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THE RANGE SEQUENCE OF AN OPERATOR

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Let T be a linear operator on a Banach space X and consider the sequence of ranges

$$X \supset TX \supset T^2X \supset \cdots \supset T^nX \supset \cdots,$$

where the inclusions are not necessarily proper. The linear subspaces $X_n = T^nX$ ($n>0$) are, in general, not closed but they have some remarkable properties [1], [2]. Let $X_0 = X$ and denote by $|x|_0$ ($x \in X_0$) the norm of X_0.

PROPOSITION 1. *For every $n \geq 0$ there is a norm $|x|_n$ on X_n such that $(X_n, |x|_n)$ is a Banach space and the topology of X_n induced by $|x|_n$ is stronger than the topology induced by the inclusion $X_n \subset X_{n-1}$ ($n \geq 1$).*

Proof. We construct $|x|_n$ by recurrence. Note that the space X_1 is algebraically isomorphic with $X/K(T)$, where $K(T)$ is the kernel of T. Indeed, if $y \in X_1$ then there is an $x \in X$ such that $y = Tx$ and the map

$$y \to x+K(T)$$

is such an isomorphism. Then we can define

$$|y|_1 = \inf_{z \in K(T)} |x+z|_0$$

and $(X_1, |y|_1)$ becomes a Banach space, as $X/K(T)$ is a Banach space. This topology is stronger than the topology induced by X. Indeed, with the same notations,

$$|y|_0 = |Tx|_0 \leq |T|_0 \, |x+z|_0 \quad \text{for any } z \in K(T),$$

hence

$$|y|_0 \leq |T|_0 \, |y|_1.$$

We need the following

LEMMA. *Let A be a continuous linear operator on X and X_1 a subspace of X, continuously embedded in X, such that $AX_1 \subset X_1$. If A_1 is the restriction of A to X_1 then A_1 is continuous for the topology of X_1.*

This result is well known and follows easily by applying, for example, the closed graph theorem.

Returning to our proposition, we get that T_1 (i.e. the restriction of T to X_1) is continuous on X_1. Hence the kernel $K(T_1)$ of T_1 is closed in X_1 and this method can be continued.

In what follows we denote by T_n the restriction of T to X_n ($n\geq 0$).

PROPOSITION. 2. *The sequence of spectra $\sigma(T_n)$ ($n\geq 0$) is decreasing.*

Proof. It is sufficient to show that $\sigma(T_1)\subset\sigma(T)$ because the proof of the other inclusions is similar. Notice that for $\lambda\notin\sigma(T)$ the operator $(\lambda-T)^{-1}$ leaves invariant the space X_1 since it commutes with T. By the lemma in the proof of Proposition 1 its restriction is continuous. It is easy to check that $(\lambda-T)_1^{-1}=(\lambda-T_1)^{-1}$. Consequently $\lambda\notin\sigma(T_1)$.

As an application of the above remarks we give a different proof of the main result contained in [3]. First we need further information.

LEMMA. 1. *If $TX=X$ then T^* is bounded below.*

Proof. By the closed graph theorem, there is an $M>0$ such that for any $y\in X$ we may choose an $x\in X$ such that $Tx=y$ and $|x|_0\leq M\,|y|_0$. Therefore we can write

$$|T^*f|_0=\sup_{|x|_0\leq 1}|f(Tx)|\geq\sup_{|y|_0\leq M^{-1}}|f(y)|=M^{-1}\,|f|_0.$$

LEMMA. 2. *If T is quasinilpotent then T cannot be bounded below.*

Proof. Indeed, if T is quasinilpotent then there is a sequence $x_n\in X$ such that $|x_n|_0=1$ and $Tx_n\to 0$.

PROPOSITION 3. *Let T be a quasinilpotent operator. Then either $T^k=0$ for a certain k or T^nX properly contains T^{n+1}, for any $n\geq 0$.*

Proof. Suppose $T^n\neq 0$ for any $n\geq 1$. By Proposition 2 we have $\sigma(T_n)\subset\sigma(T)=\{0\}$, hence T_n is quasinilpotent for any $n\geq 1$. Since T_n^* is also quasinilpotent then, according to Lemma 2, T_n^* cannot be bounded below. Then Lemma 1 implies $T_nX_n=X_{n+1}\neq X_n$.

This last result has a very elegant proof in [3], depending upon formal power series. However, our method might be an explanation of its existence. The author is grateful to the referee for some improvements.

References

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