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HOW LARGE IS THE JUMP DISCONTINUITY IN THE DIFFUSION COEFFICIENT OF A TIME-HOMOGENEOUS DIFFUSION?

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We consider high-frequency observations from a one-dimensional time-homogeneous diffusion process Y. We assume that the diffusion coefficient σ is continuously differentiable in y, but with a jump discontinuity at some level y, say $y = 0$. We first study sign-constrained kernel estimators of functions of the left and right limits of σ at 0. These functions intricately depend on both limits. We propose a method to extricate these functions by searching for bandwidths where the kernel estimators are stable by iteration. We finally provide an estimator of the discontinuity jump size. We prove its convergence in probability and discuss its rate of convergence. A Monte Carlo study shows the finite sample properties of this estimator.

1. INTRODUCTION

In recent years, the broad availability of high-frequency intraday financial data has led to a considerable collection of works on statistical modeling and inference for jumps of time-continuous stochastic processes. However, methods for estimating volatility jumps have not been as well developed from a statistical point of view. Empirical evidence of jumps in the volatility process was first obtained using econometric techniques developed for jumps in the price process and applied to an observable volatility measure such as the index of implied volatility of S&P 500 index options (VIX, Chicago Board Options Exchange Volatility Index; see Todorov and Tauchen, 2011). Considering the question of whether price and volatility jump together, Jacod and Todorov (2010) introduced left and right local volatility estimators from the neighboring high-frequency price increments at the time where a jump price is suspected. Bibinger and Winkelmann (2018) extended their approach to take into account market microstructure. We refer readers to Chapter 10.5 in Ait-Sahalia and Jacod (2014) for recent results on the almost sure convergence of estimators of the locations of volatility jumps and on the stable

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convergence for jump sizes. The best achievable rate of convergence was shown to be $n^{1/4}$ where n is the number of price increments.

In this paper, we consider a stochastic process satisfying a time-homogeneous stochastic differential equation (SDE) and are interested in the estimation of the size of a discontinuity of the diffusion coefficient function. This discontinuity may generate volatility jumps, but it is not possible to consider local volatility estimators (in time) as proposed in Jacod and Todorov (2010) due to the intricate structure of jump times which are strongly linked with the local time of the diffusion process at the level of the discontinuity. It is necessary to consider the local estimators in space, but not in time, and to take into account constraints on the value of the process itself.

There exists a large econometric literature on the estimation of time-homogeneous diffusion processes with jumps in various asymptotic setups when the drift, diffusion coefficient, jump intensity, and conditional impact of a jump satisfy local Lipschitz and growth conditions (they are very often assumed to be at least twice continuously differentiable functions of the Markov state). Based on n observations of the state variable in the time span $[0,T]$, two asymptotic setups are considered: infill asymptotics, where the distances between adjacent discretely sampled observations asymptotically decrease and T is fixed, and long span asymptotics, where the length of data increases as T tends to infinity. In absence of the jump component, the nonparametric estimation of the diffusion coefficient was first studied by Florens-Zmirou (1993) in an infill asymptotic setup, and then the approach was refined and extended to long span asymptotics in Jiang and Knight (1997), Stanton (1997), Bandi and Phillips (2003), and Renò (2008), among others. In the presence of a finite activity doubly stochastic compound Poisson jump part, the nonparametric estimation of the diffusion coefficient was first considered by Bandi and Nguyen (2003) and Johannes (2004) in a long span asymptotic setup. Mancini and Renò (2011) use the fact that it is possible to disentangle the discontinuous part of the state variable through the squared increments between observations not exceeding a suitable threshold function to propose consistent asymptotic normal estimators in the presence of both finite and infinite activity (finite variation) jumps.

In comparison, the estimation of time-homogeneous diffusion processes when the drift and the diffusion coefficients may be discontinuous has been less studied. However, many research areas are concerned with these types of diffusion processes, e.g., geophysics (LaBolle et al., 2000), population ecology (Cantrell and Cosner, 1999), and finance (Decamps, Goovaerts, and Schoutens, 2006; Rossello, 2012; Gairat and Shcherbakov, 2016).

In a recent paper in mathematical finance (Pigato, 2019), a local volatility model, taking two possible values (σ_- and σ_+) depending on the value of the underlying price with respect to a fixed threshold, has been considered. It was proved that when the threshold is taken at the money (ATM), the ATM implied volatility skew explodes for short maturities. This phenomenon has been observed for several implied volatility surfaces on financial markets. More precisely, it was established that as the time to maturity T tends to 0, the skew is equivalent to

$$-\sqrt{\frac{\pi}{2}}\theta\frac{1}{\sqrt{T}}, \quad \text{for } T \downarrow 0,$$

where $\theta = (\sigma_- - \sigma_+)/(\sigma_- + \sigma_+)$ is the relative jump size of the discontinuity. The European call option price of this model strongly depends on the values σ_-, σ_+ as well as on their difference $\delta = \sigma_- - \sigma_+$. Therefore, the question of a discontinuity in the diffusion coefficient and its size is important since it can provide a possible explanation of this empirically stylized fact.

Lejay and Pigato (2018) recently facilitated the study of the parameters of the oscillating Brownian motion (OBM) for which the diffusion coefficient can take only two values (see Keilson and Wellner, 1978). Lejay, Mordecki, and Torres (2013, 2019) investigated the estimation of the parameter of the skew Brownian motion (SBM) which is intrinsically connected to the oscillating Brownian motion by a simple increasing transformation.

As explained above, we focus on a stochastic process satisfying a time-homogeneous SDE whose diffusion coefficient is continuously differentiable in the values of the process, but with a jump discontinuity at some level. We first study estimators of the right and left limits (in space) of the diffusion coefficient at the discontinuity. However, this approach, which was developed in Lejay and Pigato (2018), cannot be used to estimate these limits because their estimators are built using the whole path of the diffusion process. In contrast, we have to focus on the process when it takes values around the level of the discontinuity. Therefore, we introduce kernel estimators. Nevertheless, these estimators are only able to estimate the right and left limits up to constant factors. Unfortunately, these factors depend on the bandwidth parameters of the kernels as well as the right and left limits themselves. This issue leads us to propose an original method that searches for bandwidth parameters for which the kernel estimators are stable by iteration and from which it is possible to infer the value of the ratio of both right and left limits, and then the size of the discontinuity of the diffusion coefficient.

The remainder of this paper is organized as follows. Section 2 introduces our settings, our estimators, and assumptions for the time homogeneous diffusion to be used in the rest of the paper. In Section 3, we explain how to reduce the study of the asymptotic convergences of the estimators of the right and left limits (in space) of the diffusion coefficient at the discontinuity to the case of the SBM. Some parts of our proofs require the adaptation to the SBM of results on the convergence toward the local time given in Jacod (1998), although some of them have already been extended in Lejay et al. (2019). We shall provide a new central limit theorem for some local time-related statistics for the SBM. In Section 4, we prove the convergence in probability of our estimator and discuss its rate of convergence. We also provide a test of whether there is a discontinuity in the diffusion coefficient. Section 5 introduces a family of alternative estimators and contains a Monte Carlo study that provides evidence of the good finite sample properties of our estimators. Proofs are deferred to Section 7. Additional proofs can be found in the Supplementary Material.

2. SETTINGS, ESTIMATORS, AND ASSUMPTIONS

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ on which a process Y is defined as the (pathwise unique) solution of the following one-dimensional time-homogeneous SDE:

$$dY_t = b(Y_{t-})\,dt + \sigma(Y_{t-})\,dW_t + \int_{|x|\geq 1} G(Y_{t-}, x)\,N(dt, dx), \quad Y_0 = y_0, \tag{1}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion, N is a Poisson random measure on $[0,\infty)\times\mathbb{R}\backslash\{0\}$ associated to a Lévy process (with a finite compensator), b is a bounded Borel measurable function, σ is a continuously differentiable positive function on $\mathbb{R}\backslash\{0\}$ such that $\lim_{y\nearrow 0-}\sigma(y) = \sigma_- < \infty$, $\lim_{y\searrow 0+}\sigma(y) = \sigma_+ < \infty$ (moreover, $\lim_{y\nearrow 0-}\sigma'(y)$ and $\lim_{y\searrow 0+}\sigma'(y)$, where σ' denotes the first derivative of σ on $\mathbb{R}\backslash\{0\}$, exist and are finite), and G is a Borel measurable function such that the mapping $y \to G(y,x)$ is continuous for all $|x|\geq 1$. We denote $\delta = \sigma_- - \sigma_+$ as the difference between the left and right limits of σ at 0.

The existence and pathwise uniqueness of the solution to equation (1) is derived from the interlacing structure of the arrival times of the compound Poisson $\int_{|x|\geq 1} xN(dt,dx)$ (following the approach proposed in Section 6 of Applebaum (2009) and the existence of unique and strong nonexploding solutions between these arrival times by using the arguments developed by Nakao (1972) or LeGall (1984)). It is noteworthy that such a construction is possible because this is the case of "finite activity" jumps. The more general case taking into account jumps with infinite activity is considered in Applebaum (2009) but only under the assumption that σ is Lipschitz (see Section 6.2 in Applebaum (2009)).

Process Y is assumed to be observed at discrete times i/n, $i = 0,\ldots,n$. Let c be a positive constant that will be considered as a bandwidth size. We first study "sign-constrained" kernel statistics (based on the absolute values of the increments of Y) that seem to be natural estimators of the right and left limits (in space) of the diffusion coefficient at the discontinuity (up to the constant factor $\sqrt{2/\pi}$ which is the mean of the absolute value of a standard Gaussian random variable). However, it is important not to include jumps of Y in these estimators. We introduce a sequence of deterministic and positive thresholds u_n such that $u_n = \gamma n^{-\varpi}$ where $\gamma > 0$ and $\varpi \in (0, 1/2)$ to disentangle jumps from the diffusive component. The estimators of the right and left limits are then defined, respectively, by

$$A_-^n(c,u_n) = \frac{\sum_{i=1}^n \mathbb{I}_{\{-c/\sqrt{n} < Y_{(i-1)/n} < 0,\, Y_{i/n} < 0,\, |Y_{i/n} - Y_{(i-1)/n}| \leq u_n\}} \sqrt{n}|Y_{i/n} - Y_{(i-1)/n}|}{\sum_{i=1}^n \mathbb{I}_{\{-c/\sqrt{n} < Y_{(i-1)/n} < 0,\, Y_{i/n} < 0,\, |Y_{i/n} - Y_{(i-1)/n}| \leq u_n\}}},$$

$$A_+^n(c,u_n) = \frac{\sum_{i=1}^n \mathbb{I}_{\{0 < Y_{(i-1)/n} < c/\sqrt{n},\, Y_{i/n} > 0,\, |Y_{i/n} - Y_{(i-1)/n}| \leq u_n\}} \sqrt{n}|Y_{i/n} - Y_{(i-1)/n}|}{\sum_{i=1}^n \mathbb{I}_{\{0 < Y_{(i-1)/n} < c/\sqrt{n},\, Y_{i/n} > 0,\, |Y_{i/n} - Y_{(i-1)/n}| \leq u_n\}}},$$

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if the denominators are positive; otherwise, they are equal to 0. We describe these kernel estimators as sign-constrained estimators because we only consider the increments of Y, i.e., $Y_{i/n} - Y_{(i-1)/n}$, when both $Y_{(i-1)/n}$ and $Y_{i/n}$ have the same sign. Note also that the kernels are therefore asymmetric.

We denote the continuous part of Y by Y^c, that is, $Y_t^c = Y_t - \int_0^t \int_{|x|\geq 1} G(Y_{t-},x)$ $N(dt,dx)$, and assume that σ is bounded (otherwise, it is possible to use a localization procedure). Then,

$$\mathbb{E}[|Y^c_{i/n} - Y^c_{(i-1)/n}|^p] \leq K_p n^{-p/2},$$

for all $p \geq 1$ (by Burkholder–Davis–Gundy inequalities), and Markov's inequality yields

$$\sum_{i=1}^{n} P\left(|Y^c_{i/n} - Y^c_{(i-1)/n}| > u_n\right) \leq K_p n^{-p/2+1}/u_n^p.$$

This quantity goes to 0, as $n \to \infty$, if $p > 1/(1/2-\varpi)$. It follows by the Borel–Cantelli lemma that, for all n large enough, $|Y^c_{i/n} - Y^c_{(i-1)/n}| \leq u_n$, for all $i = 1,\ldots,n$. Let us denote $T_1,\ldots,T_q$ as the q arrival times of the compound Poisson process $\int_{|x|\geq 1} xN(dt,dx)$ over $[0,1]$, and $\nu_q = \inf_{j=1,\ldots,q} |Y_{T_j-}| > 0$. We now impose the following mild regularity condition: the jump marks of Y almost surely do not fall on the boundary of $[0,1]\times\mathbb{R}\backslash\{0\}$ (see, e.g., Proposition 1 in Li, Todorov, and Tauchen (2017)). Given this assumption, when n is large enough, among all intervals $((i-1)/n, i/n]$, exactly q of them contain a single jump. All others contain no jumps at all, and $Y_{i/n} - Y_{(i-1)/n} = Y^c_{i/n} - Y^c_{(i-1)/n}$. Since $\nu_q > 0$, we also have that, for all n large enough, with probability approaching 1, any interval $((i-1)/n, i/n]$ for which $|Y_{(i-1)/n}| < c/\sqrt{n}$ will contain no jumps. It is therefore possible to retain only those intervals where there are no jumps. At the end of the Supplementary Material, we provide a discussion that shows that the study of the asymptotic behavior of $A^n_-(c,u_n)$ and $A^n_+(c,u_n)$ is based on the same methodology as if there were no jumps. The developments of proofs are actually minimally impacted by the presence of a finite activity jump component.

For simplicity of exposition, we therefore assume that process Y is defined as the (pathwise unique) solution of the following one-dimensional time-homogeneous SDE:

$$dY_t = b(Y_t)\,dt + \sigma(Y_t)\,dW_t, \quad Y_0 = y_0. \tag{2}$$

That is, we assume that the finite activity jump component of Y in equation (1) is equal to 0. The estimators $A^n_-(c,u_n)$ and $A^n_+(c,u_n)$ are then replaced by

$$A^n_-(c) = \frac{\sum_{i=1}^n \mathbb{I}_{\{-c/\sqrt{n}<Y_{(i-1)/n}<0,\, Y_{i/n}<0\}}\sqrt{n}|Y_{i/n} - Y_{(i-1)/n}|}{\sum_{i=1}^n \mathbb{I}_{\{-c/\sqrt{n}<Y_{(i-1)/n}<0,\, Y_{i/n}<0\}}},$$

$$A^n_+(c) = \frac{\sum_{i=1}^n \mathbb{I}_{\{0<Y_{(i-1)/n}<c/\sqrt{n},\, Y_{i/n}>0\}}\sqrt{n}|Y_{i/n} - Y_{(i-1)/n}|}{\sum_{i=1}^n \mathbb{I}_{\{0<Y_{(i-1)/n}<c/\sqrt{n},\, Y_{i/n}>0\}}}.$$

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Using volatility estimators based on the auxiliary parameter c may appear as unusual. However, the estimators $A_-^n(c)$ and $A_+^n(c)$ will not be able to estimate what they have intended to estimate, i.e., the right and left limits of the diffusion coefficient at the discontinuity. The auxiliary parameter c is necessary to achieve this goal and will be explained later. Finally, note that our approach differs from the approach developed by Lejay and Pigato (2018). In this study, it is assumed that the diffusion coefficient is constant over $(-\infty,0)$ and $(0,\infty)$ and both values of this function are estimated considering the square increments of the negative and positive parts of the process. We need to consider increments for which $Y_{(i-1)/n}$ is close to the origin (i.e., in the interval $(-c/\sqrt{n}, c/\sqrt{n})$) to locally estimate the right and left limits, and the auxiliary parameter c plays a central role. This local approach substantially changes the asymptotic properties of our estimators compared to Lejay and Pigato (2018).

Let

$$\theta = \frac{\sigma_- - \sigma_+}{\sigma_- + \sigma_+} \in (-1,1)$$

be the “relative” jump size of the discontinuity. For $c > 0$, we define

$$\varphi_\theta(c) = \frac{\int_0^c \left[\mathbb{E}[|Z|\mathbb{I}_{\{Z>-x\}}] + \theta\mathbb{E}[|Z-2x|\mathbb{I}_{\{Z>x\}}]\right] dx}{\int_0^c \left[\bar{\Phi}(-x) + \theta\bar{\Phi}(x)\right] dx},$$

where Z is a standard Gaussian random variable with cumulative and survival distribution functions denoted by Φ and $\bar{\Phi}$, respectively. We will prove that, as $n \to \infty$,

$$A_-^n(c) \overset{ucp_c}{\Longrightarrow} A_{\theta,-}(c) := \sigma_- \times \varphi_{-\theta}(c/\sigma_-) \quad \text{and}$$
$$A_+^n(c) \overset{ucp_c}{\Longrightarrow} A_{\theta,+}(c) := \sigma_+ \times \varphi_{\theta}(c/\sigma_+),$$

where $\overset{ucp_c}{\Longrightarrow}$ means that there is uniform convergence in probability over compact sets of $\mathbb{R}^+\backslash\{0\}$. However, the constants $\varphi_{-\theta}(c/\sigma_-)$ and $\varphi_\theta(c/\sigma_+)$ cannot be estimated directly since they depend on σ_- and σ_+. We propose a method that is based on the fixed points of $A_{\theta,-}$ and $A_{\theta,+}$ to find particular values, c_- and c_+, for which the ratio $A_{\theta,-}(c_-)/A_{\theta,+}(c_+)$ only depends on θ and not on σ_- or σ_+.

For $\theta \in (-1,1)$, we denote s_θ as the unique fixed point of φ_θ, i.e., the constant that satisfies $\varphi_\theta(s_\theta) = s_\theta$. The fixed point of $A_{\theta,-}$ (resp. $A_{\theta,+}$) is denoted as $c_{\theta,-}$ (resp. $c_{\theta,+}$) and satisfies $c_{\theta,-} = \sigma_- \times s_{-\theta}$ (resp. $c_{\theta,+} = \sigma_+ \times s_\theta$).

Let us now define the function $\theta \mapsto H(\theta)$ from $(-1,1)$ to $\mathbb{R}^+$ as

$$H(\theta) := \frac{c_{\theta,-}}{c_{\theta,+}} = \frac{1+\theta}{1-\theta}\frac{s_{-\theta}}{s_\theta}.$$

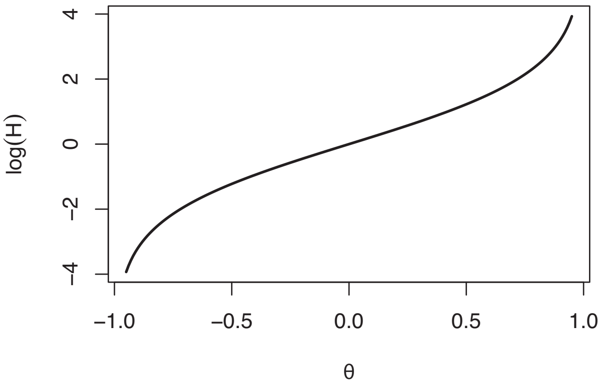

Figure 1. Graphical representation of the function $\log(H)$.

H is a positive, increasing, and one-to-one function on $(-1,1)$ (see Section 7.9 and Figure 1), and satisfies

$$\log(H(-\theta)) = -\log(H(\theta)), \quad \theta \in (-1,1).$$

Therefore, we define estimators of $c_{\theta,-}$, $c_{\theta,+}$, and θ as follows:

$$c_-^n = \arg \min_{A_-^n(c)>0} \{|A_-^n(c) - c|\} \quad \text{and} \quad c_+^n = \arg \min_{A_+^n(c)>0} \{|A_+^n(c) - c|\},$$

and

$$\hat{\theta}_n = H^{-1}\left(c_-^n / c_+^n\right).$$

The estimators of the right and left limits of the diffusion coefficient at the discontinuity are finally given by

$$\hat{\sigma}_-^n = c_-^n / s_{-\hat{\theta}_n} \quad \text{and} \quad \hat{\sigma}_+^n = c_+^n / s_{\hat{\theta}_n},$$

and an estimator of δ is then naturally derived as

$$\hat{\delta}^n = \hat{\sigma}_-^n - \hat{\sigma}_+^n.$$

The choice of the absolute values of the increments of Y has been made to provide theoretical arguments on the existence and uniqueness of the fixed point s_θ, which is the key element for the construction of the estimators of σ_- and σ_+. However, it is possible to propose alternative estimators based on other powers of absolute values of the increments of Y. Such estimators are introduced in Section 5, where we conduct a Monte Carlo study and discuss the finite sample properties of $\hat{\delta}^n$ as well as of the other alternative estimators.

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3. REDUCTION TO THE SBM CASE AND ASYMPTOTIC BEHAVIORS OF SOME LOCAL TIME-RELATED STATISTICS

3.1. Reduction to the SBM Case

It is usual for diffusion processes to consider twice differentiable transformation functions and the Ito formula. In the present case of a discontinuous diffusion coefficient, the considered transformation function will be written as the difference of two convex functions (and therefore will not be necessarily a twice differentiable function everywhere) in order to use the generalized Ito–Tanaka–Meyer formula for continuous semimartingales (see, e.g., Theorem 70 in Protter (2005) or Theorem 7.1 in Karatzas and Shreve (2000)). More specifically, let us consider the function S, which is defined as

$$S(y) = \int_0^y \frac{1}{\sigma(x)} dx, \quad y \in \mathbb{R} \backslash \{0\}, \quad S(0) = 0.$$

This transformation function is well known (Lamperti transform) and is used to obtain a diffusion coefficient equal to 1. Note that, for $y \in \mathbb{R} \backslash \{0\}$,

$$S'(y) = \frac{1}{\sigma(y)}, \quad S''(y) = -\frac{\sigma'(y)}{\sigma^2(y)}.$$

It follows that S may be written as the difference of two convex functions (see, e.g., Problem 6.24, Chapter 3, in Karatzas and Shreve (2000)).

For a process X with continuous paths, we denote $L(X)$ as its symmetric local time at level 0, which is given by

$$L_t(X) = |X_t| - |X_0| - \int_0^t sgn(X_s)\, dX_s,$$

where $sgn(x) = 1$ if $x > 0$, $= -1$ if $x < 0$, and $= 0$ if $x = 0$.

PROPOSITION 1. *By the Ito–Tanaka–Meyer formula for continuous semimartingales,*

$$S(Y_t) = S(y_0) + \int_0^t a(Y_s) ds + W_t + \theta L_t(S(Y)),$$

where

$$a(y) = \left(\frac{b(y)}{\sigma(y)} - \frac{1}{2}\sigma'(y) \right) \mathbb{I}_{\{y \neq 0\}}.$$

Let $X_t := S(Y_t)$, $x_0 := S(y_0)$, and $L_t := L_t(S(Y)) = L_t(X)$, and assume that $b(y) = \sigma'(y)\sigma(y)\mathbb{I}_{\{y\neq 0\}}/2$. Then,

$$X_t = x_0 + W_t + \theta L_t \tag{3}$$

is an SBM (see Lejay (2006) for a presentation of the properties of the SBM as well as various ways to construct this process).

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We now follow the same arguments as developed in Section 2 of Jacod (1998) to reduce the study of the asymptotic behaviors of local time-related statistics for Y to the case of the SBM X.

Let $(\mathcal{G}_t)_{t\geq 0}$ be the filtration generated by Y. First, one may assume that $\mathcal{F}_t=\mathcal{G}_t$ and that it is possible to replace the original space with the canonical space. Second, if the asymptotic results for the convergence of local time-related statistics hold for a given pair (σ,b) (satisfying the assumptions given in Section 2), then they hold for any other pair $(\sigma,\tilde{b})$. As a consequence, we can choose, without loss of generality,

$$b(y)=\frac{1}{2}\sigma'(y)\sigma(y)\mathbb{I}_{\{y\neq 0\}}$$

such that $X_t=S(Y_t)$ is an SBM. Third, if the asymptotic results hold for σ, $1/\sigma$, σ' bounded, they also hold without the boundedness of σ, $1/\sigma$, σ', via a localization procedure. We will therefore assume that σ, $1/\sigma$, σ' are thereafter bounded functions.

Now, we remark that

$$A_-^n(c)=\frac{\sum_{i=1}^n \mathbb{I}_{\{S(-c/\sqrt{n})<X_{(i-1)/n}<0,X_{i/n}<0\}}\sqrt{n}\left|S^{-1}\left(X_{i/n}\right)-S^{-1}\left(X_{(i-1)/n}\right)\right|}{\sum_{i=1}^n \mathbb{I}_{\{S(-c/\sqrt{n})<X_{(i-1)/n}<0,X_{i/n}<0\}}},$$

$$A_+^n(c)=\frac{\sum_{i=1}^n \mathbb{I}_{\{0<X_{(i-1)/n}<S(c/\sqrt{n}),X_{i/n}>0\}}\sqrt{n}\left|S^{-1}\left(X_{i/n}\right)-S^{-1}\left(X_{(i-1)/n}\right)\right|}{\sum_{i=1}^n \mathbb{I}_{\{0<X_{(i-1)/n}<S(c/\sqrt{n}),X_{i/n}>0\}}},$$

where S^{-1} denotes the inverse function of S. For $X_{(i-1)/n}X_{i/n}>0$, using the Lagrange form in the remainder of Taylor's theorem, we obtain

$$S^{-1}\left(X_{i/n}\right)-S^{-1}\left(X_{(i-1)/n}\right)=\sigma\left(Y_{i,n}\right)\left(X_{i/n}-X_{(i-1)/n}\right),$$

where $Y_{i/n}\wedge Y_{(i-1)/n}\leq Y_{i,n}\leq Y_{i/n}\vee Y_{(i-1)/n}$, and

$$S\left(-c/\sqrt{n}\right)=-\frac{1}{\sigma(u_n)}\frac{c}{\sqrt{n}}\quad\text{and}\quad S\left(c/\sqrt{n}\right)=\frac{1}{\sigma(v_n)}\frac{c}{\sqrt{n}},$$

where $-c/\sqrt{n}\leq u_n\leq 0$ and $0\leq v_n\leq c/\sqrt{n}$. Therefore, we have the following approximations when n is large:

$$A_-^n(c)\simeq\sigma_-B_-^n(c/\sigma_-)\quad\text{and}\quad A_+^n(c)\simeq\sigma_+B_+^n(c/\sigma_+),$$

where

$$B_-^n(c)=\frac{\sum_{i=1}^n \mathbb{I}_{\{-c/\sqrt{n}<X_{(i-1)/n}<0,X_{i/n}<0\}}\sqrt{n}\left|X_{i/n}-X_{(i-1)/n}\right|}{\sum_{i=1}^n \mathbb{I}_{\{-c/\sqrt{n}<X_{(i-1)/n}<0,X_{i/n}<0\}}},$$

$$B_+^n(c)=\frac{\sum_{i=1}^n \mathbb{I}_{\{0<X_{(i-1)/n}<c/\sqrt{n},X_{i/n}>0\}}\sqrt{n}\left|X_{i/n}-X_{(i-1)/n}\right|}{\sum_{i=1}^n \mathbb{I}_{\{0<X_{(i-1)/n}<c/\sqrt{n},X_{i/n}>0\}}}.$$

More precisely, we will prove (see the proof of Corollary 1) that

$$n^{1/4}\left|A_-^n(c)-\sigma_- B_-^n(c/\sigma_-)\right| \overset{ucp_c}{\Longrightarrow} 0 \quad \text{and} \quad n^{1/4}\left|A_+^n(c)-\sigma_+ B_+^n(c/\sigma_+)\right| \overset{ucp_c}{\Longrightarrow} 0.$$

Now, we focus on local time-related statistics for the SBM.

3.2. Asymptotic Behaviors of Local Time-Related Statistics for the SBM

We assume that X is the SBM defined as the strong solution of equation (3) and that θ is a constant in $(-1,1)$. The transition density of the SBM is given by

$$p_\theta(t,x,y) = p(t,x-y) + sgn(y)\theta p(t,|x|+|y|),$$

where $p(t,x) = (2\pi t)^{-1/2}\exp\left(-x^2/(2t)\right)$ (see Walsh (1978)). Note that the SBM presents the same scaling property as Brownian motion.

The joint probability density function (pdf) of (X_t, L_t) given that $X_0 = x_0$, denoted as $f^\theta_{x_0,X_t,L_t}(x,l)$, has been entirely characterized in Corollary 3.3 in Appuhamillage et al. (2011a) (see also Appuhamillage et al. (2011b)). Let us, for example, provide the pdf when $x_0 = 0$:

$$f^\theta_{0,X_t,L_t}(x,l) = \begin{cases} \frac{2\alpha(l+x)}{\sqrt{2\pi t^3}}\exp\left(-\frac{(l+x)^2}{2t}\right), & \text{if } x \geq 0, l > 0, \\ \frac{2(1-\alpha)(l-x)}{\sqrt{2\pi t^3}}\exp\left(-\frac{(l-x)^2}{2t}\right), & \text{if } x \leq 0, l > 0, \end{cases} \tag{4}$$

where $\alpha = (1+\theta)/2$. It is deduced that the pdf of L_t is given by

$$f^\theta_{0,L_t}(l) = \frac{2}{\sqrt{2\pi t}}\exp\left\{-\frac{l^2}{2t}\right\},$$

and we observe that it does not depend on θ. In particular, $\mathbb{E}_0^\theta[L_1] = \sqrt{2/\pi}$ and $\mathbb{E}_0^\theta\left[L_1^2\right] = 1$, where $\mathbb{E}_x^\theta$ denotes the expectation of X given $X_0 = x$, when the initial condition has to be explained.

In this section, we first provide Propositions 1 and 2 of Lejay et al. (2019) that extend results of Theorems 1.1 and 1.2a, respectively, of Jacod (1998) for the SBM, and propose an extension when also considering the increments of the local time L. Let us first introduce some notations. For a Lebesgue-integrable function f on $\mathbb{R}$, we let $\lambda(f) = \int f(x)dx$ and

$$\lambda_\theta(f) = (1+\theta)\int_0^{+\infty} f(x)dx + (1-\theta)\int_{-\infty}^0 f(x)dx.$$

For a Borel function f on $\mathbb{R}$ and $\gamma \geq 0$, let $\beta_\gamma(f) = \int_{-\infty}^{+\infty}|x|^\gamma |f(x)|dx$. Let g be a Borel function on $\mathbb{R}$, h be a Borel function on $\mathbb{R}^2$, and f be a Borel function on $\mathbb{R}^3$,

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and define

$$V(g)_t^n = \sum_{i=1}^{\lfloor nt \rfloor} g\left(\sqrt{n} X_{(i-1)/n}\right),$$

$$U(h)_t^n = \sum_{i=1}^{\lfloor nt \rfloor} h\left(\sqrt{n} X_{(i-1)/n}, \sqrt{n}(X_{i/n} - X_{(i-1)/n})\right),$$

$$T(f)_t^n = \sum_{i=1}^{\lfloor nt \rfloor} f\left(\sqrt{n} X_{(i-1)/n}, \sqrt{n}(X_{i/n} - X_{(i-1)/n}), \sqrt{n}(L_{i/n} - L_{(i-1)/n})\right).$$

Finally, we let

$$H_{\theta,h}(x) = \int p_\theta(1,x,y)\, h(x,y-x) dy = \mathbb{E}_x^\theta \left[h(x, X_1 - x)\right],$$

$$G_{\theta,f}(x) = \int f_{x,X_1,L_1}^\theta(y,l) f(x,y-x,l) dy dl = \mathbb{E}_x^\theta \left[f(x, X_1 - x, L_1)\right].$$

We also introduce the following four conditions.

Condition 1. *The function g is bounded on $\mathbb{R}$ and $\beta_2(g) < \infty$.*

Condition 2. *The function h is a Borel function on $\mathbb{R}^2$ such that the functions $H_{\theta,h}$, H_{θ,h^2} satisfy Condition 1.*

Condition 3. *The function f is a Borel function on $\mathbb{R}^3$ such that the functions $G_{\theta,f}$, G_{θ,f^2} satisfy Condition 1.*

Condition 4. *The function h is a Borel function on $\mathbb{R}^2$ such that $|h(x,y)| \leq \bar{h}(x)e^{a|y|}$, where $a \in \mathbb{R}$, the functions $x \to \bar{h}(x)$ and $x \to |x|\bar{h}(x)$ are positive and bounded on $\mathbb{R}$, and $\beta_\gamma(\bar{h}) < \infty$, for some $\gamma > 4$.*

We can now provide the following results for the *ucp_t* convergence (in time) of the local time related statistics $V(g)_t^n$, $U(h)_t^n$, and $T(f)_t^n$ following Propositions 2.4 and 2.7 in Lejay et al. (2019).

PROPOSITION 2. (i) *Let g be a Borel function on $\mathbb{R}$ satisfying Condition 1. Then, as $n \to \infty$, we obtain*

$$\frac{1}{\sqrt{n}} V(g)_t^n \stackrel{ucp_t}{\Longrightarrow} \lambda_\theta(g) L_t.$$

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(ii) *Let h be a Borel function on* $\mathbb{R}^2$ *satisfying Condition 2. Then, as* $n \to \infty$*, we obtain*

$$\frac{1}{\sqrt{n}} U(h)^n_t \overset{ucp_t}{\Longrightarrow} \lambda_\theta(H_{\theta,h})L_t.$$

(iii) *Let f be a Borel function on* $\mathbb{R}^3$ *satisfying Condition 3. Then, as* $n \to \infty$*, we obtain*

$$\frac{1}{\sqrt{n}} T(f)^n_t \overset{ucp_t}{\Longrightarrow} \lambda_\theta(G_{\theta,f})L_t.$$

Let us now discuss the rate of convergence of the statistic $U(h)^n_t/\sqrt{n}$ in the case when $\lambda_\theta(H_{\theta,h}) = 0$, and the weak limit of the normalized statistic. Some additional notations are necessary. We denote $P^\theta = \left(P^\theta_t\right)_{t\geq 0}$ as the semigroup of the SBM with parameter θ: $P^\theta_t f(x) = \int p_\theta(t,x,y) f(y) dy$, where f is a Lebesgue-integrable function. Note that λ_θ is the invariant measure of this semigroup. If $\lambda_\theta(f) = 0$, we let

$$F_\theta(f)(x) = \sum_{j=1}^{\infty} P^\theta_j f(x),$$

which is well defined if $\beta^\theta_2(f) < \infty$ since the series is absolutely convergent by equation (2.4) in Lemma 2.1 in Lejay et al. (2019). Let us now define

$$\Phi_h(x) = F_\theta(H_{\theta,h})(x)$$

and, for a Lebesgue-integrable function f,

$$\bar{H}_{\theta,h,f}(x) = \mathbb{E}^\theta_x\left[(h(x, X_1 - x) f(X_1))\right] = \int p_\theta(1,x,y)\, h(x, y-x) f(y)\, dy.$$

We can also define a bilinear and nonnegative function η_θ in the following way:

$$\eta_\theta(h,\tilde{h}) = \lambda_\theta\left(H_{\theta,h\tilde{h}} + \bar{H}_{\theta,h,\Phi_{\tilde{h}}} + \bar{H}_{\theta,\tilde{h},\Phi_h}\right),$$

where h and $\tilde{h}$ satisfy, respectively, $\lambda_\theta(H_{\theta,h}) = 0$ and $\lambda_\theta(H_{\theta,\tilde{h}}) = 0$, and Condition 4. The fact that η_θ exists is not obvious (see Lemma 1) and that it is nonnegative follows from the fact that this quantity is a limit of nonnegative numbers (see the proof of Proposition 3 as well as the remark at the top of page 511 in Jacod (1998)).

Following Jacod (1998), we expect a rate of convergence of $n^{1/4}$. Let $h = \left(h^i\right)_{1\leq i\leq d}$ be a d-dimensional measurable function such that, for $i = 1,\ldots,d$, $\lambda_\theta(H_{\theta,h^i}) = 0$, and let $U(h)^n = \left(U(h^i)^n\right)_{1\leq i\leq d}$. Therefore, we consider

$$Z^n = \frac{1}{n^{1/4}} U(h)^n$$

as an element of E, the space of càdlàg functions in $\mathbb{R}^d$ endowed with the Skorohod topology.

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Before stating the result, we need to recall the notion of stable convergence, which was introduced in Renyi (1963). Let E be a Polish space, Z_n be a sequence of E-valued random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$, and Z be an E-valued random variable defined on an extension, $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. Then we state that Z_n converges $\mathcal{G}$-stably to Z (and write $Z_n \overset{\mathcal{G}-stably}{\Longrightarrow} Z$) if

$$\lim_{n\to\infty} \mathbb{E}[Uf(Z_n)] = \tilde{\mathbb{E}}[Uf(Z)]$$

for all bounded continuous functions f on E and all bounded $\mathcal{G}$-measurable random variables U. This notion of convergence is stronger than convergence in law—but weaker than convergence in probability.

PROPOSITION 3. *Assume that h^i, $i = 1, \ldots, d$, are bounded Borel functions on $\mathbb{R}^2$ such that $\lambda_\theta(H_{\theta,h^i}) = 0$, $i = 1, \ldots, d$, and satisfy Condition 4. Then,*

$$Z^n \overset{\mathcal{G}_1-stably}{\Longrightarrow} Z,$$

where Z is defined on an extension of the space $(\Omega, \mathcal{G}_1, \mathbb{P})$ and is a $\mathcal{G}_1$-conditional Gaussian continuous martingale with bracket

$$\left\langle Z^i, Z^j \right\rangle = \eta_\theta \left(h^i, h^j\right) L.$$

Note that Proposition 3 is an extension of Theorem 1.2(ii-a) in Jacod (1998) to the case of the SBM when $\lambda_\theta(H_{\theta,h^i}) = 0$, for $i = 1, \ldots, d$. Recently, Mazzonetto (2021) established a more general result (without assuming that $\lambda_\theta(H_{\theta,h^i}) = 0$) using the same type of approach as proposed in Jacod (1998).

4. ASYMPTOTIC PROPERTIES OF THE ESTIMATORS AND A TEST FOR DISCONTINUITY IN THE DIFFUSION COEFFICIENT

We now present the asymptotic properties of the estimators of θ, σ_+, σ_-, and δ. Regardless of the method that is used, estimating these parameters on the basis of observations within the time interval $[0,1]$ is of course possible only if Y (or X) visits the value 0 (where σ is discontinuous) before time 1. Thus, we consider estimating the parameters only if we are inside the following equivalent subsets of Ω:

$$\Omega_0^Y = \{\omega : L_1(Y)(\omega) > 0\} = \Omega_0^X = \{\omega : L_1(X)(\omega) > 0\}.$$

Assuming, for example, that $y_0 = 0$ ensures that $\Omega = \Omega_0^Y = \Omega_0^X$.

We first study the ucp_c convergence of $B_-^n(c)$ and $B_+^n(c)$. Actually, we obtain

$$B_-^n(c) = \frac{U(h_{c,\sigma-})_t^n}{U(h_{c,k-})_t^n} \quad \text{and} \quad B_+^n(c) = \frac{U(h_{c,\sigma+})_t^n}{U(h_{c,k+})_t^n},$$

where

$$h_{c,\sigma-}(x,y) = \mathbb{I}_{\{-c<x<0,\,y+x<0\}}|y|, \quad h_{c,k-}(x,y) = \mathbb{I}_{\{-c<x<0,\,y+x<0\}},$$
$$h_{c,\sigma+}(x,y) = \mathbb{I}_{\{0<x<c,\,y+x>0\}}|y|, \quad h_{c,k+}(x,y) = \mathbb{I}_{\{0<x<c,\,y+x>0\}}.$$

By using Proposition 2, we deduce the following result.

PROPOSITION 4. *As $n \to \infty$, we obtain*

$$B_-^n(c) \overset{ucp_{\mathbb{C}}}{\Longrightarrow} \frac{\lambda_\theta\left(H_{\theta,h_{c,\sigma-}}\right)}{\lambda_\theta\left(H_{\theta,h_{c,k-}}\right)} = \varphi_{-\theta}(c) \quad \text{and} \quad B_+^n(c) \overset{ucp_{\mathbb{C}}}{\Longrightarrow} \frac{\lambda_\theta\left(H_{\theta,h_{c,\sigma+}}\right)}{\lambda_\theta\left(H_{\theta,h_{c,k+}}\right)} = \varphi_\theta(c),$$

in restriction to the set Ω_0^X.

Let us now present some properties of φ_θ.

PROPOSITION 5. *For $\theta \in (-1,1)$, there exists a unique s_θ such that $\varphi_\theta(s_\theta) = s_\theta$. Moreover, $\theta \to s_\theta$ is a differentiable function on $(-1,1)$.*

Let us finally introduce two empirical functions that depend on θ

$$\sigma_-^n(\theta) = s_{-\theta}^{-1} A_-^n\left(c_{\theta,-}\right) \quad \text{and} \quad \sigma_+^n(\theta) = s_\theta^{-1} A_+^n\left(c_{\theta,+}\right).$$

These functions converge in probability, respectively, to σ_- and σ_+. We derive from Propositions 4 and 5 the following results.

PROPOSITION 6. *As $n \to \infty$,*

$$A_-^n(c) \overset{ucp_{\mathbb{C}}}{\Longrightarrow} A_{\theta,-}(c) \quad \text{and} \quad A_+^n(c) \overset{ucp_{\mathbb{C}}}{\Longrightarrow} A_{\theta,+}(c),$$

in restriction to the set Ω_0^Y. Therefore, as $n \to \infty$,

$$\sigma_-^n(\theta) \xrightarrow{P} \sigma_- \quad \text{and} \quad \sigma_+^n(\theta) \xrightarrow{P} \sigma_+,$$

in restriction to the set Ω_0^Y. Moreover, as $n \to \infty$,

$$c_-^n \xrightarrow{P} c_{\theta,-} \quad \text{and} \quad c_+^n \xrightarrow{P} c_{\theta,+},$$

and

$$\hat{\theta}_n \xrightarrow{P} \theta, \quad \hat{\sigma}_-^n \xrightarrow{P} \sigma_-, \quad \hat{\sigma}_+^n \xrightarrow{P} \sigma_+, \quad \hat{\delta}^n \xrightarrow{P} \delta,$$

in restriction to the set Ω_0^Y.

We were not able to provide a central limit theorem for $\left(\hat{\sigma}_-^n, \hat{\sigma}_+^n\right)$. Therefore, we only provide a central limit theorem for $\left(\sigma_-^n(\theta), \sigma_+^n(\theta)\right)$. The main reason for this occurrence is that we are only able to provide such a limit theorem for $\left(B_-^n(c_-), B_+^n(c_+)\right)$ (c_- and c_+ being fixed), but not for $\left(\left(B_-^n(c_1), B_+^n(c_2)\right)\right)_{(c_1,c_2)\in(0,\infty)\times(0,\infty)}$ when considered as a multiparameter process because we failed to prove some necessary tightness conditions for this process.

Let

$$h_{c_-,\theta,-}(x,y) = \mathbb{I}_{\{-c_- < x < 0, y+x<0\}}(|y| - \varphi_{-\theta}(c_-)),$$
$$h_{c_+,\theta,+}(x,y) = \mathbb{I}_{\{0<x<c_+, y+x>0\}}(|y| - \varphi_{\theta}(c_+)).$$

PROPOSITION 7. *As $n \to \infty$, we have*

$$n^{1/4}\begin{pmatrix} B_-^n(c_-) - \varphi_{-\theta}(c_-) \\ B_+^n(c_+) - \varphi_\theta(c_+) \end{pmatrix} \overset{\mathcal{G}_1-stably}{\Longrightarrow} T_\theta(c_-,c_+),$$

in restriction to the set Ω_0^X, where $T_\theta(c_-,c_+)$ is defined on an extension of the space $(\Omega,\mathcal{G}_1,\mathbb{P})$ and is a $\mathcal{G}_1$-conditional centered Gaussian random vector with variance given by

$$L_1^{-1}\begin{pmatrix} \sigma_{11,\theta}(c_-) & \sigma_{12,\theta}(c_-,c_+) \\ \sigma_{21,\theta}(c_-,c_+) & \sigma_{22,\theta}(c_+) \end{pmatrix},$$

where

$$\sigma_{12,\theta}(c_-,c_+) = \sigma_{21,\theta}(c_-,c_+) = \lambda_\theta^{-1}(H_{\theta,h_{c_-,k-}})\lambda_\theta^{-1}(H_{\theta,h_{c_+,k-}})$$
$$\times \lambda_\theta\left(H_{\theta,h_{c_-,\theta,-}h_{c_+,\theta,+}} + \bar{H}_{\theta,h_{c_-,\theta,-},\Phi_{h_{c_+,\theta,+}}} + \bar{H}_{\theta,h_{c_+,\theta,+},\Phi_{h_{c_-,\theta,-}}}\right),$$
$$\sigma_{11,\theta}(c_-) = \lambda_\theta^{-2}(H_{\theta,h_{c_-,k-}})\lambda_\theta\left(H_{\theta,h^2_{c_-,\theta,-}} + 2\bar{H}_{\theta,h_{c_-,\theta,-},\Phi_{h_{c_-,\theta,-}}}\right),$$
$$\sigma_{22,\theta}(c_+) = \lambda_\theta^{-2}(H_{\theta,h_{c_+,k+}})\lambda_\theta\left(H_{\theta,h^2_{c_+,\theta,+}} + 2\bar{H}_{\theta,h_{c_+,\theta,+},\Phi_{h_{c_+,\theta,+}}}\right).$$

We derive the following corollary.

COROLLARY 1. *As $n \to \infty$, we have*

$$n^{1/4}\begin{pmatrix} \sigma_-^n(\theta) - \sigma_- \\ \sigma_+^n(\theta) - \sigma_+ \end{pmatrix} \overset{\mathcal{G}_1-stably}{\Longrightarrow} U_\theta(c_{\theta,-},c_{\theta,+}),$$

in restriction to the set Ω_0^Y, where $U_\theta(c_-,c_+)$ is defined on an extension of the space $(\Omega,\mathcal{G}_1,\mathbb{P})$ and is a $\mathcal{G}_1$-conditional centered Gaussian random vector with variance given by

$$L_1^{-1}$$
$$\begin{pmatrix} s_{-\theta}^{-2}\sigma_-^2\sigma_{11,\theta}(c_{\theta,-}/\sigma_-) & s_{-\theta}^{-1}s_\theta^{-1}\sigma_-\sigma_+\sigma_{12,\theta}(c_{\theta,-}/\sigma_-,c_{\theta,+}/\sigma_+) \\ s_{-\theta}^{-1}s_\theta^{-1}\sigma_-\sigma_+\sigma_{21,\theta}(c_{\theta,-}/\sigma_-,c_{\theta,+}/\sigma_+) & s_\theta^{-2}\sigma_+^2\sigma_{22,\theta}(c_{\theta,+}/\sigma_+) \end{pmatrix}.$$

Although it is not possible to obtain a central limit theorem for $\left(\hat{\sigma}_-^n, \hat{\sigma}_+^n\right)$ at rate $n^{1/4}$, we can provide an upper bound for the convergence rate for $\hat{\delta}^n$ and $\hat{\theta}_n$, $n^{1/5}/(\log n)^{1+\eta}$, for some $\eta > 0$.

PROPOSITION 8. *Let $\eta > 0$. As $n \to \infty$, we have*

$$\frac{n^{1/5}}{(\log n)^{1+\eta}} \left|\hat{\delta}^n - \delta\right| \xrightarrow{P} 0 \quad \textit{and} \quad \frac{n^{1/5}}{(\log n)^{1+\eta}} \left|\hat{\theta}_n - \theta\right| \xrightarrow{P} 0,$$

in restriction to the set Ω_0^Y.

We end this section by proposing a test to decide whether the discontinuity size δ is equal to 0 (i.e., the diffusion coefficient function is continuous): $H_0 : \delta = 0$ versus $H_1 : \delta \neq 0$. A central limit theorem for $\left(\hat{\sigma}_-^n, \hat{\sigma}_+^n\right)$ is actually not needed for such a test. Indeed, $\sigma_- = \sigma_+$ implies that $A_-(c) = A_+(c)$ for all $c > 0$, and a test can be based on the comparison of their empirical versions without considering the fixed points of these functions. We propose the following test statistic: for $c > 0$,

$$T_n(c) = \sqrt{n}\left(A_-^n(c) - A_+^n(c)\right)^2 \frac{L_1^n}{V(c, \sigma^n)},$$

where

$$V(c,\sigma) = \sigma^2\left[\sigma_{11,0}(c/\sigma) + \sigma_{22,0}(c/\sigma) - 2\sigma_{21,0}(c/\sigma, c/\sigma)\right]$$

and

$$\sigma^n = \frac{\lambda_0\left(H_{0,h_d}\right)}{\lambda_0\left(H_{0,h_u}\right)} \frac{\sum_{i=1}^n \mathbb{I}_{\{Y_{(i-1)/n} Y_{i/n} < 0\}} \sqrt{n}\left|Y_{i/n} - Y_{(i-1)/n}\right|}{\sum_{i=1}^n \mathbb{I}_{\{Y_{(i-1)/n} Y_{i/n} < 0\}}},$$

$$L_1^n = \frac{1}{\lambda_0\left(H_{0,h_d}\right)} \frac{1}{\sqrt{n}} \sum_{i=1}^n \mathbb{I}_{\{Y_{(i-1)/n} Y_{i/n} < 0\}},$$

where $h_u(x,y) = |y| \mathbb{I}_{\{x(x+y)<0\}}$ and $h_d(x,y) = \mathbb{I}_{\{x(x+y)<0\}}$.

First, note that using Proposition 2 and similar arguments as developed in the proof of Proposition 6, we can easily establish that $\sigma^n \xrightarrow{P} \sigma$ and $L_1^n \xrightarrow{P} L_1$. Second, using the same approach as in Proposition 7 and Corollary 1, we have that $n^{1/4}\left(A_-^n(c) - A_+^n(c)\right)$ stably converges to a Gaussian distribution. Under H_0, $\theta = 0$, and we deduce that $T_n(c) \xrightarrow{d} \chi^2(1)$, whereas, under H_1, $T_n \xrightarrow{P} \infty$. It is of course possible to generalize this test by considering several values $0 < c_1 < \cdots < c_d$ and the quantity $\sum_{i=1}^d \left(A_-^n(c_i) - A_-^n(c_i)\right)^2$ for the test statistic.

5. EXTENSION TO ESTIMATORS WITH OTHER POWERS AND A SIMULATION STUDY

5.1. A Family of Estimators

Rather than considering the absolute values of the increments in the definitions of $A_-^n(c)$ and $A_-^n(c)$, it is possible to use other powers of the absolute values of the increments for building other estimators. We introduce for $\alpha > 0$ the following quantities:

$$A_-^n(c,\alpha) = \left(\frac{\sum_{i=1}^n \mathbb{I}_{\{-c/\sqrt{n}<Y_{(i-1)/n}<0,\, Y_{i/n}<0\}} \sqrt{n} |Y_{i/n} - Y_{(i-1)/n}|^\alpha}{\sum_{i=1}^n \mathbb{I}_{\{-c/\sqrt{n}<Y_{(i-1)/n}<0,\, Y_{i/n}<0\}}} \right)^{1/\alpha},$$

$$A_+^n(c,\alpha) = \left(\frac{\sum_{i=1}^n \mathbb{I}_{\{0<Y_{(i-1)/n}<c/\sqrt{n},\, Y_{i/n}>0\}} \sqrt{n} |Y_{i/n} - Y_{(i-1)/n}|^\alpha}{\sum_{i=1}^n \mathbb{I}_{\{0<Y_{(i-1)/n}<c/\sqrt{n},\, Y_{i/n}>0\}}} \right)^{1/\alpha},$$

and define

$$\varphi_{\theta,\alpha}(c) = \left(\frac{\int_0^c \left[\mathbb{E}[|Z|^\alpha \mathbb{I}_{\{Z>-x\}}] + \theta \mathbb{E}[|Z-2x|^\alpha \mathbb{I}_{\{Z>x\}}] \right] dx}{\int_0^c \left[\bar{\Phi}(-x) + \theta \bar{\Phi}(x) \right] dx} \right)^{1/\alpha}.$$

Using the same approach as for $A_-^n(c)$ and $A_-^n(c)$, it is possible to prove that, as $n \to \infty$,

$$A_-^n(c,\alpha) \overset{ucp_c}{\Longrightarrow} A_{\theta,\alpha,-}(c,\alpha) := \sigma_- \times \varphi_{-\theta,\alpha}(c/\sigma_-) \quad \text{and}$$
$$A_+^n(c,\alpha) \overset{ucp_c}{\Longrightarrow} A_{\theta,\alpha,+}(c,\alpha) := \sigma_+ \times \varphi_{\theta,\alpha}(c/\sigma_+).$$

Although we were not able to prove the existence of a unique fixed point of $\varphi_{\theta,\alpha}$, denoted by $s_{\theta,\alpha}$, for $\theta \in (-1,1)$, we observe numerically that it is the case. The fixed point of $A_{\theta,\alpha,-}$ (resp. $A_{\theta,\alpha,+}$) is denoted as $c_{\theta,\alpha,-}$ (resp. $c_{\theta,\alpha,+}$) and satisfies $c_{\theta,\alpha,-} = \sigma_- \times s_{-\theta,\alpha}$ (resp. $c_{\theta,\alpha,+} = \sigma_+ \times s_{\theta,\alpha}$). We define the function $\theta \mapsto H_\alpha(\theta)$ from $(-1,1)$ to $\mathbb{R}^+$ as

$$H_\alpha(\theta) := \frac{c_{\theta,\alpha,-}}{c_{\theta,\alpha,+}} = \frac{1+\theta}{1-\theta} \frac{s_{-\theta,\alpha}}{s_{\theta,\alpha}}.$$

We observe numerically that H_α is an increasing and one-to-one function over $(-1,1)$. Therefore, we define estimators of $c_{\theta,\alpha,-}$, $c_{\theta,\alpha,+}$, and θ as follows:

$$c_{\alpha,-}^n = \arg \min_{A_-^n(c,\alpha)>0} \{|A_-^n(c,\alpha) - c|\} \quad \text{and} \quad c_{\alpha,+}^n = \arg \min_{A_+^n(c,\alpha)>0} \{|A_+^n(c,\alpha) - c|\},$$

and

$$\hat{\theta}_{\alpha,n} = H_\alpha^{-1}\left(c_{\alpha,-}^n / c_{\alpha,+}^n \right).$$

The estimators of the right and left limits of the diffusion coefficient at the discontinuity are given by

$$\hat{\sigma}_{\alpha,-}^n = c_{\alpha,-}^n / s_{-\hat{\theta}_n,\alpha} \quad \text{and} \quad \hat{\sigma}_{\alpha,+}^n = c_{\alpha,+}^n / s_{\hat{\theta}_n,\alpha},$$

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Table 1. Biases, standard deviations, and RMSEs of the estimators $\hat{\delta}^n_\alpha$.

	$n = 10^5$			$n = 10^6$		
α	Bias	Sd	RMSE	Bias	Sd	RMSE
1.0	0.0276	0.1589	0.1613	0.0360	0.1184	0.1237
1.5	0.0018	0.1499	0.1499	–0.0021	0.1044	0.1044
2.0	–0.0077	0.1471	0.1473	–0.0097	0.0968	0.0973
2.5	–0.0140	0.1400	0.1407	–0.0114	0.0955	0.0962
3.0	–0.0174	0.1299	0.1310	–0.0163	0.0908	0.0922
3.5	–0.0291	0.1213	0.1248	–0.0261	0.0865	0.0903
4.0	–0.0474	0.1190	0.1281	–0.0396	0.0769	0.0865
4.5	–0.0737	0.1118	0.1339	–0.0725	0.0758	0.1050
5.0	–0.1023	0.1064	0.1476	–0.1069	0.0739	0.1300

and new estimators of δ are then naturally given by

$$\hat{\delta}^n_\alpha = \hat{\sigma}^n_{\alpha,-} - \hat{\sigma}^n_{\alpha,+}.$$

5.2. Simulation Study

In this section, we study the finite sample properties of our estimators of the discontinuity size δ. We work with the OBM with the parameters given by $\sigma_-^2 = 0.7$, $\sigma_+^2 = 0.3$, and initial value $y_0 = 0$. Therefore, $\delta \simeq 0.289$ and $\theta \simeq 0.209$. Note that by considering the OBM, we assume that σ is a piecewise constant function, which is not necessary but simplifies the simulation scheme. This assumption leads to only studying the bias due to the structural forms of the estimators rather than incorporating an additional part due to the fact that $\lim_{y \nearrow 0-} \sigma'(y)$ and $\lim_{y \searrow 0+} \sigma'(y)$ are different from 0.

In Keilson and Wellner (1978), the OBM was constructed as a limit of discrete processes, called Oscillating Random Walks, analogously to how the Brownian motion (BM) is constructed as a limit of Random Walks (see also Lejay and Pigato (2018)). We use this approximation to generate several paths of the OBM with initial condition $Y_0 = 0$.

Then, $c_- \simeq 0.697$ and $c_+ \simeq 0.663$. When $n = 10^5$, $c_-/\sqrt{n} \simeq 2.204 \times 10^{-4}$, and when $n = 10^6$, $c_-/\sqrt{n} \simeq 6.97 \times 10^{-5}$. This implies that the assumption that σ is piecewise constant is not very restrictive here since the optimal bandwidths are small when compared to σ_- and σ_+. Therefore, the additional part of the bias due to the first derivatives of σ will only appear for very large values of $\lim_{y \nearrow 0-} \sigma'(y)$ and $\lim_{y \searrow 0+} \sigma'(y)$.

In Table 1, we provide the biases, standard deviations, and root mean square errors (RMSEs) of the estimators $\hat{\delta}^n_\alpha$ with respect to the values of α. These values have been computed using samples of size 500.

We observe that the biases and standard deviations decrease with α. The smallest absolute values of the biases are obtained when $\alpha = 1.5$, whereas the smallest RMSE values are obtained when $\alpha = 3.5$ and $n = 10^5$ and when $\alpha = 4$ and $n = 10^6$. The biases for these optimal values of α decrease with n.

The changes in the standard deviation with respect to α are the consequences of two opposing effects. On the one hand, when α increases, the one-sided truncated moments in $\varphi_{\theta,\alpha}$ increase, which leads to an increase in the standard deviations of the estimators. On the other hand, for a fixed value of c, when α increases, $A_-^n(c,\alpha)$ (resp. $A_+^n(c,\alpha)$) also increases. Since we look for the value $c_{\alpha,-}^n$ (resp. $c_{\alpha,+}^n$) such that $A_-^n\left(c_{\alpha,-}^n,\alpha\right) \simeq c_{\alpha,-}^n$ (resp. $A_-^n\left(c_{\alpha,+}^n,\alpha\right) \simeq c_{\alpha,+}^n$), we deduce that it is necessary to consider larger values for c to find the fixed point of $A_-^n(c,\alpha)$ (resp. $A_+^n(c,\alpha)$) when α increases. As a consequence, the empirical mean, $A_-^n(c,\alpha)$ (resp. $A_+^n(c,\alpha)$), has more random components which "stabilizes" the ratio and decreases the standard deviation of $c_{\alpha,-}^n$ (resp. $c_{\alpha,+}^n$). The second effect appears to be more important in the simulation study and leads to a decrease in the standard deviation of $\hat{\delta}_\alpha^n$.

We finally observe that the average values of the ratios of the standard deviations when $n = 10^5$ and $n = 10^6$ (over the different values of α) is approximately equal to 1.45 while $10^{1/4} \simeq 1.77$ and $10^{1/5} \simeq 1.58$, respectively, which may suggest that the rate of convergence of the estimators is perhaps not $n^{1/4}$.

6. CONCLUSION

We consider a one-dimensional time-homogeneous diffusion process whose diffusion coefficient is discontinuous at some known level. We develop an estimator of the discontinuity jump size by using asymmetric kernels to evaluate the left and right limits of the diffusion coefficient function at the discontinuity level (σ_- and σ_+). We can guarantee an upper bound on the rate of convergence of our estimator, but it remains to obtain the appropriate rate to show that it is asymptotically normal.

We would like to point out that, since $\lim_{c\to\infty}\varphi_\theta(c) = \sqrt{2/\pi}$ for any $\theta \in (-1,1)$, natural estimators of σ_- and σ_+ are also given by

$$\hat{\sigma}_-^n(u_n) = \sqrt{\frac{\pi}{2}}\frac{\sum_{i=1}^n \mathbb{I}_{\{-1<u_nY_{(i-1)/n}<0,\,Y_{i/n}<0\}}\sqrt{n}|Y_{i/n}-Y_{(i-1)/n}|}{\sum_{i=1}^n \mathbb{I}_{\{-1<u_nY_{(i-1)/n}<0,\,Y_{i/n}<0\}}},$$

$$\hat{\sigma}_+^n(u_n) = \sqrt{\frac{\pi}{2}}\frac{\sum_{i=1}^n \mathbb{I}_{\{0<u_nY_{(i-1)/n}<1,\,Y_{i/n}>0\}}\sqrt{n}|Y_{i/n}-Y_{(i-1)/n}|}{\sum_{i=1}^n \mathbb{I}_{\{0<u_nY_{(i-1)/n}<1,\,Y_{i/n}>0\}}},$$

where $u_n \to \infty$ and $u_n/\sqrt{n} \to 0$. Although such estimators seem to be attractive, their properties cannot be studied with the mathematical tools that we develop in this paper and that extend the results developed in Jacod (1998) to the case of the SBM. Therefore, the study of such estimators and the choice for an optimal sequence (u_n) are left for future research.

Finally, it would be very relevant to propose a statistical test to decide whether the diffusion coefficient function has a discontinuity without knowing the level where the discontinuity is located.

7. PROOFS

The proofs of Propositions 1, 2, and 5–7 and Corollary 1 are given in this section. The proofs of Propositions 3 and 8 are deferred to the Supplementary Material.

The constant K may change from line to line and will depend neither of n, x, y, c nor t.

7.1. Proof of Proposition 2

(i) Condition 1 is equivalent to the conditions of Hypothesis 2.2 in Lejay et al. (2019) since g is bounded on $\mathbb{R}$ and $\beta_\gamma(g) < +\infty$ for $\gamma = 2$, implying that $\beta_\gamma(g) < +\infty$ for $\gamma = 0, 1$. We conclude with Proposition 2.3 of Lejay et al. (2019).

(ii) First note that H_{θ,h^γ} in this paper is equal to F_γ in Lejay et al. (2019) ($f(x,y)$ in this paper has to be replaced by $h(x, y-x)$). The assumption that H_{θ,h^2} satisfies Condition 1 is equivalent to the assumption that F_2 satisfies the conditions of Hypothesis 2.2 in Lejay et al. (2019). The assumption that $H_{\theta,h}$ satisfies Condition 1 is equivalent to the assumption that F_1 satisfies the conditions of Hypothesis 2.2 in Lejay et al. (2019). Note that this last condition also implies that $(F_1)^2$ satisfies the conditions of Hypothesis 2.2 in Lejay et al. (2019). Therefore, the conditions of Hypothesis 2.6 in Lejay et al. (2019) are satisfied, and we can conclude by Proposition 2.3 of Lejay et al. (2019).

(iii) Note that Condition 3 does not appear in Lejay et al. (2019). However, the same arguments as those developed in the proof of Proposition 2.3 in Lejay et al. (2019) can be used to prove the uniform convergence in time replacing Condition 2 by Condition 3.

7.2. Lemma 1

LEMMA 1. *Assume that h is a Borel function satisfying Condition 4, then $\lambda_\theta(|H_{\theta,h}|) < \infty$ and $\lambda_\theta(|H_{\theta,h^2}|) < \infty$. Moreover, if $\lambda_\theta(H_{\theta,h}) = 0$, then $\lambda_\theta(|\bar{H}_{\theta,h,\Phi_{\tilde{h}}}|) < \infty$.*

Proof. From Condition 4, the function h is a Borel function on $\mathbb{R}^2$ such that $|h(x,y)| \le \bar{h}(x)e^{a|y|}$, where $a \in \mathbb{R}$, the function $x \to \bar{h}(x)$ is positive and bounded on $\mathbb{R}$, and $\beta_\gamma(\bar{h}) < \infty$, for some $\gamma \ge 2$.

(i) Since $H_{\theta,h}(x) = \mathbb{E}_x^\theta[h(x, X_1 - x)]$ and $\mathbb{E}_x^\theta\left[e^{a|X_1-x|}\right] < \infty$ from equation (4), we deduce that

$$\lambda_\theta(|H_{\theta,h}|) \le 2\int |H_{\theta,h}(x)|dx \le K\int |\bar{h}(x)|dx = K\beta_0(\bar{h}) < \infty.$$

(ii) Since $H_{\theta,h^2}(x) = \mathbb{E}_x^\theta\left[h^2(x,X_1-x)\right]$ and $\mathbb{E}_x^\theta\left[e^{2a|X_1-x|}\right] < \infty$ from equation (4), we deduce that

$$\lambda_\theta(|H_{\theta,h^2}|) \le 2\int |H_{\theta,h^2}(x)|dx \le K\int |\bar{h}(x)|^2 dx \le K\int |\bar{h}(x)|dx = K\beta_0(\bar{h}) < \infty.$$

(iii) Recall that $\Phi_{\tilde{h}}(x) = F_\theta(H_{\theta,\tilde{h}})(x)$. Note that $\beta_2(H_{\theta,\tilde{h}}) < \infty$, since

$$\beta_2(H_{\theta,\tilde{h}}) = \int |x|^2 |H_{\theta,\tilde{h}}(x)|dx \le K\int |x|^2|\bar{h}(x)|dx = K\beta_2(\bar{h})$$

and then

$$|\Phi_{\tilde{h}}(x)| \le K\left(\beta_2(H_{\theta,\tilde{h}}) + \beta_1(H_{\theta,\tilde{h}})|x|\right)$$

by equation (2.4) of Lemma 2.1 in Lejay et al. (2019). Since $\bar{H}_{\theta,h,\Phi_{\tilde{h}}}(x) = \mathbb{E}_x^\theta\left[\left(h(x,X_1-x)\Phi_{\tilde{h}}(X_1)\right)\right]$ and $\mathbb{E}_x^\theta\left[|X_1|e^{a|X_1-x|}\right] < \infty$ from equation (4), we deduce that

$$\lambda_\theta(|\bar{H}_{\theta,h,\Phi_{\tilde{h}}}|) \le K\int |x||\bar{h}(x)|dx = K\beta_1(\bar{h}) < \infty. \qquad \square$$

7.3. Proof of Proposition 1

Let us denote μ as the measure associated to the second derivative of S. For any once-differentiable function g with compact support, μ satisfies

$$\int_{-\infty}^{\infty} g(x)\mu(dx) = -\int_{-\infty}^{\infty} g'(x)S'(x)dx.$$

Therefore, we deduce that

$$\mu(dx) = -\frac{\sigma'(x)}{\sigma^2(x)}\mathbb{I}_{\{x\neq 0\}}dx + \delta_0(x)\left(\frac{1}{\sigma_+} - \frac{1}{\sigma_-}\right),$$

where δ_0 denotes the Dirac function at 0. By the Meyer–Ito–Tanaka formula (since Y is a continuous semimartingale), we have

$$\begin{aligned} S(Y_t) &= S(y_0) + \int_0^t \frac{1}{\sigma(Y_s)}\left(b(Y_s)ds + \sigma(Y_s)dW_s\right) - \frac{1}{2}\int_0^t \sigma'(Y_s)\mathbb{I}_{\{Y_s\neq 0\}}ds \\ &\quad + \frac{1}{2}\left(\frac{1}{\sigma_+} - \frac{1}{\sigma_-}\right)L_t(Y) \\ &= S(y_0) + \int_0^t a(Y_s)ds + W_t + \frac{1}{2}\left(\frac{1}{\sigma_+} - \frac{1}{\sigma_-}\right)L_t(Y), \end{aligned}$$

where $L(Y) = (L_t(Y))_{t\ge 0}$ is the symmetric local time of Y at 0.

By definition of the local time of $S(Y)$ at level 0, we have

$$|S(Y_t)| = |S(y_0)| + \int_0^t sgn(S(Y_s))dS(Y_s) + L_t(S(Y)).$$

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Since $sgn(S(Y_s)) = sgn(Y_s)$, we also have

$$|S(Y_t)| = |S(y_0)| + \int_0^t sgn(Y_s)a(Y_s)\,ds + \int_0^t sgn(Y_s)dW_s + L_t(S(Y)).$$

Now, we consider the function $y \to |S(y)|$ (which can also be written as the difference of two convex functions), we derive by the Ito–Tanaka–Meyer formula that

$$\begin{aligned}|S(Y_t)| &= |S(y_0)| + \int_0^t \frac{1}{\sigma(Y_s)} sgn(Y_s)\left(b(Y_s)\,ds + \sigma(Y_s)\,dW_s\right)\\ &\quad - \frac{1}{2}\int_0^t sgn(Y_s)\,\sigma'(Y_s)\mathbb{I}_{\{Y_s\neq 0\}}ds + \frac{1}{2}\left(\frac{1}{\sigma_+}+\frac{1}{\sigma_-}\right)L_t(Y)\\ &= |S(y_0)| + \int_0^t sgn(Y_s)a(Y_s)\,ds + \int_0^t sgn(Y_s)dW_s + \frac{1}{2}\left(\frac{1}{\sigma_+}+\frac{1}{\sigma_-}\right)L_t(Y).\end{aligned}$$

Therefore, we deduce that

$$L_t(S(Y)) = \frac{1}{2}\left(\frac{1}{\sigma_+}+\frac{1}{\sigma_-}\right)L_t(Y)$$

and

$$S(Y_t) = S(y_0) + \int_0^t a(Y_s)\,ds + W_t + \theta L_t(S(Y)).$$

7.4. Proof of Proposition 4

Since

$$\begin{aligned}h_{c,\sigma-}(x,y) &= \mathbb{I}_{\{-c<x<0,\,y+x<0\}}|y|, \quad & h_{c,k-}(x,y) &= \mathbb{I}_{\{-c<x<0,\,y+x<0\}},\\ h_{c,\sigma+}(x,y) &= \mathbb{I}_{\{0<x<c,\,y+x>0\}}|y|, \quad & h_{c,k+}(x,y) &= \mathbb{I}_{\{0<x<c,\,y+x>0\}},\end{aligned}$$

it is easily deduced that Condition 2 is satisfied for $h_{c,\sigma-}$, $h_{c,\sigma+}$, $h_{c,k-}$, and $h_{c,k+}$. Therefore, we can use Proposition 2 and deduce that

$$\begin{aligned}&\frac{1}{\sqrt{n}}U(h_{c,\sigma-})_1^n\\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^n \mathbb{I}_{\{-c/\sqrt{n}<X_{(i-1)/n}<0,\,X_{i/n}<0\}}\sqrt{n}\left|X_{i/n}-X_{(i-1)/n}\right| \xrightarrow{P} \lambda_\theta(H_{\theta,h_{c,\sigma-}})L_1,\\ &\frac{1}{\sqrt{n}}U(h_{c,\sigma+})_1^n\\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^n \mathbb{I}_{\{0<X_{(i-1)/n}<c/\sqrt{n},\,X_{i/n}>0\}}\sqrt{n}\left|X_{i/n}-X_{(i-1)/n}\right| \xrightarrow{P} \lambda_\theta(H_{\theta,h_{c,\sigma+}})L_1,\end{aligned}$$

$$\frac{1}{\sqrt{n}}U(h_{c,k-})_1^n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbb{I}_{\{-c/\sqrt{n}<X_{(i-1)/n}<0, X_{i/n}<0\}} \xrightarrow{P} \lambda_\theta(H_{\theta,h_{c,k-}})L_1,$$

$$\frac{1}{\sqrt{n}}U(h_{c,k+})_1^n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbb{I}_{\{0<X_{(i-1)/n}<c/\sqrt{n}, X_{i/n}>0\}} \xrightarrow{P} \lambda_\theta(H_{\theta,h_{c,k+}})L_1.$$

Let us compute the limits.
(i) Case $\sigma-$: We have

$$\begin{aligned} H_{\theta,h_{c,\sigma-}}(x) &= \int p_\theta(1,x,y)\, h_{c,\sigma-}(x,y-x)dy = \int p_\theta(1,x,y)\,\mathbb{I}_{\{-c<x<0,y<0\}}|y-x|dy \\ &= \mathbb{I}_{\{-c<x<0\}}\left[\int_{-\infty}^{0} p(1,x-y)|y-x|dy - \theta\int_{-\infty}^{0} p(1,-x-y)|y-x|dy\right] \\ &= \mathbb{I}_{\{-c<x<0\}}\left[\mathbb{E}[|Z|\mathbb{I}_{\{Z<-x\}}] - \theta\mathbb{E}[|Z-2x|\mathbb{I}_{\{Z<x\}}]\right] \end{aligned}$$

and, therefore,

$$\lambda_\theta(H_{\theta,h_{c,\sigma-}}) = (1-\theta)\int_{-c}^{0}\left[\mathbb{E}[|Z|\mathbb{I}_{\{Z<-x\}}] - \theta\mathbb{E}[|Z-2x|\mathbb{I}_{\{Z<x\}}]\right]dx.$$

In the same way, we have

$$\lambda_\theta(H_{\theta,h_{c,k-}}) = (1-\theta)\int_{-c}^{0}[\Phi(-x) - \theta\Phi(x)]\,dx,$$

and it follows that

$$\frac{\lambda_\theta(H_{\theta,h_{c,\sigma-}})}{\lambda_\theta(H_{\theta,h_{c,k-}})} = \frac{\int_{-c}^{0}\left[\mathbb{E}[|Z|\mathbb{I}_{\{Z<-x\}}] - \theta\mathbb{E}[|Z-2x|\mathbb{I}_{\{Z<x\}}]\right]dx}{\int_{-c}^{0}[\Phi(-x)-\theta\Phi(x)]\,dx}.$$

(ii) Case $\sigma+$: We have

$$\begin{aligned} H_{\theta,h_{c,\sigma+}}(x) &= \int p_\theta(1,x,y)\, h_{c,\sigma+}(x,y-x)dy = \int p_\theta(1,x,y)\,\mathbb{I}_{\{0<x<c,y>0\}}|y-x|dy \\ &= \mathbb{I}_{\{0<x<c\}}\left[\int_{0}^{+\infty} p(t,x-y)|y-x|dy + \theta\int_{0}^{+\infty} p(t,x+y)|y-x|dy\right] \\ &= \mathbb{I}_{\{0<x<c\}}\left[\mathbb{E}[|Z|\mathbb{I}_{\{Z>-x\}}] + \theta\mathbb{E}[|Z-2x|\mathbb{I}_{\{Z>x\}}]\right] \end{aligned}$$

and, therefore,

$$\lambda_\theta(H_{\theta,h_{c,\sigma+}}) = (1+\theta)\int_{0}^{c}\left[\mathbb{E}[|Z|\mathbb{I}_{\{Z>-x\}}] + \theta\mathbb{E}[|Z-2x|\mathbb{I}_{\{Z>x\}}]\right]dx.$$

In the same way, we have

$$\lambda_\theta(H_{\theta,h_{c,k+}}) = (1+\theta)\int_{0}^{c}\left[\bar{\Phi}(-x) + \theta\bar{\Phi}(x)\right]dx,$$

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and it follows that

$$\frac{\lambda_\theta(H_{\theta,h_{c,\sigma+}})}{\lambda_\theta(H_{\theta,h_{c,k+}})}=\frac{\int_0^c\left[\mathbb{E}[|Z|\mathbb{I}_{\{Z>-x\}}]+\theta\mathbb{E}[|Z-2x|\mathbb{I}_{\{Z>x\}}]\right]dx}{\int_0^c\left[\bar{\Phi}(-x)+\theta\bar{\Phi}(x)\right]dx}=\varphi_\theta(c).$$

(iii) Since Z has a symmetric distribution, we have

$$\frac{\int_{-c}^0\left[\mathbb{E}[|Z|\mathbb{I}_{\{Z<-x\}}]-\theta\mathbb{E}[|Z-2x|\mathbb{I}_{\{Z<x\}}]\right]dx}{\int_{-c}^0\left[\Phi(-x)-\theta\Phi(x)\right]dx}$$

$$=\frac{\int_0^c\left[\mathbb{E}[|Z|\mathbb{I}_{\{Z>-x\}}]-\theta\mathbb{E}[|Z-2x|\mathbb{I}_{\{Z>x\}}]\right]dx}{\int_0^c\left[\bar{\Phi}(-x)-\theta\bar{\Phi}(x)\right]dx}=\varphi_{-\theta}(c).$$

Now, since $n^{-1/2}U(h_{c,\sigma-})$, $n^{-1/2}U(h_{c,\sigma+})$, $n^{-1/2}U(h_{c,k-})$, and $n^{-1/2}U(h_{c,k+})$ are increasing functions with respect to c, with continuous limits in c, we deduce that the convergence is locally uniform in c, and finally that

$$B_-^n(c)\overset{ucp_c}{\Longrightarrow}\varphi_{-\theta}(c)\quad\text{and}\quad B_+^n(c)\overset{ucp_c}{\Longrightarrow}\varphi_\theta(c).$$

7.5. Proof of Proposition 5

We split the proof into several parts. Let us denote by φ the pdf of the standard Gaussian distribution.

(i) For $x>0$,

$$\begin{aligned}\mathbb{E}[|Z|\mathbb{I}_{\{Z>-x\}}]&=\int_{-x}^{\infty}|z|\varphi(z)dz\\&=[|z|\left(-\bar{\Phi}(z)\right)]_{-x}^{\infty}+\int_{-x}^{\infty}sgn(z)\bar{\Phi}(z)\,dz\\&=x\bar{\Phi}(-x)-\int_{-x}^{0}\bar{\Phi}(z)\,dz+\int_0^\infty\bar{\Phi}(z)\,dz\\&=x\bar{\Phi}(-x)-\int_{-x}^{0}\bar{\Phi}(z)\,dz+\frac{1}{2}\sqrt{\frac{2}{\pi}}\end{aligned}$$

and

$$\begin{aligned}\int_0^c\mathbb{E}[|Z|\mathbb{I}_{\{Z>-x\}}]dx&=\int_0^c x\bar{\Phi}(-x)\,dx+\frac{1}{2}\sqrt{\frac{2}{\pi}}c-\int_0^c\int_{-x}^0\bar{\Phi}(z)\,dzdx\\&=\int_0^c x\bar{\Phi}(-x)\,dx+\frac{1}{2}\sqrt{\frac{2}{\pi}}c-\int_{-c}^0(x+c)\,\bar{\Phi}(x)\,dx\\&=2\int_0^c x\bar{\Phi}(-x)\,dx+\frac{1}{2}\sqrt{\frac{2}{\pi}}c-c\int_0^c\bar{\Phi}(-x)\,dx.\end{aligned}$$

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(ii) For $x > 0$,

$$
\begin{aligned}
\mathbb{E}[|Z-2x|\mathbb{I}_{\{Z>x\}}] &= \int_x^\infty |z-2x|\varphi(z)dz \\
&= [|z-2x|\left(-\bar{\Phi}(z)\right)]_x^\infty + \int_x^\infty sgn(z-2x)\bar{\Phi}(z)\,dz \\
&= x\bar{\Phi}(x) - \int_x^{2x}\bar{\Phi}(z)\,dz - \int_0^{2x}\bar{\Phi}(z)\,dz + \int_0^\infty \bar{\Phi}(z)\,dz \\
&= x\bar{\Phi}(x) - 2\int_x^{2x}\bar{\Phi}(z)\,dz - \int_0^{x}\bar{\Phi}(z)\,dz + \frac{1}{2}\sqrt{\frac{2}{\pi}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^c \mathbb{E}[|Z-2x|\mathbb{I}_{\{Z>x\}}]dx \\
&= \int_0^c x\bar{\Phi}(x)\,dx + \frac{1}{2}\sqrt{\frac{2}{\pi}}c - \int_0^c\int_0^x \bar{\Phi}(z)\,dzdx - 2\int_0^c\int_x^{2x}\bar{\Phi}(z)\,dzdx \\
&= \int_0^c x\bar{\Phi}(x)\,dx + \frac{1}{2}\sqrt{\frac{2}{\pi}}c - \int_0^c (c-x)\,\bar{\Phi}(x)\,dx - 2\int_0^c\int_x^{2x}\bar{\Phi}(z)\,dzdx \\
&= 2\int_0^c x\bar{\Phi}(x)\,dx + \frac{1}{2}\sqrt{\frac{2}{\pi}}c - c\int_0^c \bar{\Phi}(x)\,dx - 2\int_0^c\int_x^{2x}\bar{\Phi}(z)\,dzdx.
\end{aligned}
$$

(iii) Let

$$
\psi_\theta(c) = \int_0^c \left[\mathbb{E}[|Z|\mathbb{I}_{\{Z>-x\}}] + \theta\mathbb{E}[|Z-2x|\mathbb{I}_{\{Z>x\}}]\right]dx - c\int_0^c \left[\bar{\Phi}(-x) + \theta\bar{\Phi}(x)\right]dx.
$$

We have

$$
\begin{aligned}
\psi_\theta(c) = &-2\left[\int_0^c (c-x)\bar{\Phi}(-x)\,dx + \theta\int_0^c (c-x)\bar{\Phi}(x)\,dx + \theta\int_0^c\int_x^{2x}\bar{\Phi}(z)\,dzdx\right] \\
&+ \frac{1}{2}\sqrt{\frac{2}{\pi}}c(1+\theta) \\
= &-2\left[\int_0^c (c-x)\bar{\Phi}(-x)\,dx + \theta\int_0^{2c}\left(c-\frac{x}{2}\right)\bar{\Phi}(x)\,dx\right] + \frac{1}{2}\sqrt{\frac{2}{\pi}}c(1+\theta) \\
= &-2\left[\int_0^c (c-x)\bar{\Phi}(-x)\,dx + 2\theta\int_0^c (c-x)\,\bar{\Phi}(2x)\,dx\right] + \frac{1}{2}\sqrt{\frac{2}{\pi}}c(1+\theta) \\
= &-2\int_0^c (c-x)\left[\bar{\Phi}(-x) + 2\theta\bar{\Phi}(2x)\right]dx + \frac{1}{2}\sqrt{\frac{2}{\pi}}c(1+\theta).
\end{aligned}
$$

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Note that

$$\varphi_\theta(s_\theta) = s_\theta \quad \Longleftrightarrow \quad \psi_\theta(s_\theta) = 0.$$

The first and second derivatives of ψ_θ with respect to c are given by

$$\psi'_\theta(c) = -2\int_0^c \left[\bar{\Phi}(-x) + 2\theta\bar{\Phi}(2x)\right]dx + \frac{1}{2}\sqrt{\frac{2}{\pi}}(1+\theta),$$

$$\psi''_\theta(c) = -2\left[\bar{\Phi}(-c) + 2\theta\bar{\Phi}(2c)\right].$$

If $\theta \in [-1/2, 1)$, then ψ_θ is a concave function. If $\theta \in (-1, -1/2)$, ψ_θ is convex over $[0, z_\theta]$ and concave over $[z_\theta, \infty)$, where z_θ satisfies $\bar{\Phi}(-z_\theta) + 2\theta\bar{\Phi}(2z_\theta) = 0$. Moreover, $\psi_\theta(0) = 0$, $\psi'_\theta(0) > 0$, and $\lim_{c\to\infty}\psi_\theta(c) = -\infty$. Therefore, there exists a unique positive s_θ such that $\psi_\theta(s_\theta) = 0$ or equivalently $\varphi_\theta(s_\theta) = s_\theta$.

(iv) We have

$$\psi_\theta(s_\theta) = 0$$

with

$$\psi_\theta(c) = -2\int_0^c (c-x)\left[\bar{\Phi}(-x) + 2\theta\bar{\Phi}(2x)\right]dx + \frac{1}{2}\sqrt{\frac{2}{\pi}}c(1+\theta).$$

Since $(\theta, c) \to \psi_\theta(c)$ is differentiable on $(-1,1)\times\mathbb{R}$, s_θ is a differentiable function in θ, and we have

$$0 = \frac{\partial\psi_\theta(s_\theta)}{\partial\theta} = \left.\frac{\partial\psi_\theta(s)}{\partial s}\right|_{s=s_\theta}\frac{\partial s_\theta}{\partial\theta} + \left.\frac{\partial\psi_\theta(s)}{\partial\theta}\right|_{s=s_\theta}.$$

Since

$$\frac{\partial\psi_\theta(s)}{\partial s} = -2\int_0^s \left[\bar{\Phi}(-x) + 2\theta\bar{\Phi}(2x)\right]dx + \frac{1}{2}\sqrt{\frac{2}{\pi}}(1+\theta),$$

$$\frac{\partial\psi_\theta(s)}{\partial\theta} = -4\int_0^s (s-x)\bar{\Phi}(2x)\,dx + \frac{1}{2}\sqrt{\frac{2}{\pi}}s,$$

we have

$$s_\theta\left.\frac{\partial\psi_\theta(s)}{\partial s}\right|_{s=s_\theta} = -2\int_0^{s_\theta} x\left[\bar{\Phi}(-x) + 2\theta\bar{\Phi}(2x)\right]dx,$$

and then

$$\frac{1}{s_\theta}\frac{\partial s_\theta}{\partial\theta} = -\frac{\left.\frac{\partial\psi_\theta(s)}{\partial\theta}\right|_{s=s_\theta}}{s_\theta\left.\frac{\partial\psi_\theta(s)}{\partial s}\right|_{s=s_\theta}} = \frac{-4\int_0^{s_\theta}(s_\theta - x)\bar{\Phi}(2x)\,dx + \frac{1}{2}\sqrt{\frac{2}{\pi}}s_\theta}{2\int_0^{s_\theta} x\left[\bar{\Phi}(-x) + 2\theta\bar{\Phi}(2x)\right]dx}. \tag{5}$$

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7.6. Proof of Proposition 6

(i) We have

$$A_-^n(c) = \sigma_- B_-^n\left(\frac{c}{\sigma(u_n)}\right) + R_-^n(c)$$

with

$$R_-^n(c) = \sum_{i=1}^n \left(\sigma\left(Y_{i,n}\right) - \sigma_-\right) \frac{\mathbb{I}_{\{S(-c/\sqrt{n})<X_{(i-1)/n}<0, X_{i/n}<0\}}\sqrt{n}\left|X_{i/n} - X_{(i-1)/n}\right|}{\sum_{i=1}^n \mathbb{I}_{\{S(-c/\sqrt{n})<X_{(i-1)/n}<0, X_{i/n}<0\}}}.$$

Now, by Taylor's theorem with a Lagrange form for the remainder, we have

$$\sigma\left(Y_{i,n}\right) - \sigma_- = \sigma'\left(W_{i,n}\right) Y_{i,n}, \qquad \text{with } X_{(i-1)/n} < 0 \text{ and } X_{i/n} < 0,$$

where $Y_{i,n} \le W_{i,n} \le 0$. Let $\underline{\sigma}, \bar{\sigma}, \underline{\sigma}', \bar{\sigma}'$ be, respectively, the lower and upper bounds of σ and σ'. Now, note that

$$\begin{aligned}\bar{\sigma}\left(X_{i/n} \vee X_{(i-1)/n}\right) &\le S^{-1}\left(X_{i/n} \vee X_{(i-1)/n}\right) \le Y_{i,n} \le S^{-1}\left(X_{i/n} \wedge X_{(i-1)/n}\right) \\ &\le \underline{\sigma}\left(X_{i/n} \wedge X_{(i-1)/n}\right)\end{aligned}$$

and then

$$\begin{aligned} R_-^n(c) &\le \underline{\sigma}\underline{\sigma}' \frac{1}{\sqrt{n}} \frac{\sum_{i=1}^n \mathbb{I}_{\{S(-c/\sqrt{n})<X_{(i-1)/n}<0, X_{i/n}<0\}} n\left|X_{i/n} - X_{(i-1)/n}\right|\left(X_{i/n} \wedge X_{(i-1)/n}\right)}{\sum_{i=1}^n \mathbb{I}_{\{S(-c/\sqrt{n})<X_{(i-1)/n}<0, X_{i/n}<0\}}}, \\ R_-^n(c) &\ge \bar{\sigma}\bar{\sigma}' \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{\mathbb{I}_{\{S(-c/\sqrt{n})<X_{(i-1)/n}<0, X_{i/n}<0\}} n\left|X_{i/n} - X_{(i-1)/n}\right|\left(X_{i/n} \vee X_{(i-1)/n}\right)}{\sum_{i=1}^n \mathbb{I}_{\{S(-c/\sqrt{n})<X_{(i-1)/n}<0, X_{i/n}<0\}}}. \end{aligned}$$

Let us recall that $S\left(-c/\sqrt{n}\right) = -c/(\sigma(u_n)\sqrt{n})$ where $-c/\sqrt{n} \le u_n \le 0$. Let

$$\begin{aligned} h_{c,\sigma-,u}(x,y) &= \mathbb{I}_{\{-c<x<0, y+x<0\}}|y|\left((x+y)\wedge x\right), \\ h_{c,\sigma-,d}(x,y) &= \mathbb{I}_{\{-c<x<0, y+x<0\}}|y|\left((x+y)\vee x\right). \end{aligned}$$

Then, using Proposition 2, we have

$$\begin{aligned} &\frac{\sum_{i=1}^n \mathbb{I}_{\{S(-c/\sqrt{n})<X_{(i-1)/n}<0, X_{i/n}<0\}} n\left|X_{i/n} - X_{(i-1)/n}\right|\left(X_{i/n} \wedge X_{(i-1)/n}\right)}{\sum_{i=1}^n \mathbb{I}_{\{S(-c/\sqrt{n})<X_{(i-1)/n}<0, X_{i/n}<0\}}} \\ &\quad \overset{ucp_c}{\Longrightarrow} \frac{\lambda_\theta\left(H_{\theta, h_{c/\sigma-,\sigma-,u}}\right)}{\lambda_\theta\left(H_{\theta, h_{c/\sigma-,k-}}\right)}, \\ &\sum_{i=1}^n \frac{\mathbb{I}_{\{S(-c/\sqrt{n})<X_{(i-1)/n}<0, X_{i/n}<0\}} n\left|X_{i/n} - X_{(i-1)/n}\right|\left(X_{i/n} \vee X_{(i-1)/n}\right)}{\sum_{i=1}^n \mathbb{I}_{\{S(-c/\sqrt{n})<X_{(i-1)/n}<0, X_{i/n}<0\}}} \\ &\quad \overset{ucp_c}{\Longrightarrow} \frac{\lambda_\theta\left(H_{\theta, h_{c/\sigma-,\sigma-,d}}\right)}{\lambda_\theta\left(H_{\theta, h_{c/\sigma-,k-}}\right)}. \end{aligned}$$

It follows that

$$R_{-}^{n}(c) \overset{ucp_{\mathcal{C}}}{\Longrightarrow} 0.$$

Since $B_{-}^{n}(c) \overset{ucp_{\mathcal{C}}}{\Longrightarrow} \varphi_{-\theta}(c)$ by Proposition 4, we deduce that

$$A_{-}^{n}(c) \overset{ucp_{\mathcal{C}}}{\Longrightarrow} A_{\theta,-}(c).$$

The same arguments hold for the following convergence:

$$A_{+}^{n}(c) \overset{ucp_{\mathcal{C}}}{\Longrightarrow} A_{\theta,+}(c).$$

(ii) Recall that

$$\sigma_{-}^{n}(\theta) = s_{-\theta}^{-1} A_{-}^{n}\left(c_{\theta,-}\right) \quad \text{and} \quad \sigma_{+}^{n}(\theta) = s_{\theta}^{-1} A_{+}^{n}\left(c_{\theta,+}\right).$$

The convergences $\sigma_{-}^{n}(\theta) \xrightarrow{P} \sigma_{-}$ and $\sigma_{+}^{n}(\theta) \xrightarrow{P} \sigma_{+}$ directly follow from (i), the fact that $A_{\theta,-}(c) = \sigma_{-} \times \varphi_{-\theta}(c/\sigma_{-})$, $A_{\theta,+}(c) = \sigma_{+} \times \varphi_{\theta}(c/\sigma_{+})$, $c_{\theta,-} = \sigma_{-} \times s_{-\theta}$, and $c_{\theta,+} = \sigma_{+} \times s_{\theta}$.

(iii) Note that $c_{-}^{n} = \arg\min_{A_{-}^{n}(c)>0}\{|A_{-}^{n}(c) - c|\}$ and $\lim_{c\to\infty} A_{\theta,-}(c) = \sigma_{-}\sqrt{2/\pi}$. Since for $c > 0$, $A_{-}^{n}(c) \overset{ucp_{\mathcal{C}}}{\Longrightarrow} A_{\theta,-}(c) > 0$, and $\left|A_{-}^{n}(c) - c\right| \overset{ucp_{\mathcal{C}}}{\Longrightarrow} \left|A_{\theta,-}(c) - c\right|$, we have

$$c_{-}^{n} = \arg\min_{A_{-}^{n}(c)>0}\{|A_{-}^{n}(c) - c|\} \xrightarrow{P} \arg\min_{c>0}\{|A_{\theta,-}(c) - c|\} = c_{\theta,-}.$$

The same arguments hold for $c_{+}^{n} \xrightarrow{P} c_{\theta,+}$.

(iv) Since H is a continuous and increasing function, we deduce by the continuous mapping theorem that

$$\hat{\theta}_n \xrightarrow{P} \theta$$

and finally that

$$\hat{\sigma}_{-}^{n} \xrightarrow{P} \sigma_{-}, \quad \hat{\sigma}_{+}^{n} \xrightarrow{P} \sigma_{+}, \quad \hat{\delta}^{n} \xrightarrow{P} \delta.$$

7.7. Proof of Proposition 7

We have the following decomposition for $n^{1/4}\left(B_{-}^{n}(c) - \varphi_{-\theta}(c)\right)$:

$$n^{1/4}\left(B_{-}^{n}(c) - \varphi_{-\theta}(c)\right)$$

$$= n^{1/4}\left(\frac{\sum_{i=1}^{n} \mathbb{I}_{\{-c/\sqrt{n}<X_{(i-1)/n}<0, X_{i/n}<0\}}\sqrt{n}\left|X_{i/n} - X_{(i-1)/n}\right|}{\sum_{i=1}^{n} \mathbb{I}_{\{-c/\sqrt{n}<X_{(i-1)/n}<0, X_{i/n}<0\}}} - \varphi_{-\theta}(c)\right)$$

$$= \frac{1}{n^{-1/2}\sum_{i=1}^{n} \mathbb{I}_{\{-c/\sqrt{n}<X_{(i-1)/n}<0, X_{i/n}<0\}}} n^{-1/4}$$

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$$\sum_{i=1}^{n}\left[\mathbb{I}_{\{-c/\sqrt{n}<X_{(i-1)/n}<0,X_{i/n}<0\}}\left\{\sqrt{n}\left|X_{i/n}-X_{(i-1)/n}\right|-\varphi_{-\theta}\left(c\right)\right\}\right]$$
$$=\frac{D_{-}^{n}\left(c\right)}{N_{-}^{n}\left(c\right)},$$

where

$$N_{-}^{n}\left(c\right)=n^{-1/2}\sum_{i=1}^{n}\mathbb{I}_{\{-c/\sqrt{n}<X_{(i-1)/n}<0,X_{i/n}<0\}}.$$

By Proposition 2, we deduce that

$$N_{-}^{n}\left(c\right)\xrightarrow{P}\lambda_{\theta}(H_{\theta,h_{c,k-}})L_{1}. \tag{6}$$

Let

$$h=\begin{pmatrix} h_1(x,y) \\ h_2(x,y)\end{pmatrix}=\begin{pmatrix} h_{c-,\sigma-}(x,y)-\varphi_{-\theta}(c_-)\,h_{c-,k-}(x,y) \\ h_{c+,\sigma+}(x,y)-\varphi_{\theta}(c_+)\,h_{c+,k+}(x,y)\end{pmatrix}$$

and note that $\lambda_\theta(H_{\theta,h_1})=0$ and $\lambda_\theta(H_{\theta,h_2})=0$ since

$$\lambda_\theta(H_{\theta,h_1})=\lambda_\theta(H_{\theta,h_{c-,\sigma-}})-\varphi_{-\theta}(c_-)\,\lambda_\theta(H_{\theta,h_{c-,k-}}),$$
$$\lambda_\theta(H_{\theta,h_2})=\lambda_\theta(H_{\theta,h_{c+,\sigma+}})-\varphi_{\theta}(c_+)\,\lambda_\theta(H_{\theta,h_{c+,k+}}),$$

and $\lambda_\theta(H_{\theta,h_{c,\sigma-}})=\varphi_{-\theta}(c)\,\lambda_\theta(H_{\theta,h_{c,k-}})$ and $\lambda_\theta(H_{\theta,h_{c,\sigma+}})=\varphi_{\theta}(c)\,\lambda_\theta(H_{\theta,h_{c,k+}})$ for $c>0$ by Proposition 4.

Since

$$D_{-}^{n}\left(c\right)=n^{-1/4}\sum_{i=1}^{n}h_1\left(\sqrt{n}X_{(i-1)/n},\sqrt{n}\left(X_{i/n}-X_{(i-1)/n}\right)\right),$$

$$D_{+}^{n}\left(c\right)=n^{-1/4}\sum_{i=1}^{n}h_2\left(\sqrt{n}X_{(i-1)/n},\sqrt{n}\left(X_{i/n}-X_{(i-1)/n}\right)\right),$$

we deduce the result of Proposition 7 from Proposition 3 and equation (6).

7.8. Proof of Corollary 1

Let us consider the case of $A_{-}^{n}\left(c\right)$

$$A_{-}^{n}\left(c\right)=\frac{\sum_{i=1}^{n}\mathbb{I}_{\{S\left(-c/\sqrt{n}\right)<X_{(i-1)/n}<0,X_{i/n}<0\}}\sqrt{n}\left|S^{-1}\left(X_{i/n}\right)-S^{-1}\left(X_{(i-1)/n}\right)\right|}{\sum_{i=1}^{n}\mathbb{I}_{\{S\left(-c/\sqrt{n}\right)<X_{(i-1)/n}<0,X_{i/n}<0\}}}.$$

We use the same notation as in the proof of Proposition 6(i). First, it is easily seen that

$$n^{1/4}R_{-}^{n}\left(c\right)\overset{ucp_{\mathbb{C}}}{\Longrightarrow}0.$$

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Then,

$$\left|B_{-}^{n}\left(\frac{c}{\sigma(u_n)}\right)-B_{-}^{n}\left(\frac{c}{\sigma_{-}}\right)\right|$$

$$\leq \frac{\sum_{i=1}^{n}\mathbb{I}_{\{-c/(\sigma(u_n)\vee\sigma_{-}\sqrt{n})<X_{(i-1)/n}<-c/(\sigma(u_n)\wedge\sigma_{-}\sqrt{n}),X_{i/n}<0\}}\sqrt{n}\left|X_{i/n}-X_{(i-1)/n}\right|}{\sum_{i=1}^{n}\mathbb{I}_{\{-c/(\sigma(u_n)\sqrt{n})<X_{(i-1)/n}<0,X_{i/n}<0\}}}$$

$$+\frac{\sum_{i=1}^{n}\mathbb{I}_{\{-c/(\sigma_{-}\sqrt{n})<X_{(i-1)/n}<0,X_{i/n}<0\}}\sqrt{n}\left|X_{i/n}-X_{(i-1)/n}\right|}{\sum_{i=1}^{n}\mathbb{I}_{\{-c/(\sigma_{-}\sqrt{n})<X_{(i-1)/n}<0,X_{i/n}<0\}}}$$

$$\times\frac{\sum_{i=1}^{n}\mathbb{I}_{\{-c/(\sigma(u_n)\vee\sigma_{-}\sqrt{n})<X_{(i-1)/n}<-c/(\sigma(u_n)\wedge\sigma_{-}\sqrt{n}),X_{i/n}<0\}}}{\sum_{i=1}^{n}\mathbb{I}_{\{-c/(\sigma(u_n)\sqrt{n})<X_{(i-1)/n}<0,X_{i/n}<0\}}}.$$

Let

$$f_n(x)=\mathbb{I}_{\{-c/(\sigma(u_n)\vee\sigma_{-})<x<-c/(\sigma(u_n)\wedge\sigma_{-})\}}.$$

We have

$$\lambda_\theta(f_n)\leq K\left|\sigma_{-}-\sigma(u_n)\right|\leq K\frac{1}{\sqrt{n}},$$

since $-c/\sqrt{n}\leq u_n\leq 0$. By using the same arguments as those for equation (3.10) in Jacod (1998), we deduce that

$$\mathbb{E}_x^\theta\left[\sum_{i=1}^{n}\mathbb{I}_{\{-c/(\sigma(u_n)\vee\sigma_{-}\sqrt{n})<X_{(i-1)/n}<-c/(\sigma(u_n)\wedge\sigma_{-}\sqrt{n})\}}\right]\leq f_n\left(\sqrt{n}x\right)+K\lambda_\theta(f_n)\sqrt{n}\leq K.$$

It follows that

$$n^{1/4}\frac{1}{\sqrt{n}}\mathbb{E}^\theta\left[\sum_{i=1}^{n}\mathbb{I}_{\{-c/(\sigma(u_n)\vee\sigma_{-}\sqrt{n})<X_{(i-1)/n}<-c/(\sigma(u_n)\wedge\sigma_{-}\sqrt{n})\}}\right]\to 0$$

and

$$n^{1/4}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbb{I}_{\{-c/(\sigma(u_n)\vee\sigma_{-}\sqrt{n})<X_{(i-1)/n}<-c/(\sigma(u_n)\wedge\sigma_{-}\sqrt{n})\}}\xrightarrow{P}0$$

and also

$$n^{1/4}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbb{I}_{\{-c/(\sigma(u_n)\vee\sigma_{-}\sqrt{n})<X_{(i-1)/n}<-c/(\sigma(u_n)\wedge\sigma_{-}\sqrt{n})\}}\overset{ucp_\zeta}{\Longrightarrow}0.$$

Therefore, we have

$$n^{1/4}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbb{I}_{\{-c/(\sigma(u_n)\vee\sigma_{-}\sqrt{n})<X_{(i-1)/n}<-c/(\sigma(u_n)\wedge\sigma_{-}\sqrt{n}),X_{i/n}<0\}}\overset{ucp_\zeta}{\Longrightarrow}0.$$

Then, for $i \geq 2$,

$$\mathbb{E}_x^\theta\left[\mathbb{I}_{\{-c/(\sigma(u_n)\vee\sigma_-\sqrt{n})<X_{(i-1)/n}<-c/(\sigma(u_n)\wedge\sigma_-\sqrt{n}),X_{i/n}<0\}}\sqrt{n}\left|X_{i/n}-X_{(i-1)/n}\right|\right]$$
$$\leq \mathbb{E}_x^\theta\left[\mathbb{I}_{\{-c/(\sigma(u_n)\vee\sigma_-\sqrt{n})<X_{(i-1)/n}<-c/(\sigma(u_n)\wedge\sigma_-\sqrt{n})\}}\sqrt{n}\left|X_{i/n}-X_{(i-1)/n}\right|\right]$$
$$= \mathbb{E}_x^\theta\left[\mathbb{I}_{\{-c/(\sigma(u_n)\vee\sigma_-\sqrt{n})<X_{(i-1)/n}<-c/(\sigma(u_n)\wedge\sigma_-\sqrt{n})\}}\sqrt{n}\mathbb{E}_x^\theta\left[\left|X_{i/n}-X_{(i-1)/n}\right||\mathcal{F}_{(i-1)/n}\right]\right]$$
$$\leq \mathbb{E}_x^\theta\left[\mathbb{I}_{\{-c/(\sigma(u_n)\vee\sigma_-\sqrt{n})<X_{(i-1)/n}<-c/(\sigma(u_n)\wedge\sigma_-\sqrt{n})\}}\right].$$

Therefore, we easily deduce that

$$n^{1/4}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbb{I}_{\{-c/(\sigma(u_n)\vee\sigma_-\sqrt{n})<X_{(i-1)/n}<-c/(\sigma(u_n)\wedge\sigma_-\sqrt{n}),X_{i/n}<0\}}\sqrt{n}\left|X_{i/n}-X_{(i-1)/n}\right|$$
$$\stackrel{ucp_c}{\Longrightarrow} 0.$$

Recall that

$$h_{c,\sigma-}(x,y)=\mathbb{I}_{\{-c<x<0,\,y+x<0\}}|y|,\qquad h_{c,k-}(x,y)=\mathbb{I}_{\{-c<x<0,\,y+x<0\}},$$
$$h_{c,\sigma+}(x,y)=\mathbb{I}_{\{0<x<c,\,y+x>0\}}|y|,\qquad h_{c,k+}(x,y)=\mathbb{I}_{\{0<x<c,\,y+x>0\}}.$$

Since

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbb{I}_{\{-c/(\sigma(u_n)\sqrt{n})<X_{(i-1)/n}<0,X_{i/n}<0\}}\stackrel{ucp_c}{\Longrightarrow}\lambda_\theta(H_{\theta,h_{c/\sigma-,k-}}),$$

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbb{I}_{\{-c/(\sigma_-\sqrt{n})<X_{(i-1)/n}<0,X_{i/n}<0\}}\sqrt{n}\left|X_{i/n}-X_{(i-1)/n}\right|\stackrel{ucp_c}{\Longrightarrow}\lambda_\theta(h_{c/\sigma-,\sigma-}),$$

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbb{I}_{\{-c/(\sigma_-\sqrt{n})<X_{(i-1)/n}<0,X_{i/n}<0\}}\stackrel{ucp_c}{\Longrightarrow}\lambda_\theta(H_{\theta,h_{c/\sigma-,k-}}),$$

we finally deduce that

$$n^{1/4}\left|B_-^n\left(\frac{c}{\sigma(u_n)}\right)-B_-^n\left(\frac{c}{\sigma_-}\right)\right|\stackrel{ucp_c}{\Longrightarrow}0.$$

7.9. Increasingness property of H

Recall that

$$H(\theta)=\frac{1+\theta}{1-\theta}\frac{s_{-\theta}}{s_\theta},\qquad \theta\in(-1,1),$$

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where s_θ and $s_{-\theta}$ are positive. It follows that $H(\theta) > 0$, and we can define

$$\log(H(\theta)) = \log(1+\theta) - \log(1-\theta) + \log s_{-\theta} - \log s_\theta.$$

Since $\theta \to s_\theta$ is differentiable, we have

$$\frac{\partial \log H(\theta)}{\partial\theta} = \frac{1}{1+\theta} + \frac{1}{1-\theta} + \frac{1}{s_{-\theta}}\frac{\partial s_{-\theta}}{\partial\theta} - \frac{1}{s_\theta}\frac{\partial s_\theta}{\partial\theta}.$$

Using equation (5), numerical evaluations provide that $\partial \log H(\theta)/\partial\theta > 0$, for $\theta \in (-1,1)$.

SUPPLEMENTARY MATERIAL

To view the online supplementary material for this article, please visit http://doi.org/10.1017/S0266466622000214

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