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On Surfaces with $p_g = 0$ and $K^2 = 5$

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Abstract. We construct new examples of surfaces of general type with $p_g = 0$ and $K^2 = 5$ as $\mathbb{Z}_2 \times \mathbb{Z}_2$-covers and show that they are genus three hyperelliptic fibrations with bicanonical map of degree two.

1 Introduction

Given a compact complex algebraic surface S, the geometric genus p_g is the dimension over $\mathbb{C}$ of the space of holomorphic two-forms and the irregularity q is the dimension of the space of holomorphic one-forms. For minimal surfaces of general type with $p_g = q = 0$, the self-intersection of the canonical class K_S satisfies $1 \leq {K_S}^2 \leq 9$. Although there are many examples of such surfaces of general type, including examples for each value of K_S^2, few general results are known.

There has been much recent work on these surfaces using either the bicanonical map or the theory of covers. As $p_g = 0$ implies that there are no canonical curves, the bicanonical system $|2K_S|$ is the next natural system of curves on S. Using careful analysis of the bicanonical map, Mendes Lopes and Pardini [6, 7] have obtained a classification of surfaces with $K^2 = 6$. In particular they prove that if the bicanonical map is not birational, then it has degree two or four. For each case they use bidouble covers to construct examples, and in the case of degree four bicanonical map they prove that such a surface must be a Burniat surface.

Burniat's bidouble cover construction [2] yields examples with $p_g = 0$ and $2 \leq K^2 \leq 6$. Catanese [3] studied how singularities of the branch curves affect the invariants of the resulting bidouble covers, with two examples of $K^2 = 5$ surfaces among his constructions. In this note we prove that the Catanese surfaces are equivalent to the Burniat example, and therefore belong to the category of surfaces with bicanonical map of degree four. We then construct two examples with degree two bicanonical map. The surfaces are constructed as bidouble covers of the plane. We also show both can be realized as genus three hyperelliptic fibrations.

First we look at some properties of the bicanonical map for $p_g = 0, K^2 = 5$ surfaces. In Section 3 we review the construction of bidouble covers and prove that the Catanese surfaces are equivalent to Burniat's. In Section 4 we find two examples with bicanonical map of degree two.

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2 The Bicanonical Map

Let S be a minimal surface of general type with $p_g = q = 0$ and $K_S^2 = 5$, and bicanonical map φ. We have the following possibilities for the degrees of φ and its image.

Lemma 2.1 *Let S be a minimal surface of general type with $p_g = q = 0$ and $K^2 = 5$ and bicanonical map $\varphi\colon S \to X$. Assume that φ is not birational. Then either*

(i) *φ has degree two and X is a degree 10 rational surface in $\mathbb{P}^5$, or*

(ii) *φ has degree four and X is a degree 5 rational surface in $\mathbb{P}^5$.*

Proof By Reider's Theorem [8], φ is a morphism, and by Xiao ([9]), the image of φ is a rational surface X. As $K_S^2 = 5$ and $\dim H^0(S, 2K_S) = 1 + \frac{1}{2}(2K_S \cdot K_S)$, we have $X \subset \mathbb{P}^5$. The product of the degree of φ and the degree of the image in $\mathbb{P}^5$ is equal to $(2K_S)^2 = 4K_S^2 = 20$. By Mendes Lopes [5], the degree of φ is at most four, thus we have the two cases. ■

We now suppose S is a minimal surface of general type with $p_g = 0$, $K_S^2 = 5$, and bicanonical map $\varphi\colon S \to \mathbb{P}^5$ of degree two onto its image. Let σ denote the involution induced by φ, and let $\pi\colon S \to \Sigma = S/\sigma$ be the quotient map.

The fixed locus of σ is the union of a smooth curve R and k isolated points $P_1, \dots, P_k$. By [6], the number of isolated fixed points of σ is $k = K_S^2 + 4 = 9$. Write $Q = \pi(R)$ and $Q_i = \pi(P_i)$. Then the Q_i are ordinary double points on the normal surface Σ. We have

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\epsilon} & S \\ \downarrow & & \downarrow \pi \\ Y & \xrightarrow{\nu} & \Sigma \end{array},$$

where $\tilde{S} \to S$ is the blowup of S at the points P_i. Then σ induces an involution on $\tilde{S}$, with fixed locus R_0, the inverse image of R, and E_i, the exceptional divisors over the P_i. We write $\tilde{S} \to Y$ for the quotient map by this involution. On Y we have 9 disjoint (-2)-curves C_i from the resolution of the nodes Q_i; write B for the image of the curve R_0 on Y. Thus we have a double cover of Y branched along $B + C_1 + \cdots + C_k$. The surface Σ is rational by Lemma 2.1, thus Y is rational as well.

To realize our surfaces as double covers, we use the following result due to Borrelli.

Theorem 2.2 ([1, Proposition 5.3]) *Let S be a smooth minimal surface of general type with $p_g = 0$ and $K_S^2 = 5$. If the bicanonical map has degree 2 and S is not a genus two pencil, then S is a minimal model of a Du Val double plane, and a hyperelliptic fibration of genus 3 with 3 or 4 double fibers.*

The double plane constructions, suggested by Du Val ([4]) as a means of constructing general type surfaces, are as follows.

First let T_1, T_2, T_3 be three lines in the plane through a point P and let C be a degree thirteen curve with an order five singularity at P and three infinitely near quadruple points P_1, P_2, P_3 with tangent T_1, T_2, T_3, respectively. These infinitely near quadruple points (or $(4, 4)$ points) are singularities of order four which remains order four after one blow-up, thus each of the four branches of C has the same tangent

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direction, that of T_i. Assume C also has three additional infinitely near triple points at P_4, P_5, P_6. (Thus at these $(3,3)$ points C requires two blowups to resolve the singularity.)

Blow up the plane at the points $P, P_1, \dots, P_6$ and then blow up the ordinary singularities above each P_i, to obtain a rational surface Y where the proper transform $\overline{C}$ of C is non-singular. In addition the transforms of the lines T_1, T_2, T_3 do not meet $\overline{C}$ on Y.

We have Y, a thirteen-fold blowup of the plane, with $K_Y^2 = -4$. Write e_i for the transform of the exceptional divisor for the first blowup of P_i and f_i for the blowup corresponding to the tangent direction. Let h denote the class of a line.

We have

$$K_Y \equiv -3h + e + \sum e_i + 2\sum f_i.$$

We add the exceptional curves e_i, $i = 1, \dots, 6$ to the curves $\overline{C}$ and $\overline{T_i}$ on Y to obtain an even branch divisor,

$$\overline{C} + \sum_1^3 \overline{T_i} + \sum_1^6 e_i \equiv 16h - 8e - 4\sum_1^3 e_i - 10\sum_1^3 f_i - 2\sum_4^6 e_i - 6\sum_4^6 f_i.$$

Thus we see the nine disjoint -2-curves $T_1, T_2, T_3, e_1, \dots, e_6$ on Y corresponding to the nodes of the quotient S/σ. We have

$$L \equiv 8h - 4e - 2\sum_1^3 e_i - 5\sum_1^3 f_i - \sum_4^6 e_i - 3\sum_4^6 f_i,$$

where $2L$ represents the branching for the double cover $\pi\colon X \to Y$. Then by standard arguments for double covers, we have

$$\pi_* \mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{O}_Y(-L)$$

and

$$K_X^2 = 2(K_Y + L)^2.$$

Thus $\chi(X) = 1$ and $K_X^2 = -4$. The canonical system on X corresponds to $H^0(Y, \mathcal{O}_Y(K_Y)) \oplus H^0(Y, \mathcal{O}_Y(K_Y + L))$; as Y is rational the first system is empty. Since $K_Y + L \equiv 5h - 3e - \sum_1^3 e_i - 3\sum_1^3 f_i - \sum_4^6 f_i \equiv T_1 + T_2 + T_3 + 2h - \sum_1^6 f_i$, we see that $p_g = 0$ provided the six points $P_1, \dots, P_6$ are not on a conic.

The lines T_i and the exceptional curves e_i are all -2 rational curves on Y; as they are components of the branch locus, they become -1 curves on X. Blowing down these nine curves yields a surface S with $K_S^2 = 5$. The bicanonical system for the cover corresponds to $|2K_Y + 2L|$, where

$$2K_Y + 2L \equiv 10h - 6e - 2\sum_1^3 e_i - 6\sum_1^3 f_i - 2\sum_4^6 f_i.$$

As each T_i is a component of this system, and

$$2K_Y + 2L - \sum T_i \equiv 7h - 3e - \sum_1^3 e_i - 4\sum_1^3 f_i - 2\sum_4^6 f_i,$$

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we see that the bicanonical system corresponds to plane curves of degree seven, with a triple point at P, tacnodes at P_1, P_2, P_3, and through P_4, P_5, P_6 with the tangent direction of C.

Consider the pencil of lines in the plane through P. This pulls back to a basepoint-free rational pencil on Y which intersects the branch locus with order eight, thus on X we have a genus three hyperelliptic pencil. There are three double fibers, corresponding to the lines T_i.

For the second Du Val double plane construction, we let T_1, T_2, T_3, T_4 be four lines through a point P and let C be a degree fourteen plane curve with an order six singularity at P and four $(4, 4)$ points P_1, P_2, P_3, P_4 with tangent T_1, T_2, T_3, T_4, respectively. Assume C also has one $(3, 3)$ point P_5 and one ordinary order four singularity at P_6. We blow up the plane at $P, P_1, \dots, P_6$, and at the singularities above $P_1, \dots, P_5$ to obtain a rational surface Y with $K_Y^2 = -3$, and

$$K_Y \equiv -3h + e + \sum_1^5 e_i + 2\sum_1^5 f_i + e_6.$$

Set $\overline{C} + \sum_1^4 \overline{T_i} + \sum_1^5 e_i = 2L$ as the branch curve for our cover, where

$$L \equiv 9h - 5e - 2\sum_1^4 e_i - 5\sum_1^4 f_i - e_5 - 3f_5 - 2e_6.$$

Then the double cover $\pi\colon X \to Y$ is a general type surface with $p_g = q = 0$ and $K_X^2 = -4$. Blowing down the nine -1 curves $T_1, \dots, T_4$ and $e_1, \dots, e_5$ on X gives a minimal surface S with $K_S^2 = 5$.

Again we consider the rational pencil of lines through P and the corresponding pencil $\pi^*(h - e)$ on S. The four lines from the branch curve of the cover give four double fibers of this pencil, corresponding to $\pi^*(T_i + 2f_i)$.

The high degree of the branch locus makes these two constructions difficult to realize. Instead we find surface with $K^2 = 5$ and degree two bicanonical map constructed as bidouble covers. We begin with a brief description of the construction of bidouble covers; see [3] for details.

3 Bidouble Covers

Recall the definition of a *bidouble cover* $f\colon X \to Y$, which is a cover with Galois group $\mathbb{Z}_2 \times \mathbb{Z}_2$. The cover $f\colon X \to Y$ is determined by divisors D_1, D_2, D_3 and line bundles L_1, L_2, L_3 on Y with $2L_i \equiv D_j + D_k$. Write $D = D_1 + D_2 + D_3 = L_1 + L_2 + L_3$ and $R = f^{-1}(D)$, the ramification divisor of f. We have $f|_R$ is degree two onto D. By standard arguments for covers,

$$\begin{aligned} f_*\mathcal{O}_X &\cong \mathcal{O}_Y \oplus \mathcal{O}_Y(-L_1) \oplus \mathcal{O}_Y(-L_2) \oplus \mathcal{O}_Y(-L_3) \\ p_g(X) &= p_g(Y) + \Sigma_1^3 h^0(Y, \mathcal{O}_Y(K_Y + L_i)) \\ K_X^2 &= (2K_Y + D)^2 \end{aligned}$$

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and

$$H^0(X, \mathcal{O}_X(2K_X)) = H^0((Y, \mathcal{O}_Y(2K_Y + D)) \oplus \bigoplus_1^3 H^0((Y, \mathcal{O}_Y(2K_Y + D - L_i)).$$

The cover X is a smooth surface when the base surface Y is smooth, the branching divisors D_1, D_2, D_3 are smooth, and D has normal crossings. Otherwise a resolution of singularities is equivalent to blowing up any singular points of D on Y and then forming the cover.

Thus for smooth bidouble covers of the plane $\pi\colon S \to \mathbb{P}^2$, we have

$$p_g(S) = \dim H^0(\mathbb{P}^2, \mathcal{O}(a)) + \dim H^0(\mathbb{P}^2, \mathcal{O}(b)) + \dim H^0(\mathbb{P}^2, \mathcal{O}(c))$$

and

$$K_S^2 = (a + b + c - 6)^2,$$

where the line bundles L_1, L_2, L_3 have degrees a, b, c, respectively. When the divisors D_i pass through a point with multiplicity s_i, we write (s_1, s_2, s_3) to denote the singularity, following the notation of Catanese [3].

Example 3.1 **(Burniat Surfaces)** Suppose the D_i are degree three curves in the plane so that each L_i has degree three. Then the cover $f\colon S \to \mathbb{P}^2$ yields a surface with $\chi_S = 4, K_S^2 = 9$. As shown below, imposing certain singularities on D reduces these invariants to $\chi = 1$ and $K^2 = 5$, resulting in a Burniat surface [2].

We consider a bidouble cover of the plane where each D_i is the union of three lines. Start with a tetrahedron of lines in $\mathbb{P}^2$ with vertices at points p_1, p_2, p_3, and p_4. Let D_1 consist of the lines p_1p_2 and p_1p_4, plus another line through p_1. Let D_2 consist of the lines p_2p_3 and p_2p_4 and another line through p_2, and D_3 the lines p_3p_1 and p_3p_4, plus another line through p_3. Thus we have total branching of degree nine, with each D_i of degree three and each line bundle L_i of degree three. The configuration has three singularities of type $(0, 1, 3)$ at the points p_1, p_2, and p_3. The point p_4 is a singularity of type $(1, 1, 1)$. The resulting bidouble cover has $p_g = 0$, $K^2 = 5$. (Similarly one can impose additional $(1, 1, 1)$ singularities to obtain surfaces with $2 \leq K^2 \leq 4$, while the case of three $(0, 1, 3)$ points and no additional singularities yields the $K^2 = 6$ example.)

To obtain the smooth Burniat surface with $K_X^2 = 5$, we first blow up the plane at p_1, p_2, p_3, and p_4 to obtain a rational surface Y with $K_Y \equiv -3h + e_1 + e_2 + e_3 + e_4$, where h is the class of a line and the e_i are the exceptional curves over each p_i. On Y we set the branch divisors for the bidouble cover

$$D_1' \equiv 3h - 3e_1 - e_2 - e_4 + e_3$$

$$D_2' \equiv 3h - 3e_2 - e_3 - e_4 + e_1$$

$$D_3' \equiv 3h - e_1 - 3e_3 - e_4 + e_2$$

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thus we have

$$L_1 \equiv 3h - e_2 - 2e_3 - e_4$$

$$L_2 \equiv 3h - 2e_1 - e_3 - e_4$$

$$L_3 \equiv 3h - e_1 - 2e_2 - e_4.$$

Note that each divisor consists of three lines together with one exceptional divisor from the blowup, with total branching $D = 9h - 3e_1 - 3e_2 - 3e_3 - 3e_4$.

To compute the bicanonical system we use

$$H^0(X, \mathcal{O}_X(2K_X)) = H^0((Y, \mathcal{O}_Y(2K_Y + D)) \oplus \bigoplus_1^3 H^0((Y, \mathcal{O}_Y(2K_Y + D - L_i)).$$

We have

$$2K_Y + D \equiv 3h - e_1 - e_2 - e_3 - e_4$$

which corresponds to $-K_Y$, the system of plane cubics through the four singular points of the branch locus, which embeds Y as a degree five surface in $\mathbb{P}^5$. For $i = 1, 2, 3$ we have $|2K_Y + D - L_i| = \varnothing$, so $H^0(X, \mathcal{O}_X(2K_X)) = H^0((Y, \mathcal{O}_Y(2K_Y + D))$. Thus the bicanonical map for the Burniat surface has degree four.

Moreover X can be realized as a hyperelliptic fibration as follows. Consider the pencil of lines in the plane through the point p_1. The corresponding genus 3 pencil on X has four reducible fibers. Three fibers are the union of two $(-1)-$elliptic curves meeting at one point, corresponding to the lines p_1p_2, p_1p_3, and p_1p_4 in D. The fourth reducible fiber is a double fiber corresponding to the line in D through p_1 only. In the same way, the pencils of lines through p_2 and lines through p_3 correspond to genus three hyperelliptic pencils on X.

Example 3.2 **(Alternate Constructions of the Burniat Surfaces)** Two examples of surfaces with $p_g = 0, K^2 = 5$ have been constructed by Catanese [3, §5] as bidouble covers. We will show each is equivalent to the Burniat construction.

First construct a bidouble cover of the plane with branch divisors of degrees $1, 3$, and 5. Set the degree five component to be a conic and a triangle of lines meeting the conic at vertices p_1, p_2, p_3. For the next component take the union of three lines through a fourth point on the conic, p_4, with one line passing through p_1 and another through p_2. Suppose the last component is a line through p_3; thus D has four singularities of type $(0, 1, 3)$. Blowing up the plane at $p_1, \dots, p_4$ and taking the bidouble cover branched along D_1, D_2, and D_3, where D_1 consists of the proper transforms of the conic and triangle, D_2 the transforms of the triple of lines, together with the exceptional divisor e_3 over p_3, and D_1 is the transform of the line through p_3 together with the exceptional divisors e_1, e_2, e_4 over p_1, p_2, and p_4. We obtain a surface with $p_g = q = 0, K^2 = 5$.

To see that this construction is equivalent to the Burniat surface, consider the Cremona transformation of the plane centered at p_1, p_3, and p_4; that is, let the map $f\colon \mathbb{P}^2 \to \mathbb{P}^2$ be the composition of the blowup of these three points with the contraction of the lines p_1p_3, p_1p_4, and p_3p_4. Write $q_0 = f(p_2)$ and let $q_1 = f(p_1p_3)$,

$q_2 = f(p_1p_4)$, and $q_3 = f(p_3p_4)$ be the contractions of those three lines. Under this transformation, a conic through p_1, p_2, p_3, p_4 is sent to a line through the point q_0; the line p_1p_2 is sent to the line q_0q_3, and the line p_2p_3 is sent to the line q_0q_2. Thus D_1 is transformed to a triple of lines through q_0, with one line through q_2 and another through q_3. The line p_1p_4 in D_2 is blown down, while the other two lines are transformed to lines q_0q_1 and another line through q_1. The other component of D_2, the exceptional divisor e_3, is sent by the Cremona transformation to the line q_1q_2. Finally the components of D_3 are sent to three lines through q_2, including q_1q_2 and q_2q_3. Thus under the quadratic transformation of the plane, the divisors D_1, D_2, D_3 are sent to the branch divisors for the Burniat bidouble cover.

If instead we blow up p_1 and p_2 and contract the line p_1p_2, we have a birational map $\mathbb{P}^2 \to \mathbb{P}^1 \times \mathbb{P}^1$. The configuration of curves is transformed to another construction of Catanese [3]: a bidouble cover of the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$ with branching D_i of bi-degree $(2, 2)$. Each D_i consists of a horizontal and a vertical section that meet at a point q_i, together with a $(1, 1)$ curve passing through q_i and q_{i+1}. Thus we have another description of the Burniat surface as a bidouble cover.

4 Examples with Bicanonical Map of Degree Two

Theorem 4.1 *Surfaces of general type with $p_g = 0$ and $K^2 = 5$ and bicanonical map of degree two can be constructed as bidouble covers of the plane, with branching divisors of degree $2, 4,$ and 6.*

We provide two constructions of surfaces, corresponding to the two cases of Du Val double planes given in Section 2. Both examples are bidouble covers of the plane determined by the three divisors D_1, D_2, D_3 and their singularities.

First, let $P_1, \dots, P_5$ be points in general position in the plane and consider four concurrent lines through P_1: $T_2 = P_1P_2$, $T_3 = P_1P_3$, $S_4 = P_1P_4$, and $S_5 = P_1P_5$. Let Q_1 and Q_2 be conics that are tangent to T_2 at P_2 and T_3 at P_3, with Q_1 through P_4 and Q_2 through P_5. Let C be a cubic, also tangent to T_2 and T_3 at P_2 and P_3, respectively, and passing through the points P_1 and P_4, with a double point at P_5. Then C and Q_1 intersect at each of P_2 and P_3 with multiplicity two, as well as at the point P_4. These two curves must also intersect at one additional point, P_6. Write S_6 for the line P_1P_6.

Set

$$\begin{aligned} D_1 &= Q_1 \\ D_2 &= T_3 + S_4 + S_5 + S_6 \\ D_3 &= T_2 + Q_2 + C. \end{aligned}$$

Thus we have the branch data for a bidouble cover of degrees $2, 4, 6$. The degrees of the line bundles L_i are therefore $3, 4, 5$, so such a cover, if non-singular, would have invariants $p_g = 10$ and $K^2 = 36$. We now compute how the singularities reduce these invariants. The D_i have six singularities, of type $(0, 2, 4)$ at P_1, $(0, 1, 3)$ at P_2, $(1, 1, 2)$ at P_3, $(1, 1, 1)$ at P_4 and P_6, and $(0, 1, 3)$ at P_5. At the points P_2 and P_3, the tangent directions of all components coincide, resulting in "infinitely near" singularities.

To resolve these singularities and obtain a smooth cover, we blow up each P_i, as well as the infinitely near points above P_2 and P_3 corresponding to the tangent directions of T_2 and T_3. Write e_i for the exceptional divisor above each P_i, and f_2, f_3 for the divisor for the second blowups above P_2, P_3. Let Y be the rational surface obtained as this eight-fold blowup of the plane. Then $K_Y^2 = 1$ and we have

$$D_1 \equiv 2h - e_2 - 2f_2 - e_3 - 2f_3 - e_4 - e_6 + e_5,$$
$$D_2 \equiv 4h - 4e_1 - e_3 - 2f_3 - e_4 - e_5 - e_6 + e_2,$$
$$D_3 \equiv 6h - 2e_1 - 3e_2 - 6f_2 - 2e_3 - 4f_3 - e_4 - 3e_5 - e_6 + e_3.$$

Note that we add the exceptional curves e_2, e_3, e_5 to the branch locus to ensure even divisors.

Thus the line bundles for the cover on Y are

$$L_1 \equiv 5h - 3e_1 - e_2 - 3f_2 - e_3 - 3f_3 - e_4 - 2e_5 - e_6,$$
$$L_2 \equiv 4h - e_1 - 2e_2 - 4f_2 - e_3 - 3f_3 - e_4 - e_5 - e_6,$$
$$L_3 \equiv 3h - 2e_1 - f_2 - e_3 - 2f_3 - e_4 - e_6,$$

and $D \equiv 12h - 6e_1 - 3e_2 - 8f_2 - 3e_3 - 8f_3 - 3e_4 - 3e_5 - 3e_6$. The canonical class on Y is $K_Y \equiv -3h + \sum_1^6 e_i + 2f_2 + 2f_3$.

Write $\pi\colon X \to Y$ for the $\mathbb{Z}_2 \times \mathbb{Z}_2$-cover. To compute the geometric genus p_g, we use

$$p_g(X) = p_g(Y) + \sum \dim H^0(Y, \mathcal{O}_Y(K_Y + L_i)).$$

We have

$$K_Y + L_1 \equiv 2h - 2e_1 - f_2 - f_3 - e_5,$$
$$K_Y + L_2 \equiv h - e_2 - 2f_2 - f_3,$$
$$K_Y + L_3 \equiv -e_1 + f_2 + e_5.$$

Clearly all three linear systems are empty, thus $p_g(X) = 0$.

We have $K_X^2 = (2K_Y + D)^2 = -3$. On X the inverse image of each of T_2, T_3, e_2, and e_3 is a disjoint union of two -1-curves. Blowing these eight curves down, we obtain a surface S with $K_S^2 = 5$.

Next we consider the bicanonical system on the cover. Recall that the bicanonical system on X corresponds to

$$H^0(Y, \mathcal{O}_Y(2K_Y + D)) \oplus \bigoplus_1^3 H^0(Y, \mathcal{O}_Y(2K_Y + D - L_i)).$$

If $\alpha_1, \alpha_2, \alpha_3$ are the non-zero elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$, with D_i the fixed divisor of the involution α_i, then $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts on $H^0(2K_Y + D - L_i)$ by the character orthogonal to α_i.

We have

$$2K_Y + D \equiv 6h - 4e_1 - e_2 - 4f_2 - e_3 - 4f_3 - e_4 - e_5 - e_6,$$

a system of plane sextics with a quadruple point at P_1, tacnodes at P_2 and P_3, through P_4, P_5, P_6. The lines T_2 and T_3 and the exceptional divisors e_2, e_3 are components of this system, thus the bicanonical divisor can be written as $2K_X \equiv \pi^*(-K_Y + T) + \pi^*(T_2 + T_3 + e_2 + e_3)$, where T is the pullback to Y of the class of lines through P_1 and $|-K_Y + T| \equiv |4h - 2e_1 - e_2 - 2f_2 - e_3 - 2f_3 - e_4 - e_5 - e_6|$. This system of quartics defines a birational map to $\mathbb{P}^4$.

The spaces $H^0(Y, \mathcal{O}_Y(2K_Y + D - L_1))$ and $H^0(Y, \mathcal{O}_Y(2K_Y + D - L_2))$ are easily seen to be empty, while $\dim H^0(2K_Y + D - L_3) = 1$, with $2K_Y + D - L_3 = T_2 + T_3 + L_{25}$, where L_{25} is the line P_2P_5. Since $\dim H^0(S, \mathcal{O}_S(2K_S)) = 6$, we see that S is a minimal surface. As $h^0(2K_Y + D - L_3) = 1$ and $h^0(2K_Y + D - L_i) = 0$ for $i = 1, 2$, the bicanonical map has degree two and coincides with the involution α_3 of the bidouble cover.

To realize this surface as a hyperelliptic fibration as in Borrelli's classification, consider the pencil of lines through P_1. This gives a rational basepoint free pencil on Y, $|h - e_1|$, which pulls back to a hyperelliptic genus three pencil on S. There are three double fibers, corresponding to the three fibers containing T_2, T_3, and e_5.

For the second construction, we let P_0 be a point in general position to $P_1, \dots, P_4$, and set

$$D_1 = Q_1$$

$$D_2 = A + T_3$$

$$D_3 = T_0 + T_2 + S_4 + S_1$$

where T_0 is the line P_0P_1, S_1 is a line through P_1, and A is a quintic through P_1 and P_4, with tacnodes at P_0, P_2, and P_3, tangent to T_0 at P_0, and tangent to Q_1 at P_2 and P_3.

Thus, again we have branch data for a bidouble cover of degrees $2, 4, 6$. The branching has five singularities, of type $(0, 1, 2)$ at P_0, $(0, 2, 4)$ at P_1, $(1, 1, 2)$ at P_2, $(0, 1, 3)$ at P_3, and $(1, 1, 1)$ at P_4. The singularities at the points P_0, P_2, and P_3 are infinitely near as the tangent directions of all components of D_1, D_2, and D_3 coincide at these points.

To resolve the singularities and obtain a smooth cover, we blow up each P_i, as well as the infinitely near points above P_0, P_2, and P_3 corresponding to the tangent directions of T_0, T_2, and T_3. Write e_i for the exceptional divisor above P_i, and f_0, f_2, f_3 for the divisor for the second blowups above P_0, P_2, P_3. Let Y be the rational surface obtained as this eight-fold blowup of the plane. Then $K_Y^2 = 1$ and we have

$$D_1 \equiv 2h - e_2 - 2f_2 - e_3 - 2f_3 - e_4,$$

$$D_2 \equiv 6h - 2e_0 - 4f_0 - 2e_1 - 2e_2 - 4f_2 - 3e_3 - 6f_3 - e_4 + e_2,$$

$$D_3 \equiv 4h - e_0 - 2f_0 - 4e_1 - e_2 - 2f_2 - e_4 + e_0 + e_3.$$

Note that we add the exceptional curves e_0, e_2, e_3 to the branch locus to ensure even divisors.

Thus the line bundles for the cover on Y are

$$L_1 \equiv 5h - e_0 - 3f_0 - 3e_1 - e_2 - 3f_2 - e_3 - 3f_3 - e_4,$$
$$L_2 \equiv 3h - f_0 - 2e_1 - e_2 - 2f_2 - f_3 - e_4,$$
$$L_3 \equiv 4h - e_0 - 2f_0 - e_1 - e_2 - 3f_2 - 2e_3 - 4f_3 - e_4,$$

and $D \equiv 12h - 2e_0 - 6f_0 - 6e_1 - 3e_2 - 8f_2 - 3e_3 - 8f_3 - 3e_4$. In this example the canonical divisor on Y is $K_Y = -3h + e_0 + 2f_0 + e_1 + e_2 + 2f_2 + e_3 + 2f_3 + e_4$.

Write $\pi \colon X \to Y$ for the $\mathbb{Z}_2 \times \mathbb{Z}_2$-cover. We have

$$K_Y + L_1 \equiv 2h - f_0 - 2e_1 - f_2 - f_3,$$
$$K_Y + L_2 \equiv e_0 + f_0 - e_1 + f_3,$$
$$K_Y + L_3 \equiv h - f_2 - e_3 - 2f_3.$$

Clearly all three linear systems are empty, thus $p_g(X) = 0$.

We have $K_X^2 = (2K_Y + D)^2 = -5$. On X the inverse image of each of T_2, T_3, e_0, e_2, and e_3 is a disjoint union of two -1-curves. Blowing these ten curves down we obtain a surface S with $K_S^2 = 5$. We note that the image of T_0 on S is a -2 rational curve, thus K_S is not ample.

Next we compute the degree of the bicanonical map. We have

$$2K_Y + D \equiv 6h - 2f_0 - 4e_1 - e_2 - 4f_2 - e_3 - 4f_3 - e_4,$$

corresponding to a system of plane sextics tangent to T_0 at P_0, with a quadruple point at P_1, tacnodes at P_2 and P_3 along T_2 and T_3, through P_4. As the line T_2 would intersect such a sextic with multiplicity four at both P_1 and P_2, T_2 must be a component of this system. Similarly T_3, e_0, e_2, and e_3 are components, so we have on Y

$$|2K_Y + D| = T_2 + T_3 + e_0 + e_2 + e_3 + |4h - e_0 - 2f_0 - 2e_1 - e_2 - 2f_2 - -e_3 - 2f_3 - e_4|.$$

This system of quartics can be written as $|-K_Y + T|$, where T corresponds to the pullback of lines through P_1, and defines a birational map to $\mathbb{P}^4$. We have $\dim H^0(Y, \mathcal{O}_Y(2K_Y + D - L_2)) = 1$, with $2K_Y + D - L_2 = T_2 + T_3 + h - f_0 - f_3$. Since the systems $|2K_Y + D - L_1|$ and $|2K_Y + D - L_3|$ are empty we have $\dim H^0(S, \mathcal{O}_S(2K_S)) = 6$. Thus S is our minimal surface with $K_S^2 = 5$. The bicanonical map has degree two and coincides with one of the involutions of the bidouble cover.

To realize this surface as a hyperelliptic fibration we again consider the rational pencil on Y of lines through P_1. In this case we have four double fibers, corresponding to the lines T_0, T_2, T_3, and T.

Remark 4.2 In the first construction, we can instead choose branch curves without the singularity at either P_4 or P_6 to obtain a surface with $K^2 = 6$. Avoiding singularities at both points results in a surface with $K^2 = 7$. In the second construction we can obtain a surface with $K^2 = 4$ by imposing an additional singularity of type $(1, 1, 1)$, by taking the line S_1 through an additional intersection point of the conic Q_1 and the quintic A.

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