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QUOTIENT RINGS OF A CLASS OF LATTICE-ORDERED-RINGS

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1. Introduction. An f-ring R with zero right annihilator is called a *qf-ring* if its Utumi maximal left quotient ring $Q = Q(R)$ can be made into an f-ring extension of R. F. W. Anderson [**2**, Theorem 3.1] has characterized unital qf-rings with the following conditions: For each $q \in Q$ and for each pair $d_1, d_2 \in R^+$ such that $d_i q \in R$,

(i) $(d_1 q)^+ \wedge (d_2 q)^- = 0$, and

(ii) $d_1 \wedge d_2 = 0$ implies $(d_1 q)^+ \wedge d_2 = 0$.

We remark that this characterization holds even when R does not have an identity element.

In this paper we are interested in left quotient rings of left convex f-rings: An f-ring R is *(left) convex* if each of its (left) ideals is an (left) l-ideal. Using Anderson's characterization it is shown that a left convex f-ring R is a qf-ring. If R has the maximum condition on polars (i.e., it has finite left Goldie dimension), then it is unital and decomposes into a direct sum of totally ordered left convex f-rings. A totally ordered left convex f-ring is, of course, a *left valuation ring* in the sense that given any two of its elements, one is a left multiple of the other. If R has regular elements, then it is a left Ore ring and also a classical quotient ring of its subring of bounded elements.

$Q(R)$ is a regular ring exactly when R has zero left singular ideal [**31**]. If the left singular ideal of the f-ring R is zero, Anderson [**2**, Theorem 4.3] has shown that R is a qf-ring if and only if it satisfies the condition

(iii) $|a| \wedge |b| = 0$ for each pair of elements $a, b \in R$ with $Ra \cap Rb = 0$.

A left convex f-ring satisfies this condition, and, in fact, left convexity arises naturally from it (see Proposition 2.1). From (iii) it can be seen that an f-ring R with zero left singular ideal is a qf-ring if and only if Q is strongly regular. (There is a purely ring theoretic generalization of this—and of 2.5 below: The maximal left quotient ring of a ring R with zero left singular ideal has no nilpotent elements, i.e., is strongly regular, if and only if $ab = 0$ for each pair of elements $a, b \in R$ with $Ra \cap Rb = 0$ [**30**]; this is proven in [**26**, Theorem 4.1] for the case that R is unital without nilpotent elements.)

Since a regular f-ring R is strongly regular each of its one-sided ideals is an ideal, and it can be seen that R is convex. In § 5 we show that a right injective f-ring is convex. In fact, there is only one totally ordered right injective ring without an identity element; and a right injective f-ring either

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has an identity element, in which case it is left convex, or it is the direct sum of a unital right injective f-ring and this totally ordered ring.

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2. Preliminaries. Let R be a ring and let X be a left R-module. A submodule Y of X is called a *complement in* X if there is a submodule Z of X such that Y is maximal with respect to $Y \cap Z = 0$. Y is called an *essential submodule of* X (and X is an *essential extension of* Y) if 0 is a complement of Y in X. It is well-known that Y is a complement in X if and only if it has no proper essential extension in X. For a subset A of X (respectively R) $l(A)$ (respectively $r(A)$ or $r_X(A)$) will denote the set of elements in R (respectively X) that annihilate A. Thus $l(A) = \{r \in R : rA = 0\}$. The submodule $Z(X) = \{x \in X : l(x)$ is an essential left ideal of $R\}$ is called the *singular submodule of* X [**23**].

The ring S is a *left quotient ring* of its subring R if for all $x, y \in S$ with $x \neq 0$, there exists $r \in R$ such that $rx \neq 0$ and $ry \in R$ ([**31**], also [**15**; **12**]). A left ideal D of the ring S is called *dense* if S is a left quotient ring of D. Equivalently [**2**, Lemma 1.1], D is dense if $r_E(D) = 0$, where E is the injective hull of ${}_SS$. If S is an f-ring, then SD^+ is dense whenever D is [**2**, Lemma 2.1].

A ring R has a left quotient ring if and only if $r(R) = 0$; and then it has a unique (up to an isomorphism that leaves the elements of R fixed—an R-isomorphism) maximal left quotient ring Q characterized by each of the following conditions [**31**, Theorem 1].

(1) Each left quotient ring of R is R-embedded in Q.

(2) If $q \in Q$ there is a dense left ideal D of R such that $Dq \subseteq R$; and further, if D is a dense left ideal of R and $\phi \in \mathrm{Hom}_R\,(D, R)$ there is a unique $q \in Q$ such that $x\phi = xq$ for each $x \in D$.

Now let R be a directed partially ordered ring (po-ring). The R-module X is an *f-module over* R if it is an l-group in which $R^+X^+ \subseteq X^+$, and as a lattice-ordered R-module X is isomorphic to a subdirect product of totally ordered R-modules [**28**; **4**, p. 54]. The *polar* of the subset A of X is the convex l-submodule

$$A_X' = \{x \in X : |x| \wedge |a| = 0 \text{ for each } a \in A\}.$$

(A_X' will usually be denoted by A'.) The set $P(X)$ of polars of X is a complete Boolean algebra with the infimum of a family of polars coinciding with the intersection of the family.

If Y is a subset of the f-module ${}_RX$, then $C_R(Y)$ will denote the convex l-submodule generated by Y. $\mathbf{Z}$ and $\mathbf{Q}$ will denote the totally ordered rings of integers and rational numbers, respectively. Unless specified otherwise a direct sum (or product) of ordered rings or modules will be the algebraic sum (or product) ordered coordinate-wise.

The following proposition gives (among other things) necessary and sufficient conditions for an f-ring with zero singular ideal to be a qf-ring.

PROPOSITION 2.1. *Let X be an f-module over the po-ring R, and assume that $r_X(R) = 0$. The following statements are equivalent.*

(a) $|a| \wedge |b| = 0$ *for all $a, b \in X$ such that $Ra \cap Rb = 0$.*

(b) *Each submodule I of X has I' as its unique complement.*

(c) *For each submodule I of X, I'' is the unique largest essential extension of I in X.*

(d) *$P(X)$ is the set of complement submodules of X.*

(e) *Each complement in X is a convex l-submodule.*

Proof. (a) $\Rightarrow$ (b): If J is a complement of I, then $|x| \wedge |y| = 0$ for each pair $(x, y) \in I \times J$. So $J \subseteq I'$. Since $I \cap I' = 0$, $J = I'$.

(b) $\Rightarrow$ (c): Since I' is the complement of I and I'' is the complement of I' containing I, I'' is an essential extension of I. If J is any essential extension of I in X, then $J \cap I' = 0$, so (b) implies that $J \subseteq I''$.

(c) $\Rightarrow$ (d): Since each polar I satisfies $I = I''$, every polar is a complement. Conversely, since a complement has no proper essential extension, (c) implies that each complement is a polar.

(d) $\Rightarrow$ (e): This is trivial.

(e) $\Rightarrow$ (a): Suppose that $Ra \cap Rb = 0$. Then $(Ra + \mathbf{Z}a) \cap (Rb + \mathbf{Z}b) = 0$. For if $0 \neq x$ is in the intersection, and $t \in R$ with $tx \neq 0$, then $tx \in Ra \cap Rb$. Let J be a complement of $Ra + \mathbf{Z}a$ containing $Rb + \mathbf{Z}b$, and let K be a complement of J containing $Ra + \mathbf{Z}a$. Then $|a| \wedge |b| \in J \cap K = 0$.

In [**31**] Utumi has defined a lattice of left ideals which includes the left complements and left annihilators, and which consists precisely of left complements when the singular ideal is zero. Let S be a left quotient ring of R, and let I be a left R-submodule of S. Define

$$M_{R-S}(I) = \{x \in S : Dx \subseteq I \text{ for some dense left ideal } D \text{ of } R\}.$$

Then $M_{R-S}(I)$ is an *M-left ideal of S*, i.e., $M_{S-S}(M_{R-S}(I)) = M_{R-S}(I)$. The set $M(R)$ of M-left ideals of R is closed under intersection and thus is a complete lattice. The mappings $I \to M_{R-S}(I)$ and $J \to J \cap R$ between $M(R)$ and $M(S)$ are inverse lattice isomorphisms and they take complements to complements. The proof of the following lemma is similar to [**29**, 2.5 and 2.6] and will therefore be omitted.

LEMMA 2.2 *Let R be an l-subring of the f-ring S, and suppose that S is a left quotient ring of R.*

(a) *If A is a non-empty subset of R, then $M_{R-S}(A_R') = A_S'$. Thus each polar of R is an M-left ideal of R, and $P(R)$ and $P(S)$ are naturally isomorphic Boolean algebras.*

(b) *Let I be a left R-submodule of S. If I is convex, a sublattice, or a prime submodule of ${}_RR$, then $M_{R-S}(I)$ is convex, a sublattice, or a prime submodule of ${}_SS$.*

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Corollary 2.3. *${}_RR$ satisfies the condition of* 2.1 *if and only if ${}_SS$ does.*

Corollary 2.4 (Anderson [**2**, 3.5]). *R is totally ordered precisely when S is.*

The element x in the ring R is called *regular* provided $l(x) = 0 = r(x)$. Let Σ be a nonempty multiplicative subset of regular elements of R. An overring R_Σ of R is the *classical left quotient ring of R with respect to* Σ in case each element of Σ is invertible in R_Σ and each element of R_Σ is of the form $a^{-1}b$ for some $(a, b) \in \Sigma \times R$. It is well-known that R_Σ exists (and is unique) if and only if R satisfies the left Ore condition with respect to Σ ([**10**] or [**21**]): If $(a, b) \in \Sigma \times R$, there exists $(c, d) \in \Sigma \times R$ such that $da = cb$. If Σ is the set of all regular elements of R and R_Σ exists, then it will be denoted by R_c, and R will be called a *left Ore ring*. If R is a left and right Ore ring, then R_c is also the classical right quotient ring of R. Since any two elements of R_Σ can be expressed with the same denominator [**21**, 1.1], R_Σ is clearly a left quotient ring of R; i.e., $R_\Sigma \subseteq Q$. If R is an f-ring, then R_Σ can be made into an f-ring extension of R [**2**, Theorem 5.1; **26**, Corollary to Proposition 1].

The R-module ${}_RX$ is said to be *finite dimensional* if it has the maximum condition on complements; equivalently, X contains no infinite direct sum of submodules [**18**, Lemma 3.5]. A *left Goldie ring* is a ring R for which ${}_RR$ is finite dimensional and which has the maximum condition on left annihilators. Goldie's theorem [**18**, Theorems 5.1 and 5.4] asserts that a ring R is a semiprime left Goldie ring exactly when R_c exists and is an artinian semiprime ring. In this case $R_c = Q(R)$.

The following theorem, due to Anderson, is just the restriction of Goldie's theorem to f-rings. (The purely ring theoretic generalization mentioned in the introduction may be obtained by replacing R by a ring without nilpotent elements, polar by annihilator, and by interpreting qf-ring to mean $Q(R)$ has no nilpotent elements.) The proof given below is slightly different and shorter than that given in [**2**].

Theorem 2.5 (Anderson [**2**, Theorems 6.1 and 6.2]). *The following statements are equivalent for the semiprime f-ring R.*

(a) *R is a qf-ring with the maximum condition on polars.*
(b) *R is a left Goldie ring.*
(c) *Q is the direct sum of totally ordered division rings.*
(d) *R is a left Ore ring and has the maximum condition on polars.*

Proof. Since a ring without nilpotent elements has zero singular ideal (iii) is a necessary and sufficient condition for R to be a qf-ring. (a) $\Rightarrow$ (b) comes from 2.1, (b) $\Rightarrow$ (c) and (d) follows from Goldie's theorem, and (c) $\Rightarrow$ (a) is a consequence of 2.3 and 2.2.

(d) $\Rightarrow$ (c): By 2.2 we may assume that $R = R_c$. Let $A_1, \ldots, A_n$ be the minimal polars of R. Then each A_i is a totally ordered domain, the sum $A = \sum A_i$ is direct, and $A' = 0$ ([**1**, Theorem 2] or [**28**, 1.7 and § 3]).

If $0 \neq a_i \in A_i$, then $a = a_1 + \ldots + a_n$ is a regular element. Thus $R = A$ and each A_i is a division ring.

3. Left convex f-rings. The f-module ${}_RX$ over the po-ring R is called *convex* if each of its submodules is a convex l-submodule. Note that a finitely generated submodule of a convex f-module is cyclic. If R is an f-ring and ${}_RR$ is convex, then R will be called a *left convex f-ring*. An f-ring in which each ideal is a convex l-subgroup need not be left convex, and a left convex f-ring need not be right convex. Examples of the former are given in 5.3 below and [**22**, Example 4.5]. An example of the latter is given in 4.2.

An f-ring is *unitable* if it can be embedded in an f-ring with an identity element [**22**; **20**]. Recall that in a unitable f-ring every idempotent is central [**20**, 2.1].

LEMMA 3.1. *Let ${}_RX$ be a convex f-module.*

(a) *X is a divisible group.*

(b) *$x \in Rx$ for each $x \in X$, and thus $r_X(R) = 0$.*

(c) *If $R = X$ is a unitable f-ring and R contains an element x with $l(x) = 0$, then R has an identity element.*

(d) *If $R = X$ is totally ordered, then R is unital.*

Proof. (a) Since nX is a submodule of X and X/nX is torsion-free, $X = nX$ for each $0 \neq n \in \mathbf{Z}$.

(b) If $x \notin Rx$, then $C_R(x) = Rx + \mathbf{Z}x = Rx + \mathbf{Q}x$ as groups, which is impossible.

(c) Since $x = bx$ for some $b \in R$, $l(x) = 0$ implies b is a right identity, hence a left identity also.

(d) If N is the set of nilpotent elements of R, then N is the lower radical of R and R/N is a totally ordered domain [**5**, p. 63]. By (b) R is not nil. Hence the proof of (c) shows that R/N has a non-zero right identity. Since an idempotent may be lifted back through a nil ideal, R contains a non-zero idempotent e. Since ${}_RR$ is indecomposable (the left ideals of R are totally ordered), $R = Re$; but $0 = r(R) = r(e)$ implies that $R = eR$, and hence e is the identity of R.

PROPOSITION 3.2. *The following statements are equivalent for the f-module ${}_RX$.*

(a) *X is convex.*

(b) *$0 \leqq x \leqq y$ implies $x \in Ry$ for all $x, y \in X$ (i.e., each submodule is a convex subset).*

(c) *$|x| \leqq |y|$ implies $x \in Ry$ for all $x, y \in X$.*

Proof. That (a) $\Rightarrow$ (b) is an immediate consequence of 3.1 (b), and that (c) $\Rightarrow$ (a) is obvious.

(b) $\Rightarrow$ (c): There exist $r_1, r_2 \in R$ such that $x^+ = r_1|x|$ and $x^- = r_2|x|$ since $x^+, x^- \leqq |x|$. Thus $x \in R|x|$. But in any f-module the equations $x = r|x|$

and $|x| = rx$ are equivalent; so $|x| \in Rx$. Thus $x \in R|x| \subseteq R|y| = Ry$ if $|x| \leqq |y|$.

COROLLARY 3.3. *The direct product or the direct sum of a family of convex f-modules (left convex f-rings) is convex (left convex) if and only if each member of the family is convex (left convex).*

Let $\mathscr{C}$ be the class of left convex f-rings, and let $\mathscr{U}$ be the variety of f-rings generated by $\mathscr{C}$. $\mathscr{C}$ misses being a variety because an l-subring of a left convex f-ring need not be left convex. The following proposition shows that this is exactly the difference between $\mathscr{C}$ and $\mathscr{U}$.

The ring T is said to have *local units* if for each $t \in T$ there is an idempotent e of T such that $t = te = et$.

PROPOSITION 3.4. (a) *Let R be a left convex f-ring, and let T be a convex l-subring of R with local units. Then T is left convex.*

(b) *$R \in \mathscr{U}$ if and only if R is an l-subring of a left convex f-ring.*

Proof. (a) If $0 \leqq s \leqq t$ where $s, t \in T$, then $s = rt$ for some $r \in R^+$. Let e be a local unit for t, and let $x = r \wedge e \in T$. Then $s = xt$.

(b) Let $R \in \mathscr{U}$. Then R is a homomorphic image of an l-subring S of a product $\Pi S_i = T$ of totally ordered left convex f-rings $S_i : R = S/A$. By 3.1 (d) each S_i has a unit. Let B_1 be the convex l-subring of T generated by $\mathbf{1}$ and S, and let A_1 be the convex l-subring of T generated by A. Then $A_1 \cap S = A$, so R is embedded in B_1/A_1, and the latter is left convex by (a).

An f-ring is called *formally real* if it is in the variety of l-rings generated by the reals. In [**20**, 4.1] Henriksen and Isbell show that each formally real f-ring R can be embedded in a formally real convex f-ring. It is their argument that we have used to prove (b). If R is semiprime it suffices to take the quotient ring $Q(R)$ for this embedding. If R is not semiprime, however, $Q(R)$ need not be convex. For an example, give the appropriate lexicographic ordering to Utumi's example [**31**, p. 2]: Let F be a totally ordered field and lexicographically order $F[x]$ so that the constant term dominates. Let R be the F-subalgebra of $F[x]/[x^4]$ generated by 1, x^2 and x^3. Then $R = Q$ is a totally ordered formally real f-ring that is not convex (in fact, R does not satisfy the conditions of 2.1).

THEOREM 3.5. *Let R be an f-ring with $r(R) = 0$. If each left ideal of R that is generated by a finite set of positive elements is principal and has a positive generator, then R is a qf-ring.*

Proof. We will show that Anderson's two conditions (i) and (ii) for R to be a qf-ring are satisfied. Let $q \in Q$ and let d_1, d_2 be elements of R^+ with $d_i q \in R$. There exists $d \in R^+$ such that

$$(Rd_1 + \mathbf{Z}d_1) + (Rd_2 + \mathbf{Z}d_2) = Rd + \mathbf{Z}d.$$

Let $r \in R^+$. Then $rd_1 = r_1 d$ and $rd_2 = r_2 d$ for some $r_1, r_2 \in R^+$.

(i) $(d_1q)^+ \wedge (d_2q)^- = 0$:

$$r[(d_1q)^+ \wedge (d_2q)^-] = (r_1dq)^+ \wedge (r_2dq)^- = r_1(dq)^+ \wedge r_2(dq)^- = 0,$$

since $dq \in R$. Hence $(d_1q)^+ \wedge (d_2q)^- = 0$.

(ii) $d_1 \wedge d_2 = 0$ implies $(d_1q)^+ \wedge d_2 = 0$:
We first note that the proof of Anderson's theorem shows that (i) implies that Q is a left f-ring (i.e., ${}_QQ$ is an f-module) extension of R. Thus

$$r[(d_1q)^+ \wedge d_2] = r_1dq^+ \wedge r_2d \leqq [r_1d(q^+ \vee 1)] \wedge [r_2d(q^+ \vee 1)] = (r_1 \wedge r_2)\, d(q^+ \vee 1) = 0,$$

since $d(q^+ \vee 1) \in R$.

According to 2.1 an f-ring R with zero singular ideal is a qf-ring exactly when each complement in ${}_RR$ is a left l-ideal. The following corollary has a similar flavour.

COROLLARY 3.6. *A left convex f-ring R is a qf-ring. If Q is the maximal left quotient ring of R, then $M(Q)$ consists of left l-ideals, and $P(Q)$ is the set of complement left ideals of Q.*

Proof. By 3.1, $r(R) = 0$, so R is a qf-ring. The remaining statements follow from 2.2 and 2.3.

If R is a unitable f-ring the set $S = S(R) = \{x \in R : |x| \leqq n \text{ for some } n \in \mathbf{Z}\}$ is a convex l-subring of R which has the property that each convex l-subgroup is an l-ideal. (Since there is a smallest unital f-ring containing a given unitable f-ring [**20**, Theorem 5.11], S is well-defined.) If R is a qf-ring with $Q = Q(R)$ let $T = S(Q)$. The next lemma generalizes the fact that a totally ordered division ring R is the classical left and right quotient ring of its subring of bounded elements.

LEMMA 3.7. *Let R be a unital f-ring in which $r \geqq 1$ implies $r^{-1} \in R$. Then R is a classical left and right quotient ring of S.*

Proof. Let $r \in R^+$. Then $r + 1 = r \vee 1 + r \wedge 1$, hence $r = a + b$ where $a = r \wedge 1 - 1$, $b = r \vee 1$, and $a, b^{-1} \in S$. So $r = (ab^{-1} + 1)\,b = b(b^{-1}a + 1)$. If $r \in R$, then $|r| = sd^{-1} = d^{-1}t$ where $d \in S^+$; so rd and dr are elements of S.

Note that even if R is an Ore domain it need not be a quotient ring of S. For an example let $R = \mathbf{Q}[x]$ be ordered lexicographically with the highest term dominating. This example also shows that R need not be left convex if S is. The converse is true, however. In fact, by 3.1(c) and [**20**, Theorem 5.12] a left convex f-ring R is an l-ideal in the smallest f-ring with unit containing R. Thus $a \wedge 1 \in R$ whenever $a \in R$; so ${}_SR$ is a convex f-module (see the proof of 3.4(a)).

THEOREM 3.8. *Let R be a left convex f-ring with regular elements. Then R is a classical quotient ring of S, and it is a left Ore ring; also R_c (in fact, any classical left quotient ring of R) is left convex.*

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Proof. That R is a classical quotient ring of S is an immediate consequence of 3.7, and it (together with the preceding remarks) allows us to assume that $R = S$. Let $a, b \in R$ with b regular. Then $ba \in Rb$, so R is left Ore. For the last statement, let $p, q \in R_c$ with $0 \leqq p \leqq q$. Then $p = a^{-1}b$, $q = a^{-1}c$ where $a, b, c \in R^+$. So $b = xc$ and $p = (a^{-1}xa)q$.

One consequence of 3.8 is that when $1 \in R$ (i.e., when R has regular elements) Q is also the maximal left quotient ring of S. This is probably true without the hypothesis $1 \in R$. We collect our evidence below.

PROPOSITION 3.9. *Let R be a left convex f-ring with maximal left quotient ring Q.*

(a) *If $p, q \in Q$ with $Sp \cap Sq = 0$, then $|p| \wedge |q| = 0$.*

(b) *${}_SS$ is essential in ${}_SQ$.*

(c) *If $Z(R) = 0$, then $Q = Q(S)$.*

(d) *If $q \in Q$ with $q \geqq 1$, then $q^{-1} \in Q$.*

(e) *If $Q = Q(S)$, then $Q = Q_c$.*

Proof. (a) Let D be a dense left ideal of R such that Dp and $Dq \subseteq R$. If $|p| \wedge |q| > 0$ there exists $d \in D^+$ with $d(|p| \wedge |q|) > 0$. Since $r(d \wedge 1) = r(d)$ we may assume that $d \in S$. Then $d(|p| \wedge |q|) \in Sp \cap Sq$.

(b) is a consequence of (a); and (c) follows from (b) since if $Z(R) = 0$, then an overring Q of the ring R is a quotient rnig of R if and only if ${}_RR$ is essential in ${}_RQ$ [**12**, p. 59].

(d) Let $D = \{d \in R : dq \in R\}$. Then $D \subseteq Dq$, so Dq is a dense left ideal of R. Therefore, the map $dq \rightarrow d$ from Dq into R is given by right multiplication by $p \in Q$. So $pq = qp = 1$.

(e) Suppose that $q \in Q$ is regular, and let $D = \{d \in S : dq \in S\}$. Then dq is an ideal of S, and $r_S(Dq) = 0$. Since the annihilator in $E({}_SS)$ of an ideal of S is a submodule of E, Dq is a dense left ideal of S. The argument in (d) now completes the proof.

An *R-value* of the non-zero element g in the f-module ${}_RX$ is a convex l-submodule maximal with respect to the exclusion of g. If g has only one R-value ($\mathbf{Z}$-value) it is called *R-special* (special). For the proof of the following theorem see [**7**] and [**28**].

THEOREM 3.10. (a) *There is a one-to-one correspondence between the set of R-values of g in X and its set of* $\mathbf{Z}$*-values.*

(b) *The following statements are equivalent:*

(1) *g is special in X.*

(2) *g is special in $C_R(g)$.*

(3) *$C_R(g)$ is a lexicographic extension of a proper convex l-submodule.*

(c) *If Y is an l-submodule of X and $g \in Y$, then g is special in Y provided it is special in X.*

(d) *The following statements are equivalent:*

(1) *g has only a finite number of R-values in X.*

(2) *$g = g_1 + \ldots + g_n$ where each g_i is special and $i \neq j$ implies $|g_i| \wedge |g_j| = 0$.*

(e) *If X has the maximum condition on polars, then each $g \in X$ has only a finite number of R-values.*

A ring with 1 is *local* if it has a unique maximal left ideal.

LEMMA 3.11. *Let Q be a unital f-ring with the property that $x \geqq 1$ implies $x^{-1} \in Q$. Then the following statements are equivalent.*

(a) *Q is local.*

(b) *1 is a special element.*

(c) *$T = C_{\mathbf{Z}}(1)$ is local.*

Proof. (a) $\Rightarrow$ (b): Let J be the maximal left ideal of Q. If $|x| \leqq |y|$ with $y \in J$ and $x \notin J$, then $|x|^{-1} \in Q$, $|y|\,|x|^{-1} \geqq 1$, and y is invertible. Consequently J is an l-ideal, and 1 is Q-special in ${}_QQ$.

(b) $\Rightarrow$ (c): Let M be the maximal convex l-subgroup of T. If $a \in T$ is not a unit of T, then $C(a) \subseteq T$, i.e., $a \in M$. Thus M is the set of non-units of T, so T is local.

(c) $\Rightarrow$ (a): Let K be the maximal left ideal of T. If M is a maximal convex l-subgroup of T, then $M \subseteq K$. So 1 is special. Let J be the Q-value of 1 in ${}_QQ$. If a is not invertible in Q, then $C_Q(a) \subseteq J$ and Q is local.

For the proof of the next theorem we will need the theory of the Johnson radical for which the reader is referred to [**22**].

THEOREM 3.12. *The following statements are equivalent for a left convex f-ring R.*

(a) *R contains a regular special element.*

(b) *S is a local ring.*

(c) *R is totally ordered.*

(d) *R is local.*

(e) *Q is local.*

(f) *T is local.*

Proof. Throughout, 3.11 will be used implicitly. Then (b) $\Leftrightarrow$ (a), (b) $\Leftrightarrow$ (d), (e) $\Leftrightarrow$ (f), and (c) $\Rightarrow$ (e) by 2.3 and 3.9 (d).

(a) $\Rightarrow$ (b): Let $0 < x$ be a regular special element of R. By 3.8, $x^{-1} \in Q$, so right multiplication by x^{-1} is an automorphism of the f-module ${}_RQ$. Since x is special in Rx, 1 is special in R.

(b) $\Rightarrow$ (c): Suppose that $x, y \in R$ with $x \wedge y = 0$. Then $Sx \oplus Sy = S(x + y)$. So $x = sx$ and $sy = 0$ for some $s \in S$. If s is invertible, then $y = 0$. Otherwise, $1 - s$ is invertible and $x = 0$.

(e) $\Rightarrow$ (d): We only have to show that $1 \in R$, by 3.10(c). Suppose that $1 \notin R$. Then R is an l-ideal in the smallest unital f-ring containing it, and the latter is $R_1 = R + \mathbf{Z} \subseteq Q$ [**20**, Theorem 5.12]. We claim that $S_1 = S + \mathbf{Z}$ is the subring of bounded elements of R_1. Let $0 \leqq x \leqq n$, where $x = r + m \in R_1$. Then $sr \in S$ for each $s \in S$. Since ${}_SR$ is left convex, $r \in Sr \subseteq S$, and hence $x \in S_1$.

Since 1 is special in Q, it is special in S_1. Let M be the maximal convex l-subgroup of S_1. Them M is an l-ideal and it is the Johnson radical of S_1. (This is true for $S(R_1)$ where R_1 is any unital f-ring in which 1 is special.) Since $S \subseteq M \subseteq S_1$ and $S_1/S \cong \mathbf{Z}$, $M = S$. Since an ideal in a left convex f-ring R is left quasi-regular if and only if it is left l-quasi-regular, the Jacobson radical $J(R)$ and the Johnson radical of R coincide. So S is a left convex Jacobson radical ring. Hence $S = 0$ by 3.1(b), which is impossible by 3.9(b).

THEOREM 3.13. *The following statements are equivalent for the left convex f-ring R.*

(a) *R has the maximum condition on polars.*

(b) *${}_RR$ is a finite dimensional module.*

(c) *R contains a regular element that has only a finite number of values in R.*

(d) *R is the direct sum of a finite number of totally ordered rings.*

Proof. The equivalence of (a) and (b) follows from 2.1; and the implication (d) $\Rightarrow$ (a) is trivial.

(a) $\Rightarrow$ (c): R is a subdirect product of a finite number of totally ordered rings $R_1, \ldots, R_n$ ([**1**, Theorem 1] or [**28**, 1.7]). Let $\phi_i : R \to R_i$ be the projections. If $e_i \in R^+$ is such that $\phi_i(e_i)$ is the unit of R_i, then $e = e_1 + \ldots + e_n$ is a regular element of R. Hence $1 \in R$ and it is finitely-valued by 3.10(e).

(c) $\Rightarrow$ (d): An argument similar to the one used in (a) $\Rightarrow$ (b) of 3.12 shows that 1 has only a finite number of values in R. Then $1 = e_1 + \ldots + e_n$ as in 3.10(d), and clearly e_i is idempotent. By 3.12, Re_i is totally ordered.

Note that because of 2.1, 2.2, 2.3 and 3.9 (a) and (b), the ring (or module) in each statement of 3.13 may be replaced by any combination of R, S, T or Q that makes sense. For example, in (b) ${}_RR$ may be replaced by ${}_TQ$.

We conclude this section by giving an example to show that Theorem 3.12 is not true for an arbitrary qf-ring. Let F be a totally ordered field and let Q_1 be the totally ordered formal power series ring with coefficients in F and positive integral exponents:

$$Q_1 = \left\{ a = \sum_{i=0}^{\infty} a_i x^i : a_i \in F \right\},$$

$$Q_1^+ = \left\{ \sum_{i=0}^{\infty} a_i x^i : a_0 = \ldots = a_{n-1} = 0, 0 < a_n \right\} \cup \{0\}.$$

Let $R = \{(f, g) \in Q_1 \oplus Q_1 : f_0 = g_0\}$. Then R is a (noetherian) local qf-ring, and $Q(R) = Q_1 \oplus Q_1$. Note that (1, 1) is special in R.

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4. Properties of left convex f-rings. In this section we collect some facts about left convex f-rings, some of which will be used in the next section. We begin by strengthening the observation that the Jacobson and Johnson radicals of a left convex f-ring coincide.

PROPOSITION 4.1. *Let R be left convex, and let $\mathscr{M}_l$, $\mathscr{M}_r$, and $\mathscr{M}$ be the sets of maximal left, right and two-sided ideals of R, respectively. Similarly, let $\mathscr{M}_l'$, $\mathscr{M}_r'$, and $\mathscr{M}'$ be the corresponding sets of l-ideals of R. Then all six sets are identical.*

Proof. Trivially, $\mathscr{M}_l = \mathscr{M}_l'$ and $\mathscr{M} = \mathscr{M}'$. By [**22**, p. 213], $\mathscr{M}' \subseteq \mathscr{M}_r' \cap \mathscr{M}_l'$. If $A \in \mathscr{M}_l' \cup \mathscr{M}_r'$ let P be a minimal prime subgroup contained in A. Then P is an ideal of R [**28**, Theorem 1.1], so A/P is a maximal one-sided l-ideal of R/P. Since R/P is local, $A \in \mathscr{M}'$ and hence $\mathscr{M}' = \mathscr{M}_l' = \mathscr{M}_r'$. Since a simple left convex f-ring is a division ring, $\mathscr{M} \subseteq \mathscr{M}_r$. Finally, suppose $A \in \mathscr{M}_r$, and let K be the annihilator of the simple right R-module R/A. Then K is a primitive ideal, hence a maximal ideal of R. If $x \in K$, then $Rx \subseteq A$. Thus $K \subseteq A$ and the two are identical.

If $J = J(R)$ is the Jacobson radical of a right injective ring R with 1, then R/J is a regular right injective ring [**12**, p. 44]. In view of this fact, Propositions 4.1 and 5.1, and the fact that a Jacobson semisimple totally ordered left convex f-ring is a division ring, one might suspect that R/J is regular when R is a left convex f-ring. There are, however, topological spaces X for which $C(X)$, the f-ring of real-valued continuous functions on X, is convex, but not regular [**17**, pp. 208–212]. It seems appropriate to note here that any Archimedean convex f-ring R is Jacobson semi-simple. For Q is Archimedean [**2**, 3.3], and thus R is an l-ideal of an Archimedean f-ring with 1. But the latter is Johnson semisimple [**22**, Theorem 2.11].

A ring is called *left* (*right*) *duo* if each of its left (right) ideals is an ideal. Conceivably, every left convex f-ring is left duo. The following example shows, however, that it need not be right duo.

Example 4.2. Let F be a totally ordered field, $\sigma : F \rightarrow F$ a 0-isomorphism that is not onto. Let R be the twisted formal power series over F with integral exponents, ordered lexicographically. Thus the underlying l-group of R is identical to that of the ring Q_1 given at the end of § 3. Multiplication in R is defined using the rule $xa = a^\sigma x$ for $a \in F$. R is a left convex domain, and if $a \in F$ is not in the image of σ, then axR is not an ideal.

The next proposition examines some chain conditions in left convex f-rings. A unital ring R is said to be *right perfect* if it has the minimum condition on principal left ideals. A ring N is *right T-nilpotent* if given any sequence $\{a_i\}$ of elements in N, there exists an integer n such that $a_n a_{n-1} \ldots a_1 = 0$. Using homological methods Bass has shown in [**3**] that R is right perfect if and only if its Jacobson radical J is right T-nilpotent and R/J is artinian.

PROPOSITION 4.3. *Let R be a left convex f-ring.*

(a) *If R has the maximum or minimum condition on principal left ideals, then R is left duo.*

(b) *If R has the maximum condition on principal left ideals, then R is left noetherian and $R_c = Q$. If S is left noetherian, then so is R.*

(c) *If R has the minimum condition on principal left ideals, then J is right T-nilpotent and the following statements are equivalent:*

(1) *R is left artinian;*

(2) *R is unital;*

(3) *R/J is left artinian.*

Proof. (a) Let $a, r \in R^+$, $r_1 = r + 1 \in Q$. Clearly Ra and Rar are contained in Rar_1. Suppose that $Ra \subsetneq Rar_1$. Then since $r_1^{-1} \in Q$,

$$\ldots \subsetneq Rar_1^{-1} \subsetneq Ra \subsetneq Rar_1 \subsetneq \ldots .$$

Thus either chain condition implies that R is left duo.

(b) Suppose $A_1 \subsetneq A_2 \subsetneq \ldots$ is a chain of left ideals of R. Take $a_1 \in A_1^+$, $b_2 \in A_2^+ \backslash A_1$, $a_2 = a_1 \vee b_2$. Then $Ra_1 \subsetneq Ra_2$. So the absence of an infinite chain of principal left ideals implies that R is left noetherian.

Assume R is left noetherian and totally ordered. If $q \in Q$ let D be a dense left ideal of R such that $Dq \subseteq R$. Then $D = Rd$ and density of D implies $r(d) = 0$. Let n be an integer such that $l(d^n) = l(d^{n+1}) = \ldots$. Then $l(d^n) \cap Rd^n = 0$, so $l(d) = 0$ and d is regular. Thus $q = d^{-1}(dq) \in R_c$; i.e. $Q = R_c$. Since $Q(R)$ preserves products [**31**, 2.1], $R_c = Q$ if R is left noetherian (3.13). If S is left noetherian, then, since S is left convex, it is unital by 3.13. Thus R is left noetherian by 3.7.

(c) Suppose that R has the minimum condition on principal left ideals. Then so does every totally ordered homomorphic image $\bar{R}$ of R: Let $\bar{R}\bar{a}_1 \supseteq \bar{R}\bar{a}_2 \supseteq \ldots$ be a chain in $\bar{R}$, where $\bar{a}_1 > \bar{a}_2 > \ldots > 0$. Then there is a sequence $\{a_i\}$ in R^+ mapping isomorphically onto $\{\bar{a}_1\}$ [**32**, p. II-62]. Since the chain $\{Ra_i\}$ is finite, so is the chain $\{\bar{R}\bar{a}_i\}$. If $\bar{B} \subsetneq \bar{A}$ left are ideals of $\bar{R}$ and $\bar{a} \in \bar{A}^+ \backslash \bar{B}$, then $\bar{B} \subsetneq \bar{R}\bar{a}$. So $\bar{R}$ is actually left artinian.

Let $\{a_i\}$ be a sequence in $J(R)$, and let $n \in \mathbf{Z}$ be such that $Ra_n \ldots a_1 = Ra_m \ldots a_1$ if $m \geqq n$. If $\bar{R}$ is a totally ordered homomorphic image of R, then $\{\bar{a}_i\} \subseteq J(\bar{R})$. Since $J(\bar{R})$ is nilpotent and $1 \in \bar{R}$, $\bar{a}_n \ldots \bar{a}_1 = 0$. Thus $a_n \ldots a_1 = 0$. So $J(R)$ is right T-nilpotent.

Clearly (1) $\Rightarrow$ (2) by 3.13 and (2) $\Rightarrow$ (3) by Bass' theorem.

(3) $\Rightarrow$ (1): Since J is nil the identity of R/J can be lifted to an idempotent e of R. Then $R = Re \oplus l(e)$ and $l(e) \subseteq J$. Since $l(e)$ is left convex and nil, $l(e) = 0$. Since R/J is left artinian, $R/J = D_1 \oplus \ldots \oplus D_n$ where each D_i is a division ring. Let $\bar{e}_i$ be the identity of D_i, and let $\{e_i\}$ be (orthogonal) idempotents of R such that $e_i + J = \bar{e}_i$. Then $R = Re_1 \oplus \ldots \oplus Re_n$ and each Re_i is local. Thus Re_i is totally ordered, and, since it has the minimum condition on principal left ideals, it is left artinian.

In general R_c is properly contained in Q. For instance if R is regular (with 1) but not injective, then $R = R_c \subsetneqq Q$. The arithmetic of a totally ordered left convex left noetherian ring is given in [**6**] and [**21**, p. 112].

We conclude this section with an examination of when a left convex f-ring is right Ore. Condition (b) of the following theorem should be contrasted with 3.8.

THEOREM 4.4. *The following statements are equivalent for the left convex f-ring R with regular elements.*

(a) *R is a right Ore ring.*

(b) *R_c is a convex left R-f-module.*

(c) *R is a convex l-subgroup of R_c.*

(d) *$a^{-1}Ra = R$ for every regular element a in R.*

(e) *$S(R_c) \subseteq R$.*

Proof. (a) $\Rightarrow$ (b): If $0 \leqq xa^{-1} \leqq ya^{-1}$ with $x, y, a \in R^+$, then $x \in Ry$, so $xa^{-1} \in Rya^{-1}$.

(b) $\Rightarrow$ (c): If $0 \leqq p \leqq r$ where $p \in R_c$ and $r \in R$, then $p = sr \in R$.

(c) $\Rightarrow$ (d): If a is a regular element of R, then conjugation by $|a|$ is an automorphism of the f-ring R_c. Since $S = S(R) = S(R_c)$, $|a|^{-1} S\,|a| = S$; so $a^{-1} Sa \subseteq S$. But then $S = a^{-2} Sa^2 \subseteq a^{-1} Sa \subseteq S$. Since R is a classical left quotient ring of S, $a^{-1} Ra = R$.

(d) $\Rightarrow$ (e): If $0 \leqq a^{-1}b \leqq n$ where $a, b \in R^+$ and $n \in \mathbf{Z}$ then $a^{-1}ba = nra$ for some $r \in R^+$; so $a^{-1}b \in R$.

(e) $\Rightarrow$ (a) is a consequence of 3.7.

A unital ring Q is said to be a *quotient ring* if each of its regular elements is invertible.

COROLLARY 4.5. *A ring R is a left convex right Ore f-ring if and only if there is a left convex quotient f-ring Q such that $S(Q) \subseteq R \subseteq Q$.*

Proof. Let Q be a left convex quotient f-ring, and suppose that R is a subring of Q containing $S = S(Q)$. If $a \in R$, then $a^+ \in Sa \subseteq R$. So R is an l-subring of Q, and just as easily it is seen that R is left convex. That R is right Ore follows from 3.7.

5. Injective f-rings. A ring R is *right* (*left*) *injective* if R_R ($_RR$) is an injective R-module. If R is also an f-ring it will be called a right (left) injective f-ring. Since a right injective f-ring with zero right singular ideal is left injective [**2**, 7.2], it is trivially a qf-ring. In this section we show that there is a unique totally ordered right injective ring that is not a qf-ring, and modulo this example every right injective f-ring is left convex. It is unknown whether or not a right injective unital f-ring is left injective.

Let $\bar{R}$ be the ring obtained by freely adjoining $\mathbf{Z}$ to R. If R is right injective, then the identity map on R can be extended to an R-homomorphism $\bar{R} \to R$.

Consequently, R has a left identity, and any R-homomorphism $I \to R$, from a right ideal of R into R, is given by left multiplication by an element of R.

If R is right injective and $(R, +)$ is a torsion-free group, then its divisible closure $\mathbf{Q} \otimes_{\mathbf{Z}} R$ is an R-module. Since the identity map on R can be extended to $\mathbf{Q} \otimes_{\mathbf{Z}} R$, R is an algebra over $\mathbf{Q}$.

PROPOSITION 5.1. *Let R be a right injective f-ring and let $A = l(R)$. Then R/A is left convex.*

Proof. Let $\bar{R} = R/A$. Suppose $x, y \in R$ with $r(y) \subseteq r(x)$. Then the map $yr \to xr$ is given by left multiplication by some element c of R; i.e., $xr = cyr$ for all $r \in R$. Thus $\bar{x} \in \bar{R}\bar{y}$. In particular, if $0 \leqq x \leqq y$ then $\bar{x} \in \bar{R}\bar{y}$.

COROLLARY 5.2. *The following statements are equivalent for the right injective f-ring R.*

(a) *R is unital.*
(b) *R is left convex.*
(c) *R is a qf-ring.*

The following example is the unique totally ordered right injective ring that is not a left qf-ring.

Example 5.3. Let

$$U = \left\{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbf{Q}\right\},$$

$$U^+ = \left\{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a > 0, \text{ or } a = 0 \text{ and } b \geqq 0\right\}.$$

Since each U-homomorphism of a right ideal of U into U is given by left multiplication by some element of U, U is right injective [**14**, 2.4]. Suppose that P is a partial order of U and (U, P) is an f-ring. Since

$$\left\{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} : b \in \mathbf{Q}\right\} = l(U)$$

is an l-ideal of (U, P), $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in P \cup -P$. Suppose that $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in P$. If $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \in P$ with $a < 0$, then

$$-a^{-1}\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -a^{-1}b \\ 0 & 0 \end{bmatrix} \in P.$$

This is impossible since the latter element is the negative of an idempotent. So $P \subseteq U^+$. If $a > 0$, then $a^{-1}\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \in P$, hence $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \in P$. So $P = U^+$.

Similarly, if $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in -P$, then

$$P = P_1 = \left\{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a > 0, \text{ or } a = 0 \text{ and } b \leqq 0\right\}.$$

Thus (U, U^+) and (U, P_1) are the only f-rings whose underlying ring is U, and clearly they are isomorphic

THEOREM 5.4. *Let R be a right injective f-ring. If R is not unital, then $R = B \oplus U$ where U is the f-ring of Example* 5.3 *and B is a unital right injective f-ring.*

Proof. Let e be a left identity of R, and let $A = l(R) = l(e)$. Since R is not unital, $A \neq 0$.

(a) A is a one-dimensional $\mathbf{Q}$-algebra: Clearly $\mathrm{Hom}_{\mathbf{Z}}(A, A) = \mathrm{Hom}_{\mathbf{Q}}(A, A)$, and since $AR = 0$, $\mathrm{Hom}_{\mathbf{Z}}(A, A) = \mathrm{Hom}_R(A_R, A_R)$. Consider the map ϕ: $Re \rightarrow \mathrm{Hom}_{\mathbf{Z}}(A, A)$ given by $t \rightarrow$ left multiplication by t. Since R is right injective ϕ is onto. Let $B = \ker \phi$. Then $B = Re \cap l(A)$ is an l-ideal of Re. So $\mathrm{Hom}_{\mathbf{Q}}(A, A)$ can be made into an f-ring and hence A is one-dimensional, say $A = \mathbf{Q}a$ with $a \in A^+$.

(b) B is a unital f-ring: $R = Re + A$ and $AB = BA = 0$, so B is an ideal of R. The projection of $B + A$ onto B is given by left multiplication by an element $t = b_0 + pa \in Re + A$. Then $b_0 b = b$ for each $b \in B$ and $b_0 a = 0$. Thus b_0 is a left identity of B, and since Re is unital, it is the identity of B.

Let $C = l(b_0) \cap Re$. Since $Re/B \cong \mathbf{Q}$, $C = \mathbf{Q}f$ where $f = e - b_0$. Let $U_1 = C + A$. Then $U_1 = l(B) = r(B) \cong U$ and $R = B \oplus U_1$.

COROLLARY 5.5. *If R is a right injective f-ring, then $P(R)$ is the set of right complements (summands) of R. Moreover, $P(R)$ is the set of left complements (summands) of R if and only if R is unital.*

Proof. If R is unital, then the summands of R_R and ${}_RR$ coincide and each summand is an l-ideal [**28**, 4.2]. Since a right complement of R is a summand, $P(R)$ is the set of right and left complements of R, by 5.2 and 2.1.

Suppose that R is not unital. Then $R = B \oplus U$ as in 5.4. Since B is unital and U has a left identity, U is the unique (not just up to isomorphism) R-injective hull of each of its non-zero ideals. If I is a right complement in R there exists a right ideal J of R such that $R = I \oplus J$ as R-modules. If $I \cap U \neq 0$, then $U = E(I \cap U) \subseteq I$ and so $I = U + (B \cap I)$. But then $J \cap U = 0$, and, since B is unital, $J \subseteq B$. Hence $B = J \oplus (B \cap I)$. Thus $B \cap I$ and hence I are polars (summands) of R. On the other hand, if $I \cap U = 0$, then I and J can be interchanged in the previous argument, so again I is a polar (summand). The first statement now follows from 2.1. To finish the proof, note that

$$B \oplus \mathbf{Q}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

is a summand of ${}_RR$ (one of its complements is $\mathbf{Q}\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$) that is not a polar.

For the remainder of this section a right injective f-ring will be unital. It is known that the class of regular rings is a radical class, i.e., each ring R has a

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largest ideal A which is a regular ring and R/A has no regular ideals. We denote the right singular ideal of R by $Z_r(R)$.

THEOREM 5.6. *If R is a right injective f-ring, then $R = R_1 \oplus R_2$ where R_1 is the regular radical of R.*

Proof. Let

$$R_2 = \{x \in R : xD \subseteq Z_r(R) \text{ for some essential right ideal } D\}.$$

Then R_2 is a right complement [**19**, 2.2], so $R = R_1 \oplus R_2$ where $R_1 = R_2'$. $Z_r(R_1) = 0$ since $Z_r(R) = Z_r(R_1) \oplus Z_r(R_2)$. But R_1 is right injective; so R_1, being its own maximal right quotient ring, is regular.

It remains to show that R_2 has no regular ideals. Suppose that A is a regular ideal of R_2. If I is an ideal of A, then $IR_2 = (IA)R_2 \subseteq IA \subseteq I$; so I is an ideal of R_2. If D is an essential right ideal of R_2, then $D \cap A$ is an essential R-submodule of A_R, hence an essential right ideal of A. Let $x \in A$, and let D be an essential right ideal of R_2 such that $xD \subseteq Z_r(R_2)$. If $d \in D$ there exists an essential right ideal D_1 of R_2 such that $xdD_1 = 0$. But then $xd(D_1 \cap A) = 0$ implies that $xd = 0$; hence $xD = 0$. So $x \in Z_r(A) = 0$.

PROPOSITION 5.7. *Let R be a right injective f-ring which has no regular ideals. Then each non-zero right (left) ideal of R contains a non-zero nilpotent right (left) ideal. Thus the Jacobson radical J of R is an essential right (left) ideal.*

Proof. Let β be the lower radical (l-radical) of R. If A is a right ideal of R, then $A \cap \beta = 0$ implies $R = E(A) \oplus F$ where $F \supseteq E(\beta)$. So $E(A)$ is semiprime, hence regular, and $E(A) = 0$. If $0 \neq B$ is a left ideal, then it is essential in B'', and the latter contains a non-zero nilpotent ideal.

Homomorphic images of ordered polynomial rings and formal power series rings are examples of right injective f-rings of the type in 5.7 (see 5.9). For a non-noetherian example consider homomorphic images of the totally ordered formal power series ring with exponents in the positive reals and coefficients in a totally ordered division ring [**25**, p. 151].

PROPOSITION 5.8. *A right injective f-ring R is the direct product of a family of totally ordered right injective rings if and only if its Boolean algebra of polars is atomic.*

Proof. Suppose that $P(R)$ is atomic, and let $\{E_\alpha : \alpha \in A\}$ be the set of maximal polars of R. Then $\cap E_\alpha = 0$ and, for each α, $R = E_\alpha \oplus R_\alpha$ where R_α is a totally ordered right injective ring. Since $\alpha \neq \beta$ implies $R_\alpha \cap R_\beta = 0$, $\sum^{\oplus} R_\alpha \subseteq R \subseteq \Pi R_\alpha$. By [**31**, 2.1] $Q_r(R) = \Pi Q_r(R_\alpha)$ where $Q_r(X)$ is the maximal right quotient ring of X. Thus $R = \Pi R_\alpha$.

A ring with identity is *quasi-Frobenius* if it is left or right noetherian (artinian) and left or right injective [**13**].

THEOREM 5.9. *The following statements are equivalent for the f-ring R.*

(a) *R is quasi-Frobenius.*

(b) *R is the direct sum of a finite number of totally ordered quasi-Frobenius rings.*

(c) *R is left and right convex and has the minimum condition on left ideals.*

(d) *R is left and right convex, left noetherian, and every prime ideal is maximal.*

Proof. The equivalence of (a) and (b) follows from 3.13, (b) $\Rightarrow$ (c) comes from 5.2, and since (c) implies R is unital, (c) $\Rightarrow$ (d) is obvious.

(d) $\Rightarrow$ (b). By 4.3, R is left duo, and by 3.13 we may assume that R is totally ordered. Since the Jacobson radical J of R is nilpotent and since R is local, ${}_RR$ has a composition series and thus R is left artinian. We claim that R is right duo. For if $Ra \not\subseteq aR$, and hence there exists an element $xa \notin aR$, then $aR \subseteq xaR$. Thus $a = xay$. If $y \notin J$, then $xa \in aR$. If $y \in J$, then $a = x^n a y^n = 0$ for some n. So R is right duo. The argument in [**25**] now completes the proof. If $f : a_i R \to R$ is an R-homomorphism, then since $f(a_i R)$ cannot have a larger composition series than $a_i R$, $f(a_i R) \subseteq a_i R = Ra_i$. So $f(a_i) = xa_i$ and R is right injective.

Note that right convexity is needed in (c) and (d) above. For if R is the f-ring of Example 4.2, then R/Rx^2 is left convex and left artinian, but not right artinian. Note also that this theorem shows again that f-rings are "more commutative" than commutative rings. For, a commutative quasi-Frobenius ring need not be a principal ideal ring [**10**].

In [**24**] Klatt and Levy have given a characterization of a commutative injective valuation ring. Since their arguments are applicable to our non-commutative situation we give this characterization below. A ring R is *left maximal* if given a set $\{J_\alpha\}$ of left ideals and a subset $\{x_\alpha\}$ of R such that the congruences $x \equiv x_\alpha (\text{mod } J_\alpha)$ are pairwise solvable, then these congruences are simultaneously solvable.

THEOREM 5.10. *A totally ordered right injective f-ring is left maximal and each principal left ideal is a left annihilator. Conversely, a totally ordered unital right duo ring satisfying these two conditions is right injective.*

Finally, we mention one other completeness property of right injective f-rings. An f-ring R is *laterally complete* provided every pairwise disjoint subset of R^+ has a least upper bound. Anderson [**2**, 7.3] has shown that a regular f-ring is injective if and only if it is laterally complete. We will prove the only if part of his theorem for an arbitrary right injective f-ring (Anderson's argument will also work in the general case).

PROPOSITION 5.11. *A right injective f-ring R (unital or not) is laterally complete.*

Proof. By Theorem 5.4 we may suppose that R is unital. For each $a \in R$ let e_a be the idempotent of R defined by $e_a R = a''$. The existence of e_a is guaranteed by 5.5. Let A be a pairwise disjoint subset of R^+. If $I = AR$, then $I' = l(\{e_a\})$ and $R = I'' \oplus I'$. By right injectivity, there exists $0 \leqq x \in I''$ such that $xe_a = a$ for each $a \in A$. We claim that x is the least upper bound of A. First, if $\bar{R}$ is any totally ordered homomorphic image of R, then $\bar{e}_a = 0$ or 1, so $\bar{x} \geqq \bar{a}$; hence $x \geqq a$ for each $a \in A$. On the other hand, if $y = y_1 + y_2$, $y_1 \in I''$, $y_2 \in I'$, is an upper bound of A, and $a \in A$, then $ye_a = y_1e_a \geqq a = xe_a$. So $(y_1 - x)e_a \geqq 0$ for each $a \in A$, and hence $y \geqq y_1 \geqq x$.

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