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SEMIPRIME GOLDIE GENERALISED MATRIX RINGS

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ABSTRACT. Necessary and sufficient conditions are given for a generalised matrix ring to be semiprime right Goldie.

Recently, Poole and Stewart [2] have proved the following result:

THEOREM 1 ([2]). *Let* $R = (R_{ij})$ *be a ring of* $n \times n$ *generalised matrices. If* R *is semiprime right Goldie, then so too are the rings* R_{ii} *for* $1 \leq i \leq n$. *Conversely, if* R *satisfies the left annihilator condition*

(A) *For all* i,j, *if* $0 \neq x \in R_{ij}$ *then* $xR_{ji} \neq 0$

and R_{ij} *is a faithful left* R_{ii}*-module for* $1 \leq i, j \leq n$ *and* R_{ii} *is semiprime right Goldie for* $1 \leq i \leq n$, *then* R *is semiprime right Goldie.*

The purpose of this note is to show that the hypothesis of faithfulness in the converse is superfluous. Our result is then

THEOREM 2. *Let* $R = (R_{ij})$ *be a ring of* $n \times n$ *generalised matrices. Then* R *is semiprime right Goldie if and only if each* R_{ii} *is semiprime right Goldie and* R *satisfies the annihilator condition (A).*

We remark that our result is valid for rings without identity, whereas the definition of generalised matrix ring in [2] implicitly assumes that the ring has identity.

The faithfulness condition appears in Proposition 6 of [2] where it is used to prove that $R = (R_{ij})$ has finite uniform dimension if each R_{ii} does. We note that it is not difficult to show that the faithfulness condition can be eliminated here under the additional assumption that R is semiprime. In view of Proposition 4 below, this is sufficient to prove Theorem 2. Rather than demonstrate this fact, however, we offer an alternative and much shorter proof of Theorem 2, the key to which is to regard $R = (R_{ij})$ as a group-graded ring.

We recall some terminology and results about group-graded rings from [1]. Let G be a group with identity e and let A be a G-graded ring. The grading is said to be *left* (resp. *right*) *non-degenerate* if $x \in A_g$ and $A_{g^{-1}}x = 0$ (resp. $xA_{g^{-1}} = 0$) implies $x = 0$. The grading is *non-degenerate* if it is left and right non-degenerate. If A is semiprime and A has finite support, the grading is necessarily non-degenerate and A_e is semiprime. If A_e is semiprime then left and right non-degeneracy are equivalent.

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PROPOSITION 3 ([1]). *Let G be a group and let A be a non-degenerate G-graded ring with finite support.*

(i) If A_e is semiprime right Goldie then A is right Goldie.

(ii) If A is right Goldie then A_e is right Goldie.

Let $R = (R_{ij})$ be a ring of $n \times n$ generalised matrices. For $k \in \mathbb{Z}$, put $R_k = \sum_{j-i=k} R_{ij}$, the k-th diagonal of R. Then $R = \bigoplus_{k \in \mathbb{Z}} R_k$ is $\mathbb{Z}$-graded. Since $R_k = 0$ for $|k| \geq n$, R has finite support as a $\mathbb{Z}$-graded ring. (Alternatively, we can regard R as $\mathbb{Z}_{2n-1}$-graded via the canonical epimorphism $\mathbb{Z} \longrightarrow \mathbb{Z}_{2n-1}$.) Note that R_0, the identity component is just the ring direct sum $R_{11} \oplus R_{22} \oplus \cdots \oplus R_{nn}$.

The annihilator condition was introduced by Wauters and Jespers [3] to characterise semiprime generalised matrix rings as follows:

PROPOSITION 4 ([3]). *Let $R = (R_{ij})$ be a ring of $n \times n$ generalised matrices. R is semiprime if and only if each R_{ii} is semiprime and R satisfies condition (A).*

We can now deduce Theorem 2 as a corollary of these two results.

PROOF OF THEOREM 2. If R is semiprime right Goldie, then the $\mathbb{Z}$-gradation is non-degenerate and Proposition 3(ii) implies that $R_0 = R_{11} \oplus R_{22} \oplus \cdots \oplus R_{nn}$ is right Goldie, so each R_{ii} is right Goldie. Condition (A) and the semiprimeness of R_{ii} follow from Proposition 4.

Conversely, suppose each R_{ii} is semiprime right Goldie and R satisfies condition (A). Then R_0 is semiprime right Goldie. Further, R is semiprime by Proposition 4 and so the $\mathbb{Z}$-gradation is non-degenerate. The result follows from Proposition 3(i). ∎

Finally, we remark that the quotient ring can be computed easily using another result of [1] which states that the classical ring of quotients of a non-degenerately G-graded semiprime right Goldie ring A with finite support is obtained by localising at the set of regular elements of the identity component A_e. In the case of a generalised matrix ring, we localise at the regular diagonal elements. It is not difficult to show that the quotient ring is a generalised matrix ring $Q = (Q_{ij})$ where Q_{ij} is the localisation of the right R_{jj}-module R_{ij} at the set of regular elements of R_{jj}. The product $Q_{ij} \times Q_{jk} \longrightarrow Q_{ik}$ is given by the rule $(rs^{-1}, pt^{-1}) \longmapsto rp'(ts')^{-1}$ where $r \in R_{ij}, p, p' \in R_{jk}$, s is a regular element of R_{jj}, s', t are regular elements of R_{kk} and $ps' = sp'$.

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