(-\frac{n}{3},\ \frac{n}{6}+\frac{\nu}{2},\ \frac{n}{6}+\frac{\nu+1}{2};\ \frac{1}{3},\ \frac{2}{3};\ 4x^3\right)$$

$$+B\,.\,F\left(-\frac{n}{3}+\frac{1}{3},\ \frac{n}{6}+\frac{\nu}{2}+\frac{1}{3},\ \frac{n}{6}+\frac{\nu+1}{2}+\frac{1}{3};\ \frac{2}{3},\ \frac{4}{3};\ 4x^3\right).(4x^3)^{\frac{1}{3}}$$

$$+C\,.\,F\left(-\frac{n}{3}+\frac{2}{3},\ \frac{n}{6}+\frac{\nu}{2}+\frac{2}{3},\ \frac{n}{6}+\frac{\nu+1}{2}+\frac{2}{3};\ \frac{4}{3},\ \frac{5}{3};\ 4x^3\right)(4x^3)^{\frac{2}{3}}$$

3. From the difference equation (3) satisfied by $P_n^\nu(x)$ it is apparent that the term in $P_n^\nu(x)$ involving the highest power of x will be that in x^n, and that the indices of the powers of x in successive terms will differ by 3; it is therefore natural to assume

$$P_n^\nu(x) = A' \sin(n-1)\frac{\pi}{3}\sin(n-2)\frac{\pi}{3}\,.\,F\left(-\frac{n}{3},\ \frac{n}{6}+\frac{\nu}{2},\ \frac{n}{6}+\frac{\nu+1}{2};\ \frac{1}{3},\ \frac{2}{3};\ 4x^3\right)$$

$$+B' \sin(n-2)\frac{\pi}{3}.\sin\frac{n\pi}{3}\,.\,F\left(-\frac{n}{3}+\frac{1}{3},\ \frac{n}{6}+\frac{\nu}{2}+\frac{1}{3},\ \frac{n}{6}+\frac{\nu+1}{2}+\frac{1}{3};\ \frac{2}{3},\ \frac{4}{3};\ 4x^3\right)(4x^3)^{\frac{1}{3}}$$

$$+C' \sin(n-1)\frac{\pi}{3}.\sin\frac{n\pi}{3}\,.\,F\left(-\frac{n}{3}+\frac{2}{3},\ \frac{n}{6}+\frac{\nu}{2}+\frac{2}{3},\ \frac{n}{6}+\frac{\nu+1}{2}+\frac{2}{3};\ \frac{4}{3},\ \frac{5}{3};\ 4x^3\right)(4x^3)^{\frac{2}{3}},$$

A', B', C' being certain constants which are to be determined.

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The coefficient of x^n in $P_n^\nu(x)$ may be obtained from (3), with the conditions

$$P_0^\nu(x) = 1$$
$$P_1^\nu(x) = 3\,\nu\,x$$
$$P_2^\nu(x) = \frac{9}{2}\,\nu(\nu+1)\,x^2$$

and may be written in the form

$$\frac{\Gamma(n+\nu)\,\Gamma(\tfrac{1}{3})\,\Gamma(\tfrac{2}{3})}{\Gamma(\nu)\,\Gamma\left(\frac{n+1}{3}\right)\Gamma\left(\frac{n+2}{3}\right)\Gamma\left(\frac{n+3}{3}\right)}.$$

On comparing coefficients in the two expansions we obtain the result

$$P_n^\nu(x) = \frac{(-1)^n\,\Gamma(n+\nu)}{3\,.\,4^{\frac{n}{3}-1}\;\Gamma(\nu)\,\Gamma\left(\frac{n}{2}+\frac{\nu}{2}\right)\Gamma\left(\frac{n}{2}+\frac{\nu+1}{2}\right)}\times$$

$$\left[\frac{\Gamma\left(\frac{n}{6}+\frac{\nu}{2}\right)\Gamma\left(\frac{n}{6}+\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{n+3}{3}\right)}\sin\,(n-1)\frac{\pi}{3}\;\sin\,(n-2)\frac{\pi}{3}\times\right.$$
$$F\left(-\frac{n}{3},\;\frac{n}{6}+\frac{\nu}{2},\;\frac{n}{6}+\frac{\nu+1}{2};\;\frac{1}{3};\;\frac{2}{3};\;4\,x^3\right)$$
$$+\frac{\Gamma\left(\frac{n}{6}+\frac{\nu}{2}+\frac{1}{3}\right)\Gamma\left(\frac{n}{6}+\frac{\nu+1}{2}+\frac{1}{3}\right)}{\frac{1}{3}\,.\,\Gamma\left(\frac{n+2}{3}\right)}\sin\,(n-2)\frac{\pi}{3}\times$$
$$\sin\frac{n\pi}{3}\,.\,F\left(-\frac{n}{3}+\frac{1}{3},\;\frac{n}{6}+\frac{\nu}{2}+\frac{1}{3},\;\frac{n}{6}+\frac{\nu+1}{2}+\frac{1}{3};\;\frac{2}{3},\;\frac{4}{3};\;4\,x^3\right)(4\,x^3)^{\frac{1}{3}}$$
$$+\frac{\Gamma\left(\frac{n}{6}+\frac{\nu}{2}+\frac{2}{3}\right)\Gamma\left(\frac{n}{6}+\frac{\nu+1}{2}+\frac{2}{3}\right)}{\frac{1}{3}\,.\,\frac{2}{3}\,\Gamma\left(\frac{n+1}{3}\right)}\sin(n-1)\frac{\pi}{3}\times$$
$$\left.\sin\frac{n\pi}{3}\,.\,F\left(-\frac{n}{3}+\frac{2}{3},\;\frac{n}{6}+\frac{\nu}{2}+\frac{2}{3},\;\frac{n}{6}+\frac{\nu+1}{2}+\frac{2}{3};\;\frac{4}{3},\;\frac{5}{3};\;4\,x^3\right)(4\,x^3)^{\frac{2}{3}}\right]$$

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In the case of Pincherle's Polynomials, where $\nu = \frac{1}{2}$, this gives

$$P_n(x) = \frac{(-1)^n \Gamma(n+\frac{1}{2})}{3 \cdot 4^{\frac{n}{3}-1}\, \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{n}{2}+\frac{1}{4}\right) \Gamma\left(\frac{n}{2}+\frac{3}{4}\right)} \times$$

$$\left[\frac{\Gamma\left(\frac{n}{6}+\frac{1}{4}\right)\Gamma\left(\frac{n}{6}+\frac{3}{4}\right)}{\Gamma\left(\frac{n+3}{3}\right)} \sin(n-1)\frac{\pi}{3} \cdot \sin(n-2)\frac{\pi}{3} \times \right.$$
$$F\left(-\frac{n}{3},\ \frac{n}{6}+\frac{1}{4},\ \frac{n}{6}+\frac{3}{4};\ \frac{1}{3},\ \frac{2}{3};\ 4x^3\right)$$
$$+ \frac{\Gamma\left(\frac{n}{6}+\frac{7}{12}\right)\Gamma\left(\frac{n}{6}+\frac{13}{12}\right)}{\frac{1}{3}\Gamma\left(\frac{n+2}{3}\right)} \sin(n-2)\frac{\pi}{3} \cdot \sin\frac{n\pi}{3} \times$$
$$F\left(-\frac{n}{3}+\frac{1}{3},\ \frac{n}{6}+\frac{7}{12},\ \frac{n}{6}+\frac{13}{12};\ \frac{2}{3},\ \frac{4}{3};\ 4x^3\right)(4x^3)^{\frac{1}{3}}$$
$$+ \frac{\Gamma\left(\frac{n}{6}+\frac{11}{12}\right)\Gamma\left(\frac{n}{6}+\frac{17}{12}\right)}{\frac{1}{3}\cdot\frac{2}{3}\cdot\Gamma\left(\frac{n+1}{3}\right)} \sin(n-1)\frac{\pi}{3} \cdot \sin\frac{n\pi}{3} \times$$
$$\left. F\left(-\frac{n}{3}+\frac{2}{3},\ \frac{n}{6}+\frac{11}{12},\ \frac{n}{6}+\frac{17}{12};\ \frac{4}{3},\ \frac{5}{3};\ 4x^3\right)(4x^3)^{\frac{2}{3}} \right]$$

4. These expressions for $P_n^\nu(x)$ and $P_n(x)$ have a meaning and satisfy the differential equations (4) and (2), respectively, even when n is not a positive integer. They may therefore be used to define functions $P_n^\nu(x)$ and $P_n(x)$ for all values of n. The functions so defined bear the same relation to the Humbert and Pincherle polynomials, respectively, that the Legendre functions bear to the Legendre polynomials.