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GROUPS WITHOUT NEARLY ABNORMAL SUBGROUPS

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1. Introduction. A subgroup M of an infinite group G is said to be *nearly maximal* if it is a maximal element of the set of all subgroups of G having infinite index; i.e. if the index $|G : M|$ is infinite but every subgroup of G properly containing M has finite index in G. The *near Frattini subgroup* $\psi(G)$ of an infinite group G can now be defined as the intersection of all nearly maximal subgroups of G, with the stipulation that $\psi(G) = G$ if G has no nearly maximal subgroups. These concepts have been introduced by Riles [**5**]. It was later proved by Lennox and Robinson [**4**] that a finitely generated soluble-by-finite group G is infinite-by-nilpotent if and only if all its nearly maximal subgroups are normal. It follows that in the class of finitely generated soluble-by-finite groups the property of being finite-by-nilpotent is inherited from the near Frattini factor group $G/\psi(G)$ to the group G itself. In the study of ordinary Frattini properties of infinite groups, some analogies exist between the behaviour of finitely generated soluble groups and soluble minimax residually finite groups (see for instance [**6**] and [**7**]). This fact could suggest that a result corresponding to that of Lennox and Robinson also holds for soluble residually finite minimax groups. Unfortunately in this case the property of being finite-by-nilpotent cannot be detected from the behaviour of nearly maximal subgroups, this phenomenon depending on the fact that infinite soluble residually finite minimax groups may be poor of such subgroups. On the other hand, Lennox and Robinson observed also that in any finitely generated group G the normality of nearly maximal subgroups is equivalent to the property that every subgroup of infinite index of G is contained in a normal subgroup of G with infinite index. This remark holds of course for all groups in which every subgroup of infinite index lies in a nearly maximal subgroup.

We shall say that a subgroup H of a group G is *nearly abnormal* in G if the index $|G : H|$ is infinite but the normal closure H^G of H has finite index in G. Thus the above quoted theorem of Lennox and Robinson is equivalent to the affirmation that a finitely generated soluble group is finite-by-nilpotent if and only if it does not contain nearly abnormal subgroups. The main result of this article will show that in this form this theorem can also be proved for soluble-by-finite minimax groups.

Most of our notation is standard and can for instance be found in [**8**]. Recall in particular that a group has *finite abelian section rank* if it does not contain infinite abelian sections of prime exponent. Moreover, a group G is said to have *torsion-free rank* n (where n is a non-negative integer) if it has a series of finite length whose factors either are periodic or infinite cyclic and the number of infinite cyclic factors in this series is n. Finally, a soluble-by-finite group G is said to be *minimax* if it has a series of finite length whose factors either are finite or infinite cyclic or of type p^∞ for some prime p. The number of infinite factors in such a series is an invariant, called the *minimax rank* of G.

2. Statements and proofs. Our first result deals with nearly abnormal subgroups of groups with finite abelian section rank.

THEOREM 1. *Let G be a finite-by-nilpotent group with finite abelian section rank. Then G does not contain nearly abnormal subgroups.*

Proof. Assume by contradiction that G has a nearly abnormal subgroup H, and let E be a finite normal subgroup of G such that the factor group G/E is nilpotent. Clearly HE/E is a nearly abnormal subgroup of G/E, and hence replacing G by G/E it can be assumed that the group G is nilpotent. Consider all nilpotent counterexamples with minimal nilpotency class, and among them choose one G with minimal torsion-free rank. As the factor group G/H_G is also a minimal counterexample, we may suppose that H does not contain non-trivial normal subgroups of G, and in particular $H \cap Z(G) = 1$. The factor group $G/Z(G)$ does not contain nearly abnormal subgroups, and hence $HZ(G)$ has finite index in G, so that the subgroup H has finitely many conjugates in G. Let T be the largest periodic normal subgroup of G, and assume first that HT has finite index in G. If z is any element of $Z(G)$, there exists a positive integer m such that $z^m = hx$, where h belongs to H and x is an element with finite order n. Then $z^{mn} = h^n$ lies in $H \cap Z(G) = 1$, so that $z^{mn} = 1$ and the centre $Z(G)$ is periodic. On the other hand, the index $|G : HZ(G)|$ is finite, and hence for any element g of G there exists a positive integer k such that $g^k \in H$. Since H has finitely many conjugates in G and $H_G = 1$, it follows that G is a periodic group. For every prime number p, let G_p and H_p denote the unique Sylow p-subgroups of G and H, respectively. As the normal closure H^G of H has finite index in G, the set π of all primes p such that H_p is properly contained in G_p is infinite, and hence the group $G_\pi = \langle G_p | p \in \pi \rangle$ satisfies the minimal condition on subgroups. Thus the commutator subgroup G'_π of G_π is finite (see [**8**] Part 1, Theorem 4.12), and so the subgroup H_π has finite index in its normal closure in G_π. It follows that H has finite index in H^G, and this contradiction shows that the index $|G : HT|$ is infinite. Then the factor group G/T is also a minimal counterexample, and hence without loss of generality it can be assumed that G is a torsion-free nilpotent group. Moreover, by the minimality of the torsion-free rank of G, also in this new counterexample the subgroup H has trivial core, and in particular $H \cap Z(G) = 1$. As the index $|G : HZ(G)|$ is finite, the centre $Z(H)$ of H is contained in the FC-centre of G. Then $Z(H)$ is also contained in $Z(G)$ (see [**8** Part 1, p.131]); that is impossible. This last contradiction completes the proof of the theorem.

In the above theorem the hypothesis that the group G has finite abelian section rank cannot be omitted, since for each prime number p there exists a nilpotent p-group of class 2 containing a nearly abnormal subgroup. To see this, let A and B be isomorphic infinite abelian groups of expotent p, and let φ be an isomorphism of A onto B. An automorphism x with order p of the group $C = A \times B$ can be defined by putting $a^x = aa^\varphi$ and $b^x = b$ for all $a \in A$ and $b \in B$. Then the semidirect product $G = \langle x \rangle \ltimes C$ is a nilpotent p-group of class 2, and A is a nearly abnormal subgroup of G.

Note also that, if K is an infinite group and E is any finite non-trivial group, then K is a subnormal nearly abnormal subgroup of the wreath product $K \wr E$. Choosing K to be polycyclic and E to be soluble, we obtain in particular that there exist polycyclic groups containing subnormal nearly abnormal subgroups. A similar conclusion also holds for metabelian Černikov p-groups (p prime), choosing K to be of type p^∞ and E of order p.

In the proof of our main theorem, the following cohomological result due to Robinson (see [**9**, Lemma 10]) will be essential.

Lemma 2. *Let G be a group and let A be an abelian normal subgroup of G with finite total rank such that the cohomology class of the extension $A \rightarrowtail G \twoheadrightarrow G/A$ has finite order m. Then there exists a subgroup X of G such that the index $|G : XA|$ and the subgroup $X \cap A$ are finite. More precisely, $|G : XA| \leq |A : A^m|$ and $|X \cap A| \leq |A[m]|$, so that in particular, $XA = G$ if A is divisible and $X \cap A = 1$ if A is torsion-free.*

Lemma 3. *Let G be a group without nearly abnormal subgroups, and let X be a subgroup of finite index of G. Then also X does not contain nearly abnormal subgroups.*

Proof. Let H be any subgroup of infinite index of X. Since H is not nearly abnormal in G, also the index $|G : H^G|$ must be infinite. Then $H^G \cap X$ has infinite index in X, so that $|X : H^X|$ is infinite, and H is not nearly abnormal in X.

We will also need the following easy result, a proof of which can be found in [**1**].

Lemma 4. *Let G be a group with finite abelian section rank, and let N be a nilpotent normal subgroup of G. If the factor group G/N' is finite-by-nilpotent, then also G is finite-by-nilpotent.*

Theorem 5. *Let G be a soluble-by-finite minimax group. Then G is finite-by-nilpotent if and only if it does not contain nearly abnormal subgroups.*

Proof. It follows from Theorem 1 that any finite-by-nilpotent minimax group cannot contain nearly abnormal subgroups. Conversely, assume that the statement is false, and let G be a counterexample with minimal minimax rank m. Suppose first that the finite residual J of G is not trivial, and let L be an infinite minimal G-invariant subgroup of J. Then the factor group G/L is finite-by-nilpotent and $[L, G] = L$ since L is a divisible abelian group. This means that $H^0(G/L, L) = 0$, so that the second cohomology group $H^2(G/L, L)$ has finite exponent (see [**3**], Theorem H), and by Lemma 2 there exists a subgroup Y of G such that $G = LY$ and $L \cap Y$ is finite. Clearly $L \cap Y$ is a normal subgroup of G, and the factor group $G/L \cap Y$ is also a counterexample with minimax rank m, so that without loss of generality it can be assumed that $L \cap Y = 1$. Then $J = (J \cap Y) \times L$ and $G/J \cap Y$ is not finite-by-nilpotent. Since $J \cap Y \simeq J/L$ is divisible, it follows that $J \cap Y = 1$ and $J = L$ has no infinite proper G-invariant subgroups. The subgroup Y has infinite index in G, so that also the index $|G : Y^G|$ is infinite and J is not contained in Y^G. Thus the divisible normal subgroup $[J, Y]$ of G is properly contained in J, so that $[J, Y] = 1$ and J is a subgroup of $Z(G)$. This contradiction shows that $J = 1$, so that the group G is residually finite and its largest periodic normal subgroup T is finite. In particular, G/T is not finite-by-nilpotent, and replacing G by G/T we may also suppose that G does not contain periodic non-trivial normal subgroups. Then the Fitting subgroup F of G is torsion-free nilpotent and the factor group G/F is finitely generated and abelian-by-finite (see [**8**] Part 2, Theorem 10.33). Moreover, it follows from Lemma 4 that G/F is not finite-by-nilpotent, so that $F' = 1$ and F is abelian. Assume that F contains an element $x \neq 1$ having only finitely many conjugates in G. Then the normal closure $\langle x\rangle^G$ is finitely generated and the factor group $G/\langle x\rangle^G$ is finite-by-nilpotent, so that there exists a non-negative integer n such that $\gamma_{n+1}(G)\langle x\rangle^G/\langle x\rangle^G$ is finite. It follows that $\gamma_{n+1}(G)$ is polycyclic-by-finite, and hence also $G/Z_{2n}(G)$ is a polycyclic-by-finite group (see [**8**] Part 1, p. 119). Then $G/Z_{2n}(G)$ is finite-by-nilpotent (see [**4**], Corollary C), and so G is finite-by-nilpotent. This contradiction shows that every non-trivial element of F has infinitely many conjugates in G. Since G is soluble-by-finite, it follows that its FC-centre is finite, and so trivial. Let K be any non-trivial G-invariant subgroup of F. Then the factor group G/K is finite-by-nilpotent and $H^0(G/K, K) = 0$, so that $H^2(G/K, K)$ has finite exponent (see [**3**], Theorem H), and by Lemma 2 there exists a subgroup X of G such that $X \cap K = 1$ and the index $|G : XK|$ is finite. Moreover, the subgroup X can be chosen in such a way that XK is normal in G. As XF has finite index in G, the centre of XF must be trivial, and so XF is not finite-by-nilpotent. Then also the group $XF/X \cap F$ is not finite-by-nilpotent. On the other hand, XF does not contain nearly abnormal subgroups by Lemma 3, so that $XF/X \cap F$ has minimax rank m, and hence $X \cap F = 1$. In particular, $K = XK \cap F$ has finite index in F. Since XF is a normal subgroup of G, we have that F is the Fitting subgroup of XF. Moreover, XF is also a counterexample with minimal minimax rank, and the above argument shows that every non-trivial XF-invariant subgroup of F has finite index in F. Obviously the subgroup $[F, X]$ is non-trivial, so that $F/[F, X]$ is finite and X^G has finite index in G; that is impossible as the index $|G : X|$ is infinite. This last contradiction completes the proof of the theorem.

In the statement of Theorem 5 the hypothesis that the group G is minimax cannot be replaced by the assumption that G has finite Prüfer rank. In fact, for every prime number p let $\langle a_p\rangle$ be a cyclic group of order p^p, and let x_p be the automorphism of $\langle a_p\rangle$ defined by $a_p^{x_p} = a_p^{1+p}$, so that the semidirect product $G_p = \langle x_p\rangle \ltimes \langle a_p\rangle$ is a metacyclic p-group of class p. Then the direct product $G = \mathrm{D}_p^r G_p$ is a metabelian periodic locally nilpotent group with Prüfer rank 2 which is not nilpotent (and so not even finite-by-nilpotent). On the other hand, it is easy to prove that G has no nearly abnormal subgroups.

Note also that a soluble minimax residually finite group in which all nearly maximal subgroups are normal need not be finite-by-nilpotent. To see this, consider the additive group A of all rational numbers whose denominators are powers of 2, and let x be the automorphism of A defined by $a^x = -a$ for all $a \in A$. Then the semidirect product $G = \langle x\rangle \ltimes A$ is a metabelian minimax residually finite group with trivial centre, and G does not contain nearly maximal subgroups.

It has already been observed that there exist nilpotent groups of class 2 containing nearly abnormal subgroups. On the other hand, in the last part of this article we will show that for a wide class of locally nilpotent groups it is true that all nearly maximal subgroups are normal. Recall here that, if H is a subgroup of a group G, the *isolator* of H in G is the subset $I_G(H)$ consisting of all elements x of G such that $x^n \in H$ for some positive integer n. It is well-known that, if G is locally nilpotent, the isolator of every subgroup of G is likewise a subgroup.

Lemma 6. *Let G be a locally nilpotent group, and let M be a nearly maximal subgroup of G such that $M < N_G(M)$. Then M is normal in G.*

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Proof. Since M is properly contained in its normalizer $N_G(M)$, the subgroup $N_G(M)$ has finite index in G and M is a nearly maximal subgroup of $N_G(M)$. As G is locally nilpotent, we have

$$N_G(I_G(M)) = I_G(N_G(M)) = G$$

(see [**2**, Lemma 4.9]), and hence the isolator $I_G(M)$ of M is normal in G. On the other hand, the group $N_G(M)/M$ is infinite cyclic (see [**8**] Part 1, Theorem 4.33), so that $I_G(M) \cap N_G(M) = M$, and M has finite index in $I_G(M)$. Therefore $M = I_G(M)$ is a normal subgroup of G.

THEOREM 7. *Let G be a locally nilpotent group whose commutator subgroup G' is hypercentral. Then every nearly maximal subgroup of G is normal.*

Proof. Assume that the theorem is false, and let M be a nearly maximal non-normal subgroup of G. Since G' is hypercentral and is not contained in M, there exists a smallest ordinal α such that $Z_\alpha(G')$ is not contained in M. Clearly α is not a limit ordinal and $Z_{\alpha-1}(G')$ lies in M. Then $G/Z_{\alpha-1}(G')$ is also a counterexample, and without loss of generality we may suppose that the centre $Z(G')$ of G' is not contained in M, so that M is a nearly maximal subgroup of $MZ(G')$. The intersection $M \cap G'$ is a normal subgroup of $MZ(G')$, and $M/M \cap G'$ is not normal in $MZ(G')/M \cap G'$ as $N_G(M) = M$ by Lemma 6. Replacing G by $MZ(G')/M \cap G'$, it can now be assumed that $G = MK$, where K is an abelian normal subgroup of G and M is abelian. Obviously we may also suppose that the core M_G of M is trivial, so that in particular $M \cap K = C_M(K) = 1$ and $C_G(K) = K$. Let N be any non-trivial G-invariant subgroup of K. Then M is properly contained in MN, so that the index $|G : MN|$ is finite and $N = K \cap MN$ has finite index in K. It follows in particular that either K is torsion-free or it is a p-group for some prime p. Let x and y be non-trivial elements of M and K, respectively. Then $\langle x, y\rangle \cap K$ is a non-trivial normal subgroup of the nilpotent group $\langle x, y\rangle$, and hence $Z(\langle x, y\rangle) \cap K \neq 1$. It follows that $Z(\langle x, K\rangle)$ is a non-trivial normal subgroup of G, and it is contained in K since $C_G(K) = K$, so that $K/Z(\langle x, K\rangle)$ is finite. Suppose first that K is torsion-free, so that also the locally nilpotent group G is torsion-free. Then also $\langle x, K\rangle/Z(\langle x, K\rangle)$ is torsion-free by a well-known result of Mal'cev, and hence $Z(\langle x, K\rangle) = K$ and $[K, x] = 1$. This contradiction shows that K is a p-group for some prime p. Let S be the subgroup consisting of all elements of order p of K. Then M is nearly maximal in MS, so that MS is also a counterexample, and replacing G by MS we may suppose that K has exponent p. As $K/Z(\langle x, K\rangle)$ is finite, there exists a positive integer n such that $[K, x^n]$ is contained in $Z(\langle x, K\rangle)$. Then $[K, x^{np}] = 1$, so that $x^{np} = 1$ and the group $\langle x, K\rangle/Z(\langle x, K\rangle)$ is also finite. It follows that the commutator subgroup $\langle x, K\rangle'$ of $\langle x, K\rangle$ is a finite non-trivial G-invariant subgroup of K (see [**8**] Part 1, Theorem 4.12). Then also K is finite, and this last contradiction completes the proof of the theorem.

If G is any group, and M is a nearly maximal subgroup of G which is also normal, the factor group G/M is infinite cyclic (see [**8** Part 1, Theorem 4.33]). Therefore Theorem 7 has the following consequence.

COROLLARY 8. *Let G be a periodic locally nilpotent group whose commutator subgroup G' in hypercentral. The G does not contain any nearly maximal subgroups.*

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Let G be a group, and let M be a nearly maximal subgroup of G. It is clear that all finite normal subgroups of G are contained in M, so that Theorem 7 also applies to finite-by-(locally nilpotent) groups with finite-by-hypercentral commutator subgroup, and hence it should be seen in relation with the corresponding remark of Lennox and Robinson [**4**] concerning finite-by-nilpotent groups.

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