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Mr W. L. THOMSON, President, in the Chair.

The Proof by Projection of the Addition Theorem in Trigonometry.

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The object of this paper is to remove the difficulty that arises in giving a general proof by projection methods of this theorem, without in any way interfering with the single-valuedness of the position of a radius vector tracing out angles from a given initial position, when the values of the trigonometrical ratios are given.

It is necessary, first of all, to give a clear statement of the definitions and theorems in Projection.

Definition: The projection of a point S on a straight line XY is the foot Z of the perpendicular from S on XY.

As the point S moves *in any manner*, the point Z moves backwards and forwards along XY. If we call the amount of a motion of Z from X towards Y a *positive segment*, and that of a motion in the opposite direction a *negative segment*, the total displacement of Z corresponding to a given motion of S is a positive or a negative segment, which is the algebraic sum of the alternately positive and negative segments which Z describes during the motion. Also, if S is given a succession of motions, the total displacement of Z is the algebraic sum of the displacements due to the several motions (*vide* Theorem IV. *infra.*).

Definition: If S moves along a straight line PQ from P to Q, the positive or negative segment MN described by Z is called the projection of PQ on XY.

The following theorems are then obvious:—

I. The projection of QP on XY = − (the projection of PQ on XY).

II. If UV is equal to, parallel to and in the same direction as PQ the projection of UV on XY = the projection of PQ on XY.

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III. If R be *any point of the unlimited line through* P and Q, so that $PR = n . PQ$ where n is *any real number*,
the projection of PR on $XY = n$ (the projection of PQ on XY);
for the projection K of R lies between M and N, on MN produced, or on NM produced, according as R lies between P and Q, on PQ produced, or on QP produced.

IV. If P and Q be joined by a succession of straight lines

$$PQ_1,\ Q_1Q_2,\ \dots\ Q_{r-1}Q,$$

the projection of PQ on XY

= the sum of the projections of PQ_1, Q_1Q_2, ... $Q_{r-1}Q$ on XY.

The generality of the proof given below of the Addition Theorem depends on Theorem III.

The Trigonometrical Ratios.

Let angles θ be described by the turning in one plane of a straight line OP about a fixed point O in it, from a fixed initial position OA. The words *positive* and *negative* can then obviously be applied to distinguish the two kinds of turning. Let OB be the position of OP when θ is *a positive right angle*, and let AO, BO produced meet the circle described by P in A′ and B′.

Definitions:

The ratio (projection of OP on B′OB: length of OP) is called the *sine* of θ.

The ratio (projection of OP on A′OA: length of OP) is called the *cosine* of θ; etc., etc.

It follows that these trigonometrical ratios are single-valued functions of the position of the vector OP, and that when $\sin\theta$ and $\cos\theta$ are given the position of OP is uniquely defined.

If OQ, OQ′, OQ″ are the positions of OP when $\theta = \alpha$, $-\alpha$ and $\left(\alpha + \frac{\pi}{2}\right)$, it is easy to obtain from consideration of the relative positions of Q, Q′, Q″ on the circle, general proofs of the formulae :

$$\sin(-\alpha) = -\sin\alpha;\ \cos(-\alpha) = \cos\alpha;$$

$$\sin\left(\alpha + \frac{\pi}{2}\right) = \cos\alpha;\ \cos\left(\alpha + \frac{\pi}{2}\right) = -\sin\alpha.$$

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The Addition Theorem. (Fig. 27.)

Let OA_1 be the position of OP when $\Theta = \alpha$, OB_1 when $\Theta = \alpha + \frac{\pi}{2}$; and let angles ϕ be measured by the turning of OP from the initial position OA_1. Let OQ be the position of OP when $\phi = \beta$; then OQ is the position of OP when $\Theta = \alpha + \beta$. Let M_1, N_1 be the projections of Q on $A_1'OA_1$ and $B_1'OB_1$.

We have then

$OQ\cos(\alpha + \beta)$

$= $ projection of OQ on A′OA

$=$ (projection of OM_1 + projection of M_1Q) on A′OA [Thm. IV.]

$=$ (projection of OM_1 + projection of ON_1) on A′OA [Thm. II.]

$= \{$projection of $(\cos\beta . OA_1)$ + projection of $(\sin\beta . OB_1)\}$ on A′OA

$= \{\cos\beta$(projection of OA_1) + $\sin\beta$(projection of OB_1)$\}$ on A′OA [Thm. III.]

$$= \cos\beta . (OA_1\cos\alpha) + \sin\beta\left\{OB_1\cos\left(\alpha + \frac{\pi}{2}\right)\right\};$$

$$\therefore \quad \cos(\alpha + \beta) = \cos\alpha\cos\beta + \cos\left(\alpha + \frac{\pi}{2}\right)\sin\beta$$

$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

Similarly, by projecting on B′OB, we get

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\left(\alpha + \frac{\pi}{2}\right)\sin\beta$$

$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

and the theorems are true whatever be the sign and whatever the magnitude of the angles α and β.