COMPOSITIO MATHEMATICA

On discrete homology of a free pro-p-group

Sergei O. Ivanov and Roman Mikhailov

Compositio Math. **154** (2018), 2195–2204.

doi:10.1112/S0010437X1800739X

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY
EST. 1865

https://doi.org/10.1112/S0010437X1800739X Published online by Cambridge University Press

Compositio Math. 154 (2018) 2195–2204
doi:10.1112/S0010437X1800739X

On discrete homology of a free pro-p-group

Sergei O. Ivanov and Roman Mikhailov

Abstract

For a prime p, let $\hat{F}_p$ be a finitely generated free pro-p-group of rank at least 2. We show that the second discrete homology group $H_2(\hat{F}_p, \mathbb{Z}/p)$ is an uncountable $\mathbb{Z}/p$-vector space. This answers a problem of A. K. Bousfield.

1. Introduction

Let p be a prime. For a profinite group G, there is a natural comparison map

$$H_2^{\mathsf{disc}}(G, \mathbb{Z}/p) \to H_2^{\mathsf{cont}}(G, \mathbb{Z}/p),$$

which connects discrete and continuous homology groups of G. Here $H_2^{\mathsf{disc}}(G, \mathbb{Z}/p) = H_2(G, \mathbb{Z}/p)$ is the second homology group of G with $\mathbb{Z}/p$-coefficients, where G is viewed as a discrete group. The continuous homology $H_2^{\mathsf{cont}}(G, \mathbb{Z}/p)$ can be defined as the inverse limit $\varprojlim H_2(G/U, \mathbb{Z}/p)$, where U runs over all open normal subgroups of G. The above comparison map $H_2^{\mathsf{disc}} \to H_2^{\mathsf{cont}}$ is the inverse limit of the coinflation maps $H_2(G, \mathbb{Z}/p) \to H_2(G/U, \mathbb{Z}/p)$ (see [FKRS08, Theorem 2.1]).

The study of the comparison map for different types of pro-p-groups is a fundamental problem in the theory of profinite groups (see [FKRS08] for discussion and references). It is well known that for a finitely generated free pro-p-group $\hat{F}_p$,

$$H_2^{\mathsf{cont}}(\hat{F}_p, \mathbb{Z}/p) = 0.$$

Bousfield posed the following question in [Bou77, Problem 4.11], (case $R = \mathbb{Z}/n$).

Problem (Bousfield). Does $H_2^{\mathsf{disc}}(\hat{F}_n, \mathbb{Z}/n)$ vanish when F is a finitely generated free group?

Here $\hat{F}_n$ is the $\mathbb{Z}/n$-completion of F, which is isomorphic to the product of pro-p-completions $\hat{F}_p$ over prime factors of n (see [Bou77, Proposition 12.3]). That is, the above problem is completely reduced to the case of homology groups $H_2^{\mathsf{disc}}(\hat{F}_p, \mathbb{Z}/p)$ for primes p and, since $H_2^{\mathsf{cont}}(\hat{F}_p, \mathbb{Z}/p) = 0$, the problem becomes a question about the non-triviality of the kernel of the comparison map for $\hat{F}_p$.

In [Bou92], Bousfield proved that, for a finitely generated free pro-p-group $\hat{F}_p$ on at least two generators, the group $H_i^{\mathsf{disc}}(\hat{F}_p, \mathbb{Z}/p)$ is uncountable for $i = 2$ or $i = 3$, or both. In particular, the wedge of two circles $S^1 \vee S^1$ is a $\mathbb{Z}/p$-bad space in the Bousfield–Kan sense.

The group $H_2^{\mathsf{disc}}(\hat{F}_p, \mathbb{Z}/p)$ plays a central role in the theory of $H\mathbb{Z}/p$-localizations developed in [Bou77]. It follows immediately from the definition of $H\mathbb{Z}/p$-localization that, for a free group F, $H_2^{\mathsf{disc}}(\hat{F}_p, \mathbb{Z}/p) = 0$ if and only if $\hat{F}_p$ coincides with the $H\mathbb{Z}/p$-localization of F. (From the point of view of profinite groups the Bousfield problem is also discussed in [Nik11, § 7] by Nikolov and in [Klo16, § 4] by Klopsch.)

In this paper we answer Bousfield's problem over $\mathbb{Z}/p$. Our main result is as follows.

Received 1 August 2017, accepted in final form 14 February 2018, published online 7 September 2018.
2010 Mathematics Subject Classification 55P60, 20E18 (primary).
Keywords: completion, profinite group, group homology.

MAIN THEOREM. *For a finitely generated free pro-p-group $\hat{F}_p$ of rank at least 2, $H_2^{\mathsf{disc}}(\hat{F}_p, \mathbb{Z}/p)$ is uncountable.*

There are two cases in Bousfield's problem, $R = \mathbb{Z}/n$ and $R = \mathbb{Q}$. We give the answer for the case of $R = \mathbb{Z}/n$. (Recently the authors gave the solution for the $R = \mathbb{Q}$ case [IM17], using completely different methods.)

The proof is organized as follows. In § 2 we consider properties of discrete and continuous homology of profinite groups. Using a result of Nikolov and Segal [NS07, Theorem 1.4], we show that for a finitely generated profinite group G and a closed normal subgroup H the cokernels of the maps $H_2^{\mathsf{disc}}(G, \mathbb{Z}/p) \to H_2^{\mathsf{disc}}(G/H, \mathbb{Z}/p)$ and $H_2^{\mathsf{cont}}(G, \mathbb{Z}/p) \to H_2^{\mathsf{cont}}(G/H, \mathbb{Z}/p)$ coincide (Theorem 2.5):

$$\begin{array}{ccccccc} H_2^{\mathsf{disc}}(G, \mathbb{Z}/p) & \longrightarrow & H_2^{\mathsf{disc}}(G/H, \mathbb{Z}/p) & \longrightarrow & Q^{\mathsf{disc}} & \longrightarrow & 0 \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \cong} & & \\ H_2^{\mathsf{cont}}(G, \mathbb{Z}/p) & \longrightarrow & H_2^{\mathsf{cont}}(G/H, \mathbb{Z}/p) & \longrightarrow & Q^{\mathsf{cont}} & \longrightarrow & 0. \end{array}$$

As a corollary we obtain (Corollary 2.6) that, for a finitely generated free pro-p-group $\hat{F}_p$, a continuous epimorphism $\pi : \hat{F}_p \twoheadrightarrow G$ to a pro-p-group induces the exact sequence

$$H_2^{\mathsf{disc}}(\hat{F}_p, \mathbb{Z}/p) \xrightarrow{\pi_*} H_2^{\mathsf{disc}}(G, \mathbb{Z}/p) \xrightarrow{\varphi} H_2^{\mathsf{cont}}(G, \mathbb{Z}/p) \longrightarrow 0. \tag{1.1}$$

That is, to prove that, for a free group F, $H_2(\hat{F}_p, \mathbb{Z}/p) \neq 0$, it is enough to find a discrete epimorphism $F \twoheadrightarrow G$ such that the comparison map of the second homology groups of the pro-p-completion of G has a non-zero kernel. Observe that the statements in § 2 significantly use the theory of profinite groups and there is no direct way to generalize them for pronilpotent groups. In particular, we do not see how to prove that $H_2(\hat{F}_{\mathbb{Z}}, \mathbb{Z}/p) \neq 0$, where $\hat{F}_{\mathbb{Z}}$ is the pronilpotent completion of F.

Section 3 follows the ideas of Bousfield from [Bou92]. Consider the ring of formal power series $\mathbb{Z}/p[\![x]\!]$, and the infinite cyclic group $C := \langle t \rangle$. We will use the multiplicative notation of the p-adic integers $C \otimes \mathbb{Z}_p = \{t^\alpha, \alpha \in \mathbb{Z}_p\}$. Consider the continuous multiplicative homomorphism $\tau : C \otimes \mathbb{Z}_p \to \mathbb{Z}/p[\![x]\!]$ sending t to $1 - x$. The main result of § 3 is Proposition 3.3, which claims that the kernel of the multiplication map

$$\mathbb{Z}/p[\![x]\!] \otimes_{\mathbb{Z}/p[C \otimes \mathbb{Z}_p]} \mathbb{Z}/p[\![x]\!] \longrightarrow \mathbb{Z}/p[\![x]\!] \tag{1.2}$$

is uncountable.

Our main example is based on the p-lamplighter group $\mathbb{Z}/p \wr \mathbb{Z}$, a finitely generated but not finitely presented group, which plays a central role in the theory of metabelian groups. The homological properties of the p-lamplighter group are considered in [Kro85]. The profinite completion of the p-lamplighter group is considered in [GK14], where it is shown that it is a semi-similar group generated by finite automaton. We consider the *double lamplighter group*,

$$(\mathbb{Z}/p)^2 \wr \mathbb{Z} = \langle a, b, c \mid [b, b^{a^i}] = [c, c^{a^i}] = [b, c^{a^i}] = b^p = c^p = 1,\ i \in \mathbb{Z} \rangle.$$

Denote by $\mathcal{DL}$ the pro-p-completion of the double lamplighter group. It follows from direct computations of homology groups that there is a diagram (in the above notation)

$$\begin{array}{ccc} \mathbb{Z}/p[\![x]\!] \otimes_{\mathbb{Z}/p[C\otimes \mathbb{Z}_p]} \mathbb{Z}/p[\![x]\!] & \longrightarrow & \mathbb{Z}/p[\![x]\!] \\ \big\downarrow \oplus & & \big\downarrow \\ H_2^{\mathsf{disc}}(\mathcal{DL}, \mathbb{Z}/p) & \longrightarrow & H_2^{\mathsf{cont}}(\mathcal{DL}, \mathbb{Z}/p) \end{array}$$

where the left vertical arrow is a split monomorphism and the upper horizontal map is the multiplication map (see proof of Theorem 4.3). This implies that, for the group $\mathcal{DL}$, the comparison map $H_2^{\mathsf{disc}}(\mathcal{DL}, \mathbb{Z}/p) \to H_2^{\mathsf{cont}}(\mathcal{DL}, \mathbb{Z}/p)$ has an uncountable kernel. Since the double lamplighter group is 3-generated, the sequence (1.1) implies that, for a free group F with at least three generators, $H_2(\hat{F}_p, \mathbb{Z}/p)$ is uncountable. Finally, we use [Bou92, Lemma 11.2] to get the same result for a 2-generated free group F.

In [Bou77], Bousfield formulated the following generalization of the above problem for the class of finitely presented groups (see [Bou77, Problem 4.10], the case $R = \mathbb{Z}/n$). Let G be a finitely presented group. Is it true that $H\mathbb{Z}/p$-localization of G equals its pro-p-completion $\hat{G}_p$? (The problem is formulated for $H\mathbb{Z}/n$-localization, but it is reduced to the case of a prime $n = p$.) It follows immediately from the definition of $H\mathbb{Z}/p$-localization that this problem can be reformulated as follows: is it true that, for a finitely presented group G, the natural homomorphism $H_2(G, \mathbb{Z}/p) \to H_2(\hat{G}_p, \mathbb{Z}/p)$? It is shown in [Bou77] that this is true for the class of polycyclic groups. The same is true for finitely presented metabelian groups [IM16]. The main theorem of the present paper implies that, for any finitely presented group P, which maps epimorphically onto the double lamplighter group, the natural map $H_2(P, \mathbb{Z}/p) \to H_2(\hat{P}_p, \mathbb{Z}/p)$ has an uncountable cokernel.

2. Discrete and continuous homology of profinite groups

For a profinite group G and a normal subgroup H, denote by $\overline{H}$ the closure of H in G in profinite topology.

Theorem 2.1 [NS07, Theorem 1.4]. *Let G be a finitely generated profinite group and H be a closed normal subgroup of G. Then the subgroup $[H, G]$ is closed in G.*

Corollary 2.2. *Let G be a finitely generated profinite group and H be a closed normal subgroup of G. Then the subgroup $[H, G] \cdot H^p$ is closed in G.*

Proof. Consider the abelian profinite group $H/[H, G]$. Then the p-power map $H/[H, G] \to H/[H, G]$ is continuous and its image is equal to $([H, G] \cdot H^p)/[H, G]$. Hence $([H, G] \cdot H^p)/[H, G]$ is a closed subgroup of $H/[H, G]$. Using the fact that the preimage of a closed set under continuous function is closed, we obtain that $[H, G] \cdot H^p$ is closed. □

Observe that, in the proof of Corollary 2.2, [NS07, Theorem 1.4] is not used in full generality. We only need it in the case of pro-p groups, and in this particular case the proof of this theorem is quite elementary.

Lemma 2.3 (mod-p Hopf formula). *Let G be a (discrete) group and H be its normal subgroup. Then there is a natural exact sequence*

$$H_2(G,\mathbb{Z}/p) \longrightarrow H_2(G/H,\mathbb{Z}/p) \longrightarrow \frac{H\cap([G,G]G^p)}{[H,G]H^p} \longrightarrow 0.$$

Proof. This follows from the five-term exact sequence

$$H_2(G,\mathbb{Z}/p) \longrightarrow H_2(G/H,\mathbb{Z}/p) \longrightarrow H_1(H,\mathbb{Z}/p)_G \longrightarrow H_1(G,\mathbb{Z}/p)$$

and the equations $H_1(H,\mathbb{Z}/p)_G = H/([H,G]H^p)$ and $H_1(G,\mathbb{Z}/p) = G/([G,G]G^p)$. □

Lemma 2.4 (Profinite mod-p Hopf formula). *Let G be a profinite group and H be its closed normal subgroup. Then there is a natural exact sequence*

$$H_2^{\mathsf{cont}}(G,\mathbb{Z}/p) \longrightarrow H_2^{\mathsf{cont}}(G/H,\mathbb{Z}/p) \longrightarrow \frac{H\cap\overline{([G,G]G^p)}}{\overline{([H,G]H^p)}} \longrightarrow 0.$$

Proof. For the sake of simplicity we set $H_*(-) = H_*^{\mathsf{discr}}(-,\mathbb{Z}/p)$ and $H_*^{\mathsf{cont}}(-) := H_*^{\mathsf{cont}}(-,\mathbb{Z}/p)$. Consider the five-term exact sequence [RZ00, Corollary 7.2.6]

$$H_2^{\mathsf{cont}}(G) \longrightarrow H_2^{\mathsf{cont}}(G) \longrightarrow H_0^{\mathsf{cont}}(G, H_1^{\mathsf{cont}}(H)) \longrightarrow H_1^{\mathsf{cont}}(G).$$

Continuous homology and cohomology of profinite groups are Pontryagin dual to each other [RZ00, Proposition 6.3.6]. There are isomorphisms

$$H^1_{\mathsf{cont}}(G) = \mathrm{Hom}(G/\overline{[G,G]G^p},\mathbb{Z}/p) = \mathrm{Hom}(G/\overline{[G,G]G^p},\mathbb{Q}/\mathbb{Z}),$$

where Hom denotes the set of continuous homomorphisms (see [Ser02, I.2.3]). It follows that $H_1^{\mathsf{cont}}(G) = G/\overline{[G,G]G^p}$. Similarly, $H_1^{\mathsf{cont}}(H) = H/\overline{[H,H]H^p}$. [RZ00, Lemma 6.3.3] implies that $H_0^{\mathsf{cont}}(G,M) = M/\overline{\langle m - mg \mid m\in M, g\in G\rangle}$ for any profinite $(\mathbb{Z}/p[G])^\wedge$-module M. Therefore $H_0^{\mathsf{cont}}(G,H_1^{\mathsf{cont}}(H)) = H/\overline{[H,H]H^p}$. The assertion follows. □

We denote by φ the comparison map

$$\varphi : H_2^{\mathsf{disc}}(G,\mathbb{Z}/p) \to H_2^{\mathsf{cont}}(G,\mathbb{Z}/p).$$

Theorem 2.5. *Let G be a finitely generated profinite group and H a closed normal subgroup of G. Denote*

$$Q^{\mathsf{disc}} := \mathsf{Coker}(H_2(G,\mathbb{Z}/p) \to H_2(G/H,\mathbb{Z}/p)),$$
$$Q^{\mathsf{cont}} := \mathsf{Coker}(H_2^{\mathsf{cont}}(G,\mathbb{Z}/p) \to H_2^{\mathsf{cont}}(G/H,\mathbb{Z}/p)).$$

Then the comparison maps φ induce an isomorphism $Q^{\mathsf{disc}} \cong Q^{\mathsf{cont}}$:

$$\begin{array}{ccccccc} H_2^{\mathsf{disc}}(G,\mathbb{Z}/p) & \longrightarrow & H_2^{\mathsf{disc}}(G/H,\mathbb{Z}/p) & \longrightarrow & Q^{\mathsf{disc}} & \longrightarrow & 0 \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\cong} & & \\ H_2^{\mathsf{cont}}(G,\mathbb{Z}/p) & \longrightarrow & H_2^{\mathsf{cont}}(G/H,\mathbb{Z}/p) & \longrightarrow & Q^{\mathsf{cont}} & \longrightarrow & 0 \end{array}$$

Proof. This follows from Lemmas 2.3, 2.4 and Corollary 2.2. □

Corollary 2.6. *Let G be a finitely generated pro-p-group and $\pi : \hat{F}_p \twoheadrightarrow G$ be a continuous epimorphism from the pro-p-completion of a finitely generated free group F. Then the sequence*

$$H_2^{\mathsf{disc}}(\hat{F}_p,\mathbb{Z}/p) \xrightarrow{\pi_*} H_2^{\mathsf{disc}}(G,\mathbb{Z}/p) \xrightarrow{\varphi} H_2^{\mathsf{cont}}(G,\mathbb{Z}/p) \longrightarrow 0$$

is exact.

Proof. This follows from Theorem 2.5 and the fact that $H_2^{\mathsf{cont}}(\hat{F}_p,\mathbb{Z}/p) = 0$. □

3. Technical results about the ring of power series $\mathbb{Z}/p[\![x]\!]$

In this section we follow to ideas of Bousfield written in [Bou92, Lemmas 10.6, 10.7]. The goal of this section is to prove Proposition 3.3.

We use the following notation: $C = \langle t \rangle$ is the infinite cyclic group; $C \otimes \mathbb{Z}_p$ is the group of p-adic integers written multiplicatively as powers of the generator $C \otimes \mathbb{Z}_p = \{t^\alpha \mid \alpha \in \mathbb{Z}_p\}$; $\mathbb{Z}/p[\![x]\!]$ is the ring of power series; $\mathbb{Z}/p(\!(x)\!)$ is the field of formal Laurent series.

Lemma 3.1. *Let* A *be a subset of* $\mathbb{Z}/p[\![x]\!]$. *Denote by* A^i *the image of* A *in* $\mathbb{Z}/p[x]/(x^{p^i})$. *Assume that*

$$\lim_{i \to \infty} |\mathsf{A}^i|/p^{p^i} = 0.$$

Then the interior of $\mathbb{Z}/p[\![x]\!] \setminus \mathsf{A}$ *is dense in* $\mathbb{Z}/p[\![x]\!]$.

Proof. Take any power series f and any its neighbourhood of the form $f + (x^{p^s})$. Then for any i the open set $f + (x^{p^s})$ is the disjoint union of smaller open sets $\bigcup_{t=1}^{p^{p^i}} f + f_t + (x^{p^{s+i}})$, where f_t runs over representatives of $(x^{p^s})/(x^{p^{s+i}})$. Chose i so that $|\mathsf{A}^{s+i}|/p^{p^{i+s}} \leqslant p^{-p^s}$. Then $|\mathsf{A}^{s+i}| \leqslant p^{p^i}$. Hence the number of elements in A^{i+s} is less than the number of open sets $f + f_t + (x^{p^{s+i}})$. It follows that there exists t such that $\mathsf{A} \cap (f + f_t + (x^{p^{s+i}})) = \varnothing$. The assertion follows. □

Denote by

$$\tau : C \otimes \mathbb{Z}_p \to \mathbb{Z}/p[\![x]\!]$$

the continuous multiplicative homomorphism sending t to $1 - x$. It is well defined because $(1-x)^{p^i} = 1 - x^{p^i}$.

Lemma 3.2. *Let* K *be the subfield of* $\mathbb{Z}/p(\!(x)\!)$ *generated by the image of* τ. *Then the degree of the extension* $[\mathbb{Z}/p(\!(x)\!) : K]$ *is uncountable.*

Proof. Denote the image of the map $\tau : C \otimes \mathbb{Z}_p \to \mathbb{Z}/p[\![x]\!]$ by A. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$, where $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \mathbb{Z}/p$, and $k \geqslant 1$. Denote by $\mathsf{A}_{\alpha,\beta,k}$ the subset of $\mathbb{Z}/p[\![x]\!]$ consisting of elements that can be written in the form

$$\frac{\alpha_1 a_1 + \dots + \alpha_n a_n}{\beta_1 b_1 + \dots + \beta_n b_n}, \tag{3.1}$$

where $a_1, \dots, a_n, b_1, \dots, b_n \in \mathsf{A}$ and $\beta_1 b_1 + \dots + \beta_n b_n \notin (x^{p^k})$. Then $K \cap \mathbb{Z}/p[\![x]\!] = \bigcup_{\alpha,\beta,k} \mathsf{A}_{\alpha,\beta,k}$.

Fix some α, β, k. Take $i \geqslant k$ and consider the images of A and $\mathsf{A}_{\alpha,\beta,k}$ in $\mathbb{Z}/p[x]/(x^{p^i})$. Denote them by A^i and $\mathsf{A}^i_{\alpha,\beta,k}$. Obviously A^i is the image of the map $C/C^{p^i} \to \mathbb{Z}/p[x]/(x^{p^i})$ that sends t to $1-x$. Then A^i consists of p^i elements. Fix some elements $\bar{a}_1, \dots, \bar{a}_n, \bar{b}_1, \dots, \bar{b}_n \in \mathsf{A}^i$ that have preimages $a_1, \dots, a_n, b_1, \dots, b_n \in \mathsf{A}$ such that the ratio (3.1) is in $\mathsf{A}_{\alpha,\beta,k}$. For any such preimages $a_1, \dots, a_n, b_1, \dots, b_n \in \mathsf{A}$ the image $\bar{r}$ of the ratio (3.1) satisfies the equation

$$\bar{r} \cdot (\beta_1 \bar{b}_1 + \dots + \beta_n \bar{b}_n) = \alpha_1 \bar{a}_1 + \dots + \alpha_n \bar{a}_n.$$

Since $\beta_1 \bar{b}_1 + \dots + \beta_n \bar{b}_n \notin (x^{p^k})$, the annihilator of $\beta_1 \bar{b}_1 + \dots + \beta_n \bar{b}_n$ consists of no more than p^{p^k} elements and the equation has no more than p^{p^k} solutions. Then we have no more than p^{2in}

https://doi.org/10.1112/S0010437X1800739X Published online by Cambridge University Press

variants of collections $\bar{a}_1, \dots, \bar{a}_n, \bar{b}_1, \dots, \bar{b}_n \in \mathsf{A}^i$, and for any such variant there are no more than p^{p^k} variants for the image of the ratio. Therefore

$$|\mathsf{A}^i_{\alpha,\beta,k}| \leqslant p^{2in+p^k}. \tag{3.2}$$

Take any sequence of elements $v_1, v_2, \dots \in \mathbb{Z}/p(\!(x)\!)$ and prove that $\sum_{m=1}^\infty K v_m \neq \mathbb{Z}/p(\!(x)\!)$. Note that $x \in K$ because $t \mapsto 1-x$. Multiplying the elements $v_1, v_2, v_3, \dots$ by powers of x, we can assume that $v_1, v_2, \dots \in \mathbb{Z}/p[\![x]\!]$. Fix some α, β, k as above. Set

$$\mathsf{A}_{\alpha,\beta,k,l} = \mathsf{A}_{\alpha,\beta,k} \cdot v_1 + \dots + \mathsf{A}_{\alpha,\beta,k} \cdot v_l.$$

Then $\sum_{m=1}^\infty K v_m = \bigcup_{\alpha,\beta,k,l,j} \mathsf{A}_{\alpha,\beta,k,l} \cdot x^{-j}$. Denote by $\mathsf{A}^i_{\alpha,\beta,k,l}$ the image of $\mathsf{A}_{\alpha,\beta,k,l}$ in $\mathbb{Z}/p[x]/(x^i)$. Then (3.2) implies $|\mathsf{A}^i_{\alpha,\beta,k,l}| \leqslant p^{(2in+p^k)l}$. Therefore

$$\lim_{i\to\infty} |\mathsf{A}^i_{\alpha,\beta,k,l}|/p^{p^i} = 0.$$

By Lemma 3.1 the interior of the complement of $\mathsf{A}_{\alpha,\beta,k,l}$ is dense in $\mathbb{Z}/p[\![x]\!]$. By the Baire theorem $\sum_{m=1}^\infty K v_m = \bigcup_{\alpha,\beta,k,l,j} \mathsf{A}_{\alpha,\beta,k,l} \cdot x^{-j}$ has empty interior. In particular, $\sum_{m=1}^\infty K v_m \neq \mathbb{Z}/p(\!(x)\!)$. □

Proposition 3.3. *Consider the ring homomorphism $\mathbb{Z}/p[C \otimes \mathbb{Z}_p] \to \mathbb{Z}/p[\![x]\!]$ induced by τ. Then the kernel of the multiplication map*

$$\mathbb{Z}/p[\![x]\!] \otimes_{\mathbb{Z}/p[C\otimes\mathbb{Z}_p]} \mathbb{Z}/p[\![x]\!] \longrightarrow \mathbb{Z}/p[\![x]\!] \tag{3.3}$$

is uncountable.

Proof. As in Lemma 3.2, we denote by K the subfield of $\mathbb{Z}/p(\!(x)\!)$ generated by the image of $C \otimes \mathbb{Z}_p$. Since $t \mapsto 1-x$, we have $x, x^{-1} \in K$. Set $R := K \cap \mathbb{Z}/p[\![x]\!]$. Note that the image of $\mathbb{Z}/p[C \otimes \mathbb{Z}_p]$ lies in R. Consider the multiplication map

$$\mu : \mathbb{Z}/p[\![x]\!] \otimes_R K \to \mathbb{Z}/p(\!(x)\!).$$

We claim that this is an isomorphism. Construct the map in the inverse direction

$$\kappa : \mathbb{Z}/p(\!(x)\!) \to \mathbb{Z}/p[\![x]\!] \otimes_R K$$

given by

$$\kappa\Big(\sum_{i=-n}^{\infty} \alpha_i x^i\Big) = \sum_{i=0}^{\infty} \alpha_{i+n} x^i \otimes x^{-n}.$$

Since we have $ax \otimes b = a \otimes xb$, κ does not depend on the choice of n, we just have to chose it big enough. Using this, we get that κ is well defined. Obviously $\mu\kappa = \mathsf{id}$. Chose $a \otimes b \in \mathbb{Z}/p[\![x]\!] \otimes_R K$. Then $b = b_1 b_2^{-1}$, where $b_1, b_2 \in R$. Since b_2 is a power series, we can chose n such that $b_2 = x^n b_3$, where b_3 is a power series with non-trivial constant term. Then b_3 is invertible in the ring of power series and $b_3, b_3^{-1} \in R$ because $x \in K$. Hence $a \otimes b = ab_3^{-1} b_3 \otimes b = ab_3^{-1} \otimes x^{-n} b_1 = ab_1 b_3^{-1} \otimes x^{-n}$. Using this presentation, we see that $\kappa\mu = \mathsf{id}$. Therefore

$$\mathbb{Z}/p[\![x]\!] \otimes_R K \cong \mathbb{Z}/p(\!(x)\!). \tag{3.4}$$

Since the image of $\mathbb{Z}/p[C \otimes \mathbb{Z}_p]$ lies in R, the tensor product $\mathbb{Z}/p[\![x]\!] \otimes_R \mathbb{Z}/p[\![x]\!]$ is a quotient of the tensor product $\mathbb{Z}/p[\![x]\!] \otimes_{\mathbb{Z}/p[C\otimes\mathbb{Z}_p]} \mathbb{Z}/p[\![x]\!]$ and it is enough to prove that the kernel of

$$\mathbb{Z}/p[\![x]\!] \otimes_R \mathbb{Z}/p[\![x]\!] \longrightarrow \mathbb{Z}/p[\![x]\!] \tag{3.5}$$

is uncountable.

For any ring homomorphism $R \to S$ and any R-modules M, N there is an isomorphism $(M \otimes_R N) \otimes_R S = (M \otimes_R S) \otimes_S (N \otimes_R S)$. Using this and the isomorphism (3.4), we obtain that after application of $- \otimes_R K$ to (3.5) we have

$$\mathbb{Z}/p(\!(x)\!) \otimes_K \mathbb{Z}/p(\!(x)\!) \longrightarrow \mathbb{Z}/p(\!(x)\!). \tag{3.6}$$

Assume to the contrary that the kernel of the map (3.5) is countable (countable = countable or finite). It follows that the linear map (3.6) has countable-dimensional kernel. Finally, note that the homomorphism

$$\Lambda^2_K \mathbb{Z}/p(\!(x)\!) \to \mathbb{Z}/p(\!(x)\!) \otimes_K \mathbb{Z}/p(\!(x)\!)$$

given by $a \wedge b \mapsto a \otimes b - b \otimes a$ is a monomorphism, its image lies in the kernel and the dimension of $\Lambda^2_K \mathbb{Z}/p(\!(x)\!)$ over K is uncountable because $[\mathbb{Z}/p(\!(x)\!) : K]$ is uncountable (Lemma 3.2). A contradiction follows. □

4. Double lamplighter pro-p-group

Let A be a finitely generated free abelian group written multiplicatively; $\mathbb{Z}/p[A]$ be its group algebra; I be its augmentation ideal; and M be a $\mathbb{Z}/p[A]$-module. Then denote by $\hat{M} = \varprojlim M/MI^i$ its I-adic completion. We embed A into the pro-p-group $A \otimes \mathbb{Z}_p$. We use the 'multiplicative' notation $a^\alpha := a \otimes \alpha$ for $a \in A$ and $\alpha \in \mathbb{Z}_p$. Note that for any $a \in A$ the power a^{p^i} acts trivially on M/MI^{p^i} because $1 - a^{p^i} = (1-a)^{p^i} \in I^{p^i}$. Then we can extend the action of A on $\hat{M}$ to the action of $A \otimes \mathbb{Z}_p$ on $\hat{M}$ in a continuous way.

The proof of the following lemma can be found in [IM16], but we include it here for completeness.

LEMMA 4.1. *Let A be a finitely generated free abelian group and M be a finitely generated $\mathbb{Z}/p[A]$-module. Then*

$$H_*(A, M) \cong H_*(A, \hat{M}) \cong H_*(A \otimes \mathbb{Z}_p, \hat{M}).$$

Proof. The first isomorphism is proven in [BD75]. Since $\mathbb{Z}/p[A]$ in Noetheran, it follows that $H_n(A, \hat{M})$ is a finite $\mathbb{Z}/p$-vector space for any n. Prove the second isomorphism. The action of $A \otimes \mathbb{Z}_p$ on $\hat{M}$ gives an action of $A \otimes \mathbb{Z}_p$ on $H_*(A, \hat{M})$ such that A acts trivially on $H_*(A, \hat{M})$. Then we have a homomorphism from $A \otimes \mathbb{Z}/p$ to a finite group of automorphisms of $H_n(A, \hat{M})$, whose kernel contains A. Since any subgroup of finite index in $A \otimes \mathbb{Z}_p$ is open (see [RZ00, Theorem 4.2.2]) and A is dense in $A \otimes \mathbb{Z}_p$, we obtain that the action of $A \otimes \mathbb{Z}_p$ on $H_*(C, \hat{M})$ is trivial. Note that $\mathbb{Z}_p/\mathbb{Z}$ is a divisible torsion free abelian group, and hence $A \otimes (\mathbb{Z}_p/\mathbb{Z}) \cong \mathbb{Q}^{\oplus \mathbf{c}}$, where $\mathbf{c}$ is the continuous cardinal. Then the second page of the spectral sequence of the short exact sequence $A \rightarrowtail A \otimes \mathbb{Z}_p \twoheadrightarrow \mathbb{Q}^{\oplus \mathbf{c}}$ with coefficients in $\hat{M}$ is $H_n(\mathbb{Q}^{\oplus \mathbf{c}}, H_m(A, \hat{M}))$, where $L_m := H_m(A, \hat{M})$ is a trivial $\mathbb{Z}/p[\mathbb{Q}^{\oplus \mathbf{c}}]$-module. Then by universal coefficient theorem we have

$$0 \longrightarrow \Lambda^n(\mathbb{Q}^{\oplus \mathbf{c}}) \otimes L_m \longrightarrow H_n(\mathbb{Q}^{\oplus \mathbf{c}}, L_m) \longrightarrow \mathsf{Tor}(\Lambda^{n-1}(\mathbb{Q}^{\oplus \mathbf{c}}), L_m) \longrightarrow 0.$$

Since $\Lambda^n(\mathbb{Q}^{\oplus \mathbf{c}})$ is torsion free and L_m is a $\mathbb{Z}/p$-vector space, we get $\Lambda^n(\mathbb{Q}^{\oplus \mathbf{c}}) \otimes L_m = 0$ and $\mathsf{Tor}(\Lambda^{n-1}(\mathbb{Q}^{\oplus \mathbf{c}}), L_m) = 0$. It follows that $H_n(\mathbb{Q}^{\oplus \mathbf{c}}, L_m) = 0$ for $n \geqslant 1$ and $H_0(\mathbb{Q}^{\oplus \mathbf{c}}, L_m) = L_m$. Then the spectral sequence consists of only one column, and hence $H_*(A \otimes \mathbb{Z}_p, \hat{M}) = H_*(A, \hat{M})$. □

https://doi.org/10.1112/S0010437X1800739X Published online by Cambridge University Press

Lemma 4.2. *Let A be an abelian group, M be a $\mathbb{Z}[A]$-module and $\sigma_M : M \to M$ be an automorphism of the underlying abelian group such that $\sigma_M(ma) = \sigma_M(m)a^{-1}$ for any $m \in M$ and $a \in A$. Then there is an isomorphism*

$$(M \otimes M)_A \cong M \otimes_{\mathbb{Z}[A]} M$$

given by

$$m \otimes m' \leftrightarrow m \otimes \sigma_M(m').$$

Proof. Consider the isomorphism $\Phi : M \otimes M \to M \otimes M$ given by $\Phi(m \otimes m') = m \otimes \sigma(m')$. The group of coinvariants $(M \otimes M)_A$ is the quotient of $M \otimes M$ by the subgroup R generated by elements $ma \otimes m'a - m \otimes m'$, where $a \in A$ and $m, m' \in M$. We can write the generators of R in the following form: $ma \otimes m' - m \otimes m'a^{-1}$. Then $\Phi(R)$ is generated by $ma \otimes \sigma_M(m') - m \otimes \sigma_M(m')a$. Using the fact that σ_M is an automorphism, we can rewrite the generators of R as follows: $ma \otimes m' - m \otimes m'a$. Taking linear combinations of the generators of $\Phi(R)$, we obtain that $\Phi(R)$ is generated by elements $m\lambda \otimes m' - m \otimes m'\lambda$, where $\lambda \in \mathbb{Z}[A]$. Then $(M \otimes M)/\Phi(R) = M \otimes_{\mathbb{Z}[A]} M$. □

The group $C = \langle t \rangle$ acts on $\mathbb{Z}/p[\![x]\!]$ by multiplication on $1 - x$. As above, we can extend the action of C on $\mathbb{Z}[\![x]\!]$ to the action of $C \otimes \mathbb{Z}_p$ in a continuous way. The group

$$\mathbb{Z}/p \wr C = \mathbb{Z}/p[C] \rtimes C = \langle a, b \mid [b, b^{a^i}] = b^p = 1, i \in \mathbb{Z} \rangle$$

is called the lamplighter group. We consider the 'double version' of this group, the *double lamplighter group*:

$$(\mathbb{Z}/p[C] \oplus \mathbb{Z}/p[C]) \rtimes C = \langle a, b, c \mid [b, b^{a^i}] = [c, c^{a^i}] = [b, c^{a^i}] = b^p = c^p = 1, i \in \mathbb{Z} \rangle.$$

Its pro-p-completion is equal to the semidirect product

$$\mathcal{DL} = (\mathbb{Z}/p[\![x]\!] \oplus \mathbb{Z}/p[\![x]\!]) \rtimes (C \otimes \mathbb{Z}_p),$$

with the action of $C \otimes \mathbb{Z}_p$ on $\mathbb{Z}/p[\![x]\!]$ described above (see [IM16, Proposition 4.12]). We call the group $\mathcal{DL}$ the *double lamplighter pro-p-group*.

Theorem 4.3. *The kernel of the comparison homomorphism for the double lamplighter pro-p-group,*

$$\varphi : H_2^{\mathsf{disc}}(\mathcal{DL}, \mathbb{Z}/p) \longrightarrow H_2^{\mathsf{cont}}(\mathcal{DL}, \mathbb{Z}/p),$$

is uncountable.

Proof. For the sake of simplicity we set $H_2(-) = H_2(-, \mathbb{Z}/p)$ and $H_2^{\mathsf{cont}}(-) = H_2^{\mathsf{cont}}(-, \mathbb{Z}/p)$. Consider the homological spectral sequence E of the short exact sequence $\mathbb{Z}/p[\![x]\!]^2 \rightarrowtail \mathcal{DL} \twoheadrightarrow C \otimes \mathbb{Z}_p$. Then the zero line of the second page is trivial: $E^2_{k,0} = H_k(C \otimes \mathbb{Z}_p) = (\Lambda^k \mathbb{Z}_p) \otimes \mathbb{Z}/p = 0$ for $k \geqslant 2$. Using Lemma 4.1, we obtain $H_k(C \otimes \mathbb{Z}_p, \mathbb{Z}/p[\![x]\!]) = H_k(C, \mathbb{Z}/p[C]) = 0$ for $k \geqslant 1$, and hence $E^2_{k,1} = 0$ for $k \geqslant 1$. It follows that

$$H_2(\mathcal{DL}) = E^2_{0,2}. \tag{4.1}$$

For any $\mathbb{Z}/p$-vector space V, the Künneth formula gives a natural isomorphism

$$H_2(V \oplus V) \cong (V \otimes V) \oplus H_2(V)^2.$$

https://doi.org/10.1112/S0010437X1800739X Published online by Cambridge University Press

Then we have a split monomorphism,

$$(\mathbb{Z}/p[\![x]\!] \otimes \mathbb{Z}/p[\![x]\!])_{C\otimes\mathbb{Z}_p} \rightarrowtail E^2_{0,2} = H_2(\mathcal{DL}). \tag{4.2}$$

It easy to see that the groups $\mathcal{DL}_{(i)} = ((x^i) \oplus (x^i)) \rtimes (C \otimes p^i\mathbb{Z}_p)$ form a fundamental system of open normal subgroups. Consider the quotients $\mathcal{DL}^{(i)} = \mathcal{DL}/\mathcal{DL}_{(i)}$. Then

$$H_2^{\mathsf{cont}}(\mathcal{DL}) = \varprojlim H_2(\mathcal{DL}^{(i)}).$$

The short exact sequence $\mathbb{Z}/p[\![x]\!]^2 \rightarrowtail \mathcal{DL} \twoheadrightarrow C \otimes \mathbb{Z}_p$ maps onto the short exact sequence $(\mathbb{Z}/p[x]/(x^i))^2 \rightarrowtail \mathcal{DL}^{(i)} \twoheadrightarrow C/C^{p^i}$. Consider the morphism of corresponding spectral sequences $E \to {}^{(i)}E$. Using (4.1), we obtain

$$\mathrm{Ker}(H_2(\mathcal{DL}) \to H_2(\mathcal{DL}^{(i)})) \supseteq \mathrm{Ker}(E^2_{2,0} \to {}^{(i)}E^2_{2,0}).$$

Similarly to (4.2), we have a split monomorphism

$$(\mathbb{Z}/p[x]/(x^i) \otimes \mathbb{Z}/p[x]/(x^i))_{C\otimes\mathbb{Z}_p} \rightarrowtail {}^{(i)}E^2_{2,0}.$$

Then we need to prove that the kernel of the map

$$(\mathbb{Z}/p[\![x]\!] \otimes \mathbb{Z}/p[\![x]\!])_{C\otimes\mathbb{Z}_p} \longrightarrow \varprojlim(\mathbb{Z}/p[x]/(x^i) \otimes \mathbb{Z}/p[x]/(x^i))_{C\otimes\mathbb{Z}_p} \tag{4.3}$$

is uncountable.

Consider the antipod $\sigma : \mathbb{Z}/p[C] \to \mathbb{Z}/p[C]$, that is, the ring homomorphism given by $\sigma(t^n) = t^{-n}$. The antipod induces a homomorphism $\sigma : \mathbb{Z}/p[x]/(x^i) \to \mathbb{Z}/p[x]/(x^i)$ such that $\sigma(1-x) = 1 + x + x^2 + \cdots$. It induces the continuous homomorphism $\sigma : \mathbb{Z}/p[\![x]\!] \to \mathbb{Z}/p[\![x]\!]$ such that $\sigma(x) = -x - x^2 - \cdots$. Moreover, we consider the antipode σ on $\mathbb{Z}/p[C \otimes \mathbb{Z}_p]$. Note that the homomorphisms

$$\mathbb{Z}/p[C] \to \mathbb{Z}/p[C \otimes \mathbb{Z}_p] \to \mathbb{Z}/p[\![x]\!] \to \mathbb{Z}/p[x]/(x^i)$$

commute with the antipodes.

By Lemma 4.2 the correspondence $a \otimes b \leftrightarrow a \otimes \sigma(b)$ gives isomorphisms

$$(\mathbb{Z}/p[\![x]\!] \otimes \mathbb{Z}/p[\![x]\!])_{C\otimes\mathbb{Z}_p} \cong \mathbb{Z}/p[\![x]\!] \otimes_{\mathbb{Z}/p[C\otimes\mathbb{Z}_p]} \mathbb{Z}/p[\![x]\!],$$
$$(\mathbb{Z}/p[x]/(x^i) \otimes \mathbb{Z}/p[x]/(x^i))_{C\otimes\mathbb{Z}_p} \cong \mathbb{Z}/p[x]/(x^i) \otimes_{\mathbb{Z}/p[C\otimes\mathbb{Z}_p]} \mathbb{Z}/p[x]/(x^i).$$

Moreover, since $\mathbb{Z}/p[C \otimes \mathbb{Z}_p] \to \mathbb{Z}/p[x]/(x^i)$ is an epimorphism, we obtain

$$\mathbb{Z}/p[x]/(x^i) \otimes_{\mathbb{Z}/p[C\otimes\mathbb{Z}_p]} \mathbb{Z}/p[x]/(x^i) \cong \mathbb{Z}/p[x]/(x^i).$$

Therefore the homomorphism (4.3) is isomorphic to the multiplication homomorphism

$$\mathbb{Z}/p[\![x]\!] \otimes_{\mathbb{Z}/p[C\otimes\mathbb{Z}_p]} \mathbb{Z}/p[\![x]\!] \longrightarrow \mathbb{Z}/p[\![x]\!],$$

whose kernel is uncountable by Proposition 3.3. □

5. Proof of main theorem

Since the double lamplighter pro-p-group is 3-generated, we have a continuous epimorphism $\hat{F}_p \twoheadrightarrow \mathcal{DL}$, where F is the 3-generated free group. Then the statement of the theorem for the 3-generated free group follows from Proposition 4.3 and Corollary 2.6. Using the fact that the 3-generated free group is a retract of the k-generated free group for $k \geqslant 3$, we obtain the result for $k \geqslant 3$. The result for the 2-generated free group follows from [Bou92, Lemma 11.2].

ACKNOWLEDGEMENT
The research is supported by the Russian Science Foundation grant N 16-11-10073.

REFERENCES

Bou77 A. K. Bousfield, *Homological localization towers for groups and π-modules*, Mem. Amer. Math. Soc. **186** (1977).

Bou92 A. K. Bousfield, *On the p-adic completions of nonnilpotent spaces*, Trans. Amer. Math. Soc. **331** (1992), 335–359.

BD75 K. S. Brown and E. Dror, *The Artin–Rees property and homology*, Israel J. Math. **22** (1975), 93–109.

FKRS08 G. A. Fernandez-Alcober, I. V. Kazatchkov, V. N. Remeslennikov and P. Symonds, *Comparison of the discrete and continuous cohomology groups of a pro-p-group*, St. Petersburg Math. J. **19** (2008), 961–973.

GK14 R. Grigorchuk and R. Kravchenko, *On the lattice of subgroups of the lamplighter group*, Internat. J. Algebra Comput. **24** (2014), 837–877.

IM16 S. O. Ivanov and R. Mikhailov, *On a problem of Bousfield for metabelian groups*, Adv. Math. **290** (2016), 552–589.

IM17 S. O. Ivanov and R. Mikhailov, *A finite $\mathbb{Q}$-bad space*, Preprint (2017), arXiv:1708.00282.

Klo16 B. Klopsch, *Abstract quotients of profinite groups, after Nikolov and Segal*, Preprint (2016), arXiv:1601.00343.

Kro85 P. H. Kropholler, *A note on the cohomology of metabelian groups*, Math. Proc. Cambridge Philos. Soc. **98** (1985), 437–445.

Nik11 N. Nikolov, *Algebraic properties of profinite groups*, Preprint (2011), arXiv:1108.5130.

NS07 N. Nikolov and D. Segal, *On finitely generated profinite groups, I: Strong completeness and uniform bounds*, Ann. of Math. (2) **165** (2007), 171–238.

RZ00 L. Ribes and P. Zalesskii, *Profinite groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, vol. 40 (Springer, Berlin, 2000).

Ser02 J.-P. Serre, *Galois cohomology*, Springer Monographs in Mathematics (Springer, Berlin, 2002).

Sergei O. Ivanov ivanov.s.o.1986@gmail.com
Laboratory of Modern Algebra and Applications, St. Petersburg State University,
14th Line, 29b, Saint Petersburg, 199178 Russia

Roman Mikhailov rmikhailov@mail.ru
Laboratory of Modern Algebra and Applications, St. Petersburg State University,
14th Line, 29b, Saint Petersburg, 199178 Russia

and

St. Petersburg Department of Steklov Mathematical Institute, Russia