Glasgow Math. J. **43** (2001) 295–309. © Glasgow Mathematical Journal Trust 2001. Printed in the United Kingdom

DISJOINTNESS PRESERVING AND LOCAL OPERATORS ON ALGEBRAS OF DIFFERENTIABLE FUNCTIONS

ROBERT KANTROWITZ

Department of Mathematics, Hamilton College, Clinton, NY 13323, USA
e-mail: rkantrow@hamilton.edu

and MICHAEL M. NEUMANN

Department of Mathematics and Statistics, Mississippi State University, Mississippi State, MS 39762, USA
e-mail: neumann@math.msstate.edu

(Received 8 November, 1999)

Abstract. This article is to discuss the automatic continuity properties and the representation of disjointness preserving linear mappings on certain normal Fréchet algebras of complex-valued functions. This class of operators is defined by the condition that any pair of functions with disjoint cozero sets is mapped to functions with disjoint cozero sets, and subsumes the class of local operators. It turns out that such operators are always continuous outside some finite singularity set of the underlying topological space. Our main emphasis is on disjointness preserving operators from Fréchet algebras of differentiable functions. Such operators are shown to admit a canonical representation that involves weighted composition for the derivatives. This result extends the classical characterization of local operators as linear partial differential operators.

1991 *Mathematics Subject Classification.* Primary 47B38. Secondary 47B48, 46H40, 46J15.

1. Introduction. Given two linear spaces $A(\Omega)$ and $B(\Gamma)$ of complex-valued functions on the non-empty sets Ω and Γ, respectively, a linear mapping $T : A(\Omega) \to B(\Gamma)$ is said to be *disjointness preserving* provided that

$$(Tf)(Tg) = 0 \qquad \text{for all } f, g \in A(\Omega) \text{ with } fg = 0.$$

This condition means precisely that T maps any two functions with disjoint cozero sets to functions with disjoint cozero sets. Here, the cozero set, $\operatorname{coz} f$, of a function $f \in A(\Omega)$ is, as usual, defined as the set of all $\omega \in \Omega$ for which $f(\omega) \neq 0$. Disjointness preserving mappings are also known as *Lamperti operators* [**6**] and as *separating mappings* [**7**].

In the case in which $\Gamma = \Omega$ and $A(\Omega) \subseteq B(\Omega)$, it is easily seen that a linear mapping $T : A(\Omega) \to B(\Omega)$ is disjointness preserving if it is *local*, in the sense that

$$(Tf)g = 0 \qquad \text{for all } f \in A(\Omega) \text{ and } g \in B(\Omega) \text{ with } fg = 0.$$

Examples of local mappings include multiplication and differential operators.

If the spaces $A(\Omega)$ and $B(\Gamma)$ are both algebras with respect to pointwise multiplication, then it is clear that all algebra homomorphisms from $A(\Omega)$ into $B(\Gamma)$ are disjointness preserving. Moreover, if $A(\Omega)$ and $B(\Gamma)$ are vector lattices, then every lattice homomorphism $T : A(\Omega) \to B(\Gamma)$ is disjointness preserving, since this condition may also be expressed as $|Tf| \wedge |Tg| = 0$, for all $f, g \in A(\Omega)$ with $|f| \wedge |g| = 0$.

https://doi.org/10.1017/S0017089501020134 Published online by Cambridge University Press

We mention that the local mappings on a vector lattice $A(\Omega)$ are precisely the band preserving linear operators on $A(\Omega)$; see [**16**, Proposition 3.1.2].

Further typical examples of disjointness preserving mappings are provided by weighted composition operators of the form

$$(Tf)(\gamma) = h(\gamma)f(\varphi(\gamma)) \qquad \text{for all } f \in A(\Omega) \text{ and } \gamma \in \Gamma,$$

for suitable functions $\varphi : \Gamma \to \Omega$ and $h : \Gamma \to \mathbb{C}$. Such operators arise naturally in connection with the Banach–Stone theorem. In fact, it is known that, in certain important special cases, all disjointness preserving operators are weighted composition operators. For pertinent results in the case of vector lattices, we refer to Abramovich [**1**], Arendt [**6**], and Luxemburg [**14**], while disjointness preserving operators between certain Banach algebras of continuous functions have been studied by Araujo and Jarosz [**5**], Beckenstein, Narici, and Todd [**7**], Font [**9**], [**10**], Font and Hernández [**11**], and Jarosz [**12**]. For a recent survey and further references, we direct the reader to Narici and Beckenstein [**17**].

In this article, our main interest is in disjointness preserving mappings on complete metrizable topological algebras of differentiable functions on an open subset Ω of $\mathbb{R}^n$. We first show that such a mapping is, in a natural sense, continuous outside some finite subset of Ω, and then proceed to its representation as a certain kind of differential operator.

To describe our principal results, let $C^m(\Omega)$ denote, for a given integer $m \geq 0$, the algebra of all m times continuously differentiable complex-valued functions on Ω, endowed, as usual, with the topology of locally uniform convergence for the functions and all their partial derivatives up to the order m. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ of non-negative integers, let $|\alpha| := \alpha_1 + \cdots + \alpha_n$ be the order of α, and, for $|\alpha| \leq m$, let $D^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ represent the corresponding linear partial differential operator acting on $C^m(\Omega)$. Also, let $C(\Gamma)$ denote the algebra of all continuous complex-valued functions on a locally compact Hausdorff space Γ, and let $C_0(\Gamma)$ consist of those functions in $C(\Gamma)$ that vanish at infinity.

Under appropriate conditions, we shall prove that every disjointness preserving operator T from $C^m(\Omega)$ into $C(\Gamma)$ or $C_0(\Gamma)$ admits a representation of the form

$$(Tf)(\gamma) = \sum_{|\alpha| \leq m} h_\alpha(\gamma)(D^\alpha f)(\varphi(\gamma)) \qquad \text{for all } f \in C^m(\Omega) \text{ and } \gamma \in \Gamma,$$

with suitable continuous functions $\varphi : \Gamma \to \Omega$ and $h_\alpha : \Gamma \to \mathbb{C}$, for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$. As a special case, we shall obtain the algebraic characterization of linear partial differential operators as local operators in the sense of Peetre [**19**].

2. Automatic continuity. If $A(\Omega)$ is a space of analytic functions on a domain Ω in the complex plane, then the identity theorem implies that every linear mapping on $A(\Omega)$ is local, and that every linear mapping from $A(\Omega)$ into another space of functions is disjointness preserving. This shows that additional conditions are needed to obtain meaningful results. Here we shall focus on disjointness preserving operators on spaces which admit partitions of unity.

Given a normal Hausdorff space Ω, a linear subspace $A(\Omega)$ of the space $C(\Omega)$ of all continuous complex-valued functions on Ω is said to be *normal* if, for all disjoint

https://doi.org/10.1017/S0017089501020134 Published online by Cambridge University Press

closed subsets F and G of Ω, there exists a function $f \in A(\Omega)$ that satisfies $f \equiv 1$ on F and $f \equiv 0$ on G. We first show that, on such spaces, local operators are precisely those which shrink the support. As usual, the support, $\operatorname{supp} f$, of a function $f \in C(\Omega)$ is defined as the closure of the cozero set of f.

LEMMA 1. *Let $A(\Omega)$ and $B(\Omega)$ be normal linear subspaces of $C(\Omega)$, and suppose that $A(\Omega) \subseteq B(\Omega)$. Then, for every linear mapping $T : A(\Omega) \to B(\Omega)$, the following assertions are equivalent:*

(a) *T is local;*
(b) *$(Tf)g = 0$ for all $f, g \in A(\Omega)$ with $fg = 0$;*
(c) *$\operatorname{supp}(Tf) \subseteq \operatorname{supp} f$ for all $f \in A(\Omega)$.*

Proof. The implication (a) $\Rightarrow$ (b) is trivial. To show (b) $\Rightarrow$ (c), let $f \in A(\Omega)$ be given, and consider a point $\omega \in \Omega \setminus \operatorname{supp} f$. Then, by the normality of $A(\Omega)$, there exists some $g \in A(\Omega)$ for which $g(\omega) = 1$ and $g \equiv 0$ on $\operatorname{supp} f$. It follows that $fg = 0$; hence $(Tf)g = 0$ by condition (b), and therefore $(Tf)(\omega) = 0$. Thus $\operatorname{coz}(Tf) \subseteq \operatorname{supp} f$, which implies (c).

Finally, to establish (c) $\Rightarrow$ (a), let $f \in A(\Omega)$ and $g \in B(\Omega)$ satisfy $fg = 0$, and consider a point $\omega \in \Omega$ for which $(Tf)(\omega) \neq 0$. Since $\omega \in \operatorname{supp}(Tf) \subseteq \operatorname{supp} f$ by condition (c), there exists a net $(\omega_\iota)_{\iota \in J}$ in Ω with the property that $\omega_\iota \to \omega$ and $f(\omega_\iota) \neq 0$ for all $\iota \in J$. But then $g(\omega_\iota) = 0$ for all $\iota \in J$, so that $g(\omega) = 0$, by the continuity of g. Thus $(Tf)(\omega)g(\omega) = 0$, and therefore $(Tf)g = 0$, as desired. □

To investigate the continuity properties of disjointness preserving mappings, we shall appeal to the automatic continuity theory for generalized local operators from [**2**] and [**4**]. Let $\mathfrak{F}(\Omega)$ denote the collection of all closed subsets of Ω, and, for a topological linear space X, let $\mathcal{S}(X)$ be the family of all closed linear subspaces of X. A mapping $\mathcal{E} : \mathfrak{F}(\Omega) \to \mathcal{S}(X)$ is called a *precapacity* if $\mathcal{E}(\emptyset) = \{0\}$ and $\mathcal{E}(F) \subseteq \mathcal{E}(G)$, for all $F, G \in \mathfrak{F}(\Omega)$ with $F \subseteq G$. The following result is contained in [**4**, Theorem 4.3].

THEOREM 2. *Let $\Theta : X \to Y$ be a (possibly discontinuous) linear mapping from a complete metrizable topological linear space X into a Banach space Y, let Ω be a normal Hausdorff space, and let $\mathcal{E}_X : \mathfrak{F}(\Omega) \to \mathcal{S}(X)$ and $\mathcal{E}_Y : \mathfrak{F}(\Omega) \to \mathcal{S}(Y)$ be precapacities such that the following conditions are fulfilled:*

(a) *$X = \mathcal{E}_X(\overline{U}) + \mathcal{E}_X(\overline{V})$ for all open sets $U, V \in \Omega$ for which $U \cup V = \Omega$;*
(b) *$\mathcal{E}_Y(\bigcap F_\iota) = \bigcap \mathcal{E}_Y(F_\iota)$ for every collection of sets $F_\iota \in \mathfrak{F}(\Omega)$;*
(c) *$\Theta\mathcal{E}_X(F) \subseteq \mathcal{E}_Y(F)$ for all $F \in \mathfrak{F}(\Omega)$.*

Then the singularity set

$$\Lambda(\Theta) := \{\omega \in \Omega : \Theta \mid \mathcal{E}_X(\overline{U}) \text{ is discontinuous for all open sets } U \subseteq \Omega \text{ with } \omega \in U\}$$

is finite, and $\Theta \mid \mathcal{E}_X(F)$ is continuous for all $F \in \mathfrak{F}(\Omega)$ for which $F \cap \Lambda(\Theta) = \emptyset$. Moreover, if $\mathcal{E}_Y(F) = \{0\}$ for all finite sets $F \subseteq \Omega$, then Θ is automatically continuous.

From general topology we recall that a topological space Ω is said to be a *k-space* if Ω is compactly generated, in the sense that a subset U of Ω is open whenever $U \cap K$ is open in K for every compact subset K of Ω. Evidently, each locally compact and each first countable topological space is a k-space. Hence the next theorem applies, for instance, to every metric space and, in particular, to every open subset of $\mathbb{R}^n$.

It is well known and easily seen that, for every k-space Ω, the space $C(\Omega)$ is complete with respect to the topology of uniform convergence on the compact subsets of Ω. Moreover, $C(\Omega)$ is metrizable whenever Ω is the union of a sequence of compact sets $K_n \subseteq \Omega$ such that each compact subset of Ω is contained in K_n for some $n \in \mathbb{N}$.

THEOREM 3. *Let Ω be a normal Hausdorff space that is a k-space, and let $A(\Omega)$ and $B(\Omega)$ be normal subalgebras of $C(\Omega)$ such that $A(\Omega) \subseteq B(\Omega)$. Suppose that $A(\Omega)$ and $B(\Omega)$ are each endowed with the topology of a complete metrizable topological algebra. Then $A(\Omega)$ is continuously embedded in $B(\Omega)$, and $B(\Omega)$ is continuously embedded in $C(\Omega)$. Moreover, every local linear mapping $T : A(\Omega) \to B(\Omega)$ is automatically continuous.*

Proof. (i) For every point $\omega \in \Omega$, let $\delta_\omega : C(\Omega) \to \mathbb{C}$ denote the evaluation functional given by $\delta_\omega(g) := g(\omega)$ for all $g \in C(\Omega)$. The crucial part of the proof consists of showing that, for every local linear mapping $T : A(\Omega) \to B(\Omega)$, the composition $\delta_\omega \circ T$ is continuous for all $\omega \in \Omega$.

Indeed, if this is known, then we obtain, in particular, that the topology of $B(\Omega)$ is finer than the topology of pointwise convergence on $B(\Omega)$, as the identity mapping on $B(\Omega)$ is local. Hence, given an arbitrary null sequence $(f_n)_{n\in\mathbb{N}}$ in $A(\Omega)$ for which $(Tf_n)_{n\in\mathbb{N}}$ converges in the topology of $B(\Omega)$ to some $g \in B(\Omega)$, we obtain, for each $\omega \in \Omega$, that $(Tf_n)(\omega) \to g(\omega)$ and $(Tf_n)(\omega) = (\delta_\omega \circ T)(f_n) \to 0$ as $n \to \infty$. It follows that $g = 0$, and therefore, by the closed graph theorem, that T is continuous.

Since the inclusion mapping from $A(\Omega)$ into $B(\Omega)$ is local, we conclude that $A(\Omega)$ is continuously embedded in $B(\Omega)$. Moreover, another straightforward application of the closed graph theorem confirms that, for each compact subset K of Ω, the restriction mapping $R_K : B(\Omega) \to C(K)$ given by $R_K f := f \mid K$ for all $f \in B(\Omega)$, is continuous with respect to the supremum norm of the Banach space $C(K)$. This shows that $B(\Omega)$ is indeed continuously embedded in $C(\Omega)$.

(ii) To prepare for the main part of the proof, we introduce the space $X := A(\Omega)$, and claim that the definition

$$\mathcal{E}_X(F) := \{f \in A(\Omega) : \operatorname{supp} f \subseteq F\} \qquad \text{for all } F \in \mathfrak{F}(\Omega),$$

yields a precapacity. Evidently, $\mathcal{E}_X$ is monotone with respect to inclusion and satisfies $\mathcal{E}_X(\emptyset) = \{0\}$. To see that $\mathcal{E}_X(F)$ is a closed linear subspace of $A(\Omega)$, it suffices, by the continuity of the algebra multiplication in $A(\Omega)$, to establish the identity

$$\mathcal{E}_X(F) = \left\{f \in A(\Omega) : fg = 0 \text{ for all } g \in A(\Omega) \text{ with } \operatorname{supp} g \cap F = \emptyset\right\},$$

for every $F \in \mathfrak{F}(\Omega)$. The inclusion $\subseteq$ is trivial. To see the converse, let $f \in A(\Omega)$ be a function for which $\operatorname{supp} f \not\subseteq F$. Then there exists a point $\omega \in \Omega \setminus F$ with $f(\omega) \neq 0$. We choose an open neighborhood U of ω such that $\omega \in U \subseteq \overline{U} \subseteq \Omega \setminus F$, and then, by normality, a function $g \in A(\Omega)$ that satisfies $g(\omega) = 1$ and $g \equiv 0$ on $\Omega \setminus U$. It follows that $\operatorname{supp} g \subseteq \overline{U} \subseteq \Omega \setminus F$ and $(fg)(\omega) \neq 0$. Thus $fg \neq 0$, for some $g \in A(\Omega)$ with $\operatorname{supp} g \cap F = \emptyset$, as desired.

(iii) It is now easily seen that $\delta_\omega \circ T$ is continuous for every isolated point $\omega \in \Omega$. Indeed, for such a point, we obtain, again by the normality of $A(\Omega)$, the decomposition

$$A(\Omega) = \mathcal{E}_X(\{\omega\}) + \mathcal{E}_X(\Omega \setminus \{\omega\}).$$

Since the latter spaces are closed, the continuity of $\delta_\omega \circ T$ follows from a standard application of the open mapping theorem, once the continuity of the two restrictions $(\delta_\omega \circ T) \,|\, \mathcal{E}_X(\{\omega\})$ and $(\delta_\omega \circ T) \,|\, \mathcal{E}_X(\Omega \setminus \{\omega\})$ has been established. But this is immediate, since $\mathcal{E}_X(\{\omega\})$ is one-dimensional, while, by locality and Lemma 1, $\delta_\omega \circ T$ vanishes on $\mathcal{E}_X(\Omega \setminus \{\omega\})$.

(iv) It remains to prove the continuity of $\delta_\omega \circ T$ for a point $\omega \in \Omega$ that is not isolated in Ω. Since Ω is a k-space, we obtain a compact set $K \subseteq \Omega$ such that ω is not isolated in K. We then introduce the Banach space $Y := C(K)$, and consider the precapacity $\mathcal{E}_Y : \mathfrak{F}(\Omega) \to \mathcal{S}(Y)$ given by

$$\mathcal{E}_Y(F) := \{f \in C(K) : \operatorname{supp} f \subseteq F\} \qquad \text{for all } F \in \mathfrak{F}(\Omega).$$

Also, let $\Theta : X \to Y$ be given by $\Theta f := (Tf) \,|\, K$ for all $f \in A(\Omega)$. Since T is local, Lemma 1 ensures that condition (c) of Theorem 2 is fulfilled. Moreover, condition (b) is obvious, and condition (a) follows from the normality of $A(\Omega)$. Indeed, given an arbitrary open cover $\{U, V\}$ of Ω, the sets $\Omega \setminus U$ and $\Omega \setminus V$ are closed and disjoint, so that there exists some $f \in A(\Omega)$ with $f \equiv 0$ on $\Omega \setminus U$ and $f \equiv 1$ on $\Omega \setminus V$. Because $\operatorname{supp}(fg) \subseteq \overline{U}$ and $\operatorname{supp}((1-f)g) \subseteq \overline{V}$ for all $g \in A(\Omega)$, it follows that $A(\Omega) = \mathcal{E}_X(\overline{U}) + \mathcal{E}_X(\overline{V})$, as desired. Hence we conclude from Theorem 2 that $\Lambda(\Theta)$ is finite.

We next observe that $\delta_\tau \circ T$ is continuous for each $\tau \in K \setminus \Lambda(\Theta)$. Indeed, we obtain, by the definition of $\Lambda(\Theta)$, an open neighborhood U of τ in Ω such that $(\delta_\tau \circ T) \,|\, \mathcal{E}_X(\overline{U})$ is continuous, and then choose an open set $V \subseteq \Omega$ such that $U \cup V = \Omega$ and $\tau \notin \overline{V}$. Then, by normality, $A(\Omega) = \mathcal{E}_X(\overline{U}) + \mathcal{E}_X(\overline{V})$, and, by locality, $\delta_\tau \circ T = 0$ on $\mathcal{E}_X(\overline{V})$. Hence another application of the open mapping theorem confirms the continuity of $\delta_\tau \circ T$ on $A(\Omega)$.

Since ω is not isolated in K, and $\Lambda(\Theta)$ is finite, there exists a net $(\tau_\iota)_{\iota \in J}$ in $K \setminus \Lambda(\Theta)$ that converges to ω. Because $\left|(\delta_{\tau_\iota} \circ T)(f)\right| \le \sup\left|(Tf)(K)\right| < \infty$ for all $\iota \in J$ and $f \in A(\Omega)$, an application of the uniform boundedness theorem leads to a neighborhood W of zero in $A(\Omega)$ such that $\left|(Tf)(\tau_\iota)\right| \le 1$, for all $\iota \in J$ and all $f \in W$. Since the functions in $B(\Omega)$ are continuous, we conclude that

$$\left|(Tf)(\omega)\right| \le 1 \qquad \text{for all } f \in W.$$

This establishes the continuity of $\delta_\omega \circ T$, and hence completes the proof. $\square$

The preceding theorem complements automatic continuity results for operators of local type from [**2**], [**3**], [**4**], and [**18**]. Theorem 3 shows, in particular, that, for every open subset Ω of $\mathbb{R}^n$ and every $m \in \mathbb{N}_0 \cup \{\infty\}$, all local operators from the Fréchet algebra $C^m(\Omega)$ into $C(\Omega)$ are necessarily continuous. The same holds for local operators on the Banach algebra $C_b(\Omega)$ of all bounded continuous functions on a metrizable space Ω. It also follows that, for every compact Hausdorff space Ω, all local operators on the Banach algebra $C(\Omega)$ are continuous.

This is in remarkable contrast to the case of disjointness preserving operators. Indeed, Jarosz [**12**] has shown that, for every infinite compact Hausdorff space Ω, there exists a discontinuous disjointness preserving operator on $C(\Omega)$. In fact, if the continuum hypothesis is assumed, then it is even possible to construct discontinuous algebra homomorphisms from such Banach algebras [**8**].

On the other hand, it will be seen in the next result that all disjointness preserving operators on a large class of Fréchet algebras are continuous outside some finite singularity set. This will follow again from Theorem 2, since, perhaps somewhat surprisingly, every disjointness preserving operator may be viewed, in the sense of condition (c) of Theorem 2, as a generalized local operator for a suitable pair of precapacities.

THEOREM 4. *Let Ω be a normal Hausdorff space, and let $A(\Omega)$ be a normal subalgebra of $C(\Omega)$ that is endowed with the topology of a complete metrizable topological algebra. Moreover, suppose that $T : A(\Omega) \to B$ is a linear mapping from $A(\Omega)$ into a complex Banach algebra B such that the following two conditions are fulfilled*:

(a) $(Tf)(Tg) = 0$ *for all* $f, g \in A(\Omega)$ *with* $fg = 0$;

(b) *if* $u \in B$ *satisfies* $uTf = 0$, *for all* $f \in A(\Omega)$ *with compact support, then* $u = 0$.

Then the set $\Lambda(T)$ of all $\omega \in \Omega$ for which $T \,|\, \{f \in A(\Omega) : \operatorname{supp} f \subseteq \overline{U}\}$ is discontinuous, for every open neighborhood U of ω, consists only of finitely many points, and

$$T \,|\, \{f \in A(\Omega) : \operatorname{supp} f \subseteq F\}$$

is continuous, for every closed set $F \subseteq \Omega$ for which $F \cap \Lambda(T) = \emptyset$. In particular, if $\Lambda(T)$ is empty, then T is automatically continuous.

Proof. From part (ii) of the proof of Theorem 3 we know that the definition

$$\mathcal{E}_X(F) := \{f \in A(\Omega) : \operatorname{supp} f \subseteq F\} \qquad \text{for all } F \in \mathfrak{F}(\Omega)$$

yields a precapacity $\mathcal{E}_X : \mathfrak{F}(\Omega) \to \mathcal{S}(X)$, where again $X := A(\Omega)$. Moreover, as shown in part (iv) of that proof, this precapacity satisfies condition (a) of Theorem 2. Also let

$$\mathcal{E}_B(F) := \big\{u \in B : uTf = 0 \text{ for all } f \in A(\Omega) \text{ with compact support in } \Omega \setminus F\big\},$$

for all $F \in \mathfrak{F}(\Omega)$. Then $\mathcal{E}_B : \mathfrak{F}(\Omega) \to \mathcal{S}(B)$ is a precapacity for which $T\mathcal{E}_X(F) \subseteq \mathcal{E}_B(F)$ for every $F \in \mathfrak{F}(\Omega)$, since T is disjointness preserving. Hence the assertion will follow from Theorem 2, once we have shown that

$$\bigcap \mathcal{E}_B(F_\iota) \subseteq \mathcal{E}_B\big(\bigcap F_\iota\big)$$

for an arbitrary collection of closed sets $F_\iota \subseteq \Omega$.

To this end, let $u \in \bigcap \mathcal{E}_B(F_\iota)$, and consider a function $f \in A(\Omega)$ with compact support contained in $\Omega \setminus \bigcap F_\iota$. By compactness, there exist finitely many $\iota_1, \dots, \iota_r$ such that

$$\Omega = (\Omega \setminus \operatorname{supp} f) \cup \big(\Omega \setminus F_{\iota_1}\big) \cup \dots \cup \big(\Omega \setminus F_{\iota_r}\big).$$

As in the proof of [**20**, Theorem 2.13], it follows easily from the normality of the algebra $A(\Omega)$ that there are functions $g_0, g_1, \dots, g_r \in A(\Omega)$ such that $g_0 + g_1 + \dots + g_r \equiv 1$ on Ω, $\operatorname{supp} g_0 \subseteq \Omega \setminus \operatorname{supp} f$, and $\operatorname{supp} g_k \subseteq \Omega \setminus F_{\iota_k}$ for $k = 1, \dots, r$. We conclude that

https://doi.org/10.1017/S0017089501020134 Published online by Cambridge University Press

$$uTf = uT(fg_0) + uT(fg_1) + \cdots + uT(fg_r) = 0,$$

since $fg_0 = 0$ and, for $k = 1, \ldots, r$, the support of fg_k is compact and contained in $\Omega \setminus F_{\iota_k}$. Thus $u \in \mathcal{E}_B(\bigcap F_\iota)$. □

Without condition (b), Theorem 4 is bound to fail in general, since condition (a) could be vacuously satisfied by endowing an arbitrary Banach space B with the zero multiplication.

On the other hand, if one requires only that, for every non-zero $u \in B$, there is some $f \in A(\Omega)$, not necessarily with compact support, such that $uTf \neq 0$, then it is still possible to deduce an automatic continuity result for the disjointness preserving operator T. Even in this case, [**4**, Theorem 4.3] ensures that T has a finite singularity set $\Lambda(T)$, but here $T|\{f \in A(\Omega) : \operatorname{supp} f \subseteq K\}$ will be continuous only for each *compact* set $K \subseteq \Omega$ with $K \cap \Lambda(T) = \emptyset$. This conclusion is remarkably weaker than the one provided by Theorem 4. In fact, the following result, in the spirit of Jarosz [**12**], illustrates that there are discontinuous disjointness preserving operators with empty singularity set.

Proposition 5. *There exists a discontinuous disjointness preserving linear functional $T : C_b(\mathbb{R}) \to \mathbb{C}$ such that $T1 = 1$ and $Tf = 0$, for all $f \in C_b(\mathbb{R})$ with compact support.*

Proof. Let $\beta\mathbb{N}$ be the Stone–Čech compactification of $\mathbb{N}$, viewed as the spectrum of the Banach algebra ℓ^∞ of all bounded sequences of complex numbers and, for each $a \in \ell^\infty$, let $\widehat{a} \in C(\beta\mathbb{N})$ denote the corresponding Gelfand transform. We fix an element $x \in \beta\mathbb{N} \setminus \mathbb{N}$, and introduce the linear subspace $V := \{a \in \ell^\infty : x \notin \operatorname{supp} \widehat{a}\}$ of ℓ^∞. Since $\mathbb{N}$ is dense in $\beta\mathbb{N}$, it is clear that V does not contain the sequences e and u given by $e_n := 1$ and $u_n := 1/n$, for all $n \in \mathbb{N}$. Hence, by Zorn's lemma, there exists a linear functional φ on ℓ^∞ such that $\varphi \equiv 0$ on V, while $\varphi(e) = 1$ and $\varphi(u) \neq 0$. Let $S : C_b(\mathbb{R}) \to \ell^\infty$ be given by

$$Sf := (f(n))_{n\in\mathbb{N}} \qquad \text{for all } f \in C_b(\mathbb{R}),$$

and define $T := \varphi \circ S$. Then $T : C_b(\mathbb{R}) \to \mathbb{C}$ is a linear mapping for which $T1 = \varphi(e) = 1$. Since, for each $a \in \ell^\infty$, the closure of the set $\{n \in \mathbb{N} : a_n \neq 0\}$ in $\beta\mathbb{N}$ is equal to $\operatorname{supp} \widehat{a}$, we have $Sf \in V$ and therefore $Tf = 0$, for all $f \in C_b(\mathbb{R})$ with compact support. Moreover, since the function $h \in C_b(\mathbb{R})$ given by $h(t) := \min\{1, 1/|t|\}$, for all $t \in \mathbb{R}$ is the uniform limit of a sequence of functions in $C_b(\mathbb{R})$ with compact support, we conclude from $T(h) = \varphi(u) \neq 0$ that T is discontinuous.

Finally, to see that T is disjointness preserving, let $f, g \in C_b(\mathbb{R})$ be functions with $fg = 0$. Then $F := \{n \in \mathbb{N} : f(n) \neq 0\}$ and $G := \{n \in \mathbb{N} : g(n) \neq 0\}$ are disjoint. Let $a \in \ell^\infty$ be given by $a_n := 3$, for all $n \in F$, and $a_n := 0$ otherwise. Since the sets $\{u \in \beta\mathbb{N} : |\widehat{a}(u)| \geq 2\}$ and $\{u \in \beta\mathbb{N} : |\widehat{a}(u)| \leq 1\}$ are disjoint and closed in $\beta\mathbb{N}$, it follows that F and G have disjoint closures in $\beta\mathbb{N}$. This ensures that x cannot belong to both $\overline{F}$ and $\overline{G}$. If $x \notin \overline{F}$, then we infer from $\operatorname{supp} \widehat{Sf} = \overline{F}$ that $Sf \in V$, and therefore $Tf = \varphi(Sf) = 0$, while, similarly, $Tg = 0$ if $x \notin \overline{G}$. Thus $(Tf)(Tg) = 0$, as desired. □

It is interesting to note that Theorem 4 may be applied to the disjointness preserving mapping considered in Proposition 5, once $C_b(\mathbb{R})$ is identified with the

Banach algebra $C(\beta\mathbb{R})$. Hence it follows that this mapping has a non-empty finite singularity set in the Stone–Čech compactification of $\mathbb{R}$.

As in Theorem 4, let $A(\Omega)$ be a normal subalgebra of $C(\Omega)$, for some normal Hausdorff space Ω, and let B be a complex vector space. In the following, we are interested in linear mappings $T: A(\Omega) \to B$ that vanish outside some compact subset K of Ω, in the sense that $Tf = 0$, for all $f \in A(\Omega)$ for which $\operatorname{supp} f \cap K = \emptyset$. If this condition holds, and if one assumes that B is a commutative Banach algebra that is generated by the range of T, then condition (b) of Theorem 4 means precisely that B is *without order*, in the sense that $u = 0$ is the only element $u \in B$ that satisfies $uv = 0$, for all $v \in B$. This condition is known from the theory of multipliers, and holds, for instance, when B is semi-simple or has an approximate identity. Note that, by elementary Gelfand theory, every semi-simple commutative Banach algebra is continuously embedded in the Banach algebra $C_0(\Gamma)$, for some locally compact Hausdorff space Γ.

Evidently, every continuous linear mapping from the Fréchet algebra $C^m(\Omega)$ into the Banach algebra $C_0(\Gamma)$ vanishes outside some compact subset of Ω. Moreover, Theorem 4 leads to the following characterization.

COROLLARY 6. *Let $m \in \mathbb{N}_0 \cup \{\infty\}$ be given, let $\Omega \subseteq \mathbb{R}^n$ be an open set, and let Γ be a locally compact Hausdorff space. Then, for every disjointness preserving linear mapping $T: C^m(\Omega) \to C_0(\Gamma)$, the following assertions are equivalent*:

(a) *T vanishes outside some compact subset of Ω;*

(b) *the singularity set $\Lambda(T)$ is finite, and $T \mid \{f \in C^m(\Omega) : \operatorname{supp} f \subseteq F\}$ is continuous for every closed set $F \subseteq \mathbb{R}^n$ for which $F \cap \Lambda(T) = \emptyset$.*

Proof. To show (a) $\Rightarrow$ (b), let K be a compact subset of Ω such that T vanishes in $\Omega \setminus K$, and choose, by the normality of $C^m(\Omega)$, a function $g \in C^m(\Omega)$ with compact support such that $g \equiv 1$ on some open neighborhood of K. Then clearly $Tf = T(fg)$, for all $f \in C^m(\Omega)$. Moreover, if $\Gamma_* := \{\gamma \in \Gamma : (Tf)(\gamma) \neq 0$, for some $f \in C^m(\Omega)\}$, then Γ_* is open, and the definition $T_* f := (Tf) \mid \Gamma_*$ yields a disjointness preserving operator $T_* : C^m(\Omega) \to C_0(\Gamma_*)$. Since, for each $\gamma \in \Gamma_*$, there exists some function $f \in C^m(\Omega)$ with compact support for which $(T_* f)(\gamma) \neq 0$, Theorem 4 applies to T_*, and ensures that condition (b) holds.

To prove (b) $\Rightarrow$ (a), we recall that the topology of the Fréchet algebra $C^m(\Omega)$ is generated by the submultiplicative semi-norms $\sigma_{m,K}$ given by

$$\sigma_{m,K}(f) := \sum_{|\alpha| \leq m} \frac{1}{\alpha!} \sup\left\{ \left|(D^\alpha f)(\omega)\right| : \omega \in K \right\} \qquad \text{for all } f \in C^m(\Omega),$$

where, as usual, $\alpha! := \alpha_1! \cdots \alpha_n!$ for all $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, and K runs over all compact subsets of Ω. We choose a compact neighborhood L of $\Lambda(T)$ in Ω, and define F to be the complement of the interior of L. By the continuity condition, there exist a constant $c > 0$ and a compact subset K of Ω such that

$$\left\| Tf \right\|_\infty \leq c\, \sigma_{m,K}(f) \qquad \text{for all } f \in C^m(\Omega) \text{ with } \operatorname{supp} f \subseteq F,$$

where $\| \cdot \|_\infty$ stands for the supremum norm on $C_0(\Gamma)$. It is then immediate that T vanishes outside the compact set $K \cup L$. $\square$

https://doi.org/10.1017/S0017089501020134 Published online by Cambridge University Press

To conclude this section, we note that Theorem 2 may also be used to obtain similar automatic continuity results for disjointness preserving operators on the Banach algebra $C_0(\Omega)$ for an arbitrary locally compact Hausdorff space Ω, although these algebras are normal only when Ω is compact. For pertinent results in this direction, see **[9]**, **[10]**, **[11]**.

Moreover, some of the preceding results extend to a more general setting, since Theorem 2 remains valid when the range space Y is the dual of a Fréchet space, endowed with the strong topology. This leads to certain automatic continuity results for local operators between spaces of distributions.

For instance, as already noted in **[2]** and **[19]**, it can be shown that, for every open subset Ω of $\mathbb{R}^n$ and every local linear mapping $T : \mathcal{D}(\Omega) \to \mathcal{D}'(\Omega)$ from the space of test functions into the space of distributions on Ω, there exists a countable subset Λ of Ω without any cluster point in Ω such that $T \,|\, \mathcal{D}(\Omega \setminus \Lambda)$ is continuous.

In remarkable contrast to the result of Theorem 3, in the latter case, the singularity set Λ need not be empty. In fact, given an arbitrary countable subset Λ in Ω without cluster points in Ω, it is possible to construct a local linear mapping $T : \mathcal{D}(\Omega) \to \mathcal{D}'(\Omega)$ such that T vanishes on $\mathcal{D}(\Omega \setminus \Lambda)$, whereas $T \,|\, \mathcal{D}(U)$ is discontinuous, for every open subset U of Ω for which $U \cap \Lambda \neq \emptyset$.

Even worse, there are examples of local linear operators on the space $\mathcal{E}'(\Omega)$ of all distributions with compact support that are discontinuous on $\mathcal{E}'(U)$, for every non-empty open subset U of Ω. Similar phenomena occur in the theory of ultradistributions; for details, we refer to **[3]**.

3. Representation. Throughout this section, we consider an integer $m \geq 0$, an open set $\Omega \subseteq \mathbb{R}^n$, and a locally compact Hausdorff space Γ. To obtain a suitable representation of disjointness preserving linear mappings from $C^m(\Omega)$ into $C_0(\Gamma)$ or $C(\Gamma)$, we first collect some preliminaries on support points, a standard tool in the theory of disjointness preserving operators. Our approach to support points is modelled after **[7]**.

Let $T : C^m(\Omega) \to C(\Gamma)$ be a disjointness preserving linear mapping that vanishes outside some compact subset of Ω, let $\gamma \in \Gamma$ be given, and let δ_γ denote the corresponding evaluation functional on $C(\Gamma)$. A point $\omega \in \Omega$ is called a *support point* for the composition $\delta_\gamma \circ T$ if, for every open neighborhood U of ω, there exists a function $f \in C^m(\Omega)$ such that $\operatorname{supp} f \subseteq U$ and $(Tf)(\gamma) \neq 0$.

It is easily seen that there exists at most one support point for $\delta_\gamma \circ T$. Indeed, if ω_1 and ω_2 are two distinct support points, then we obtain disjoint open neighborhoods U_1 and U_2 of ω_1 and ω_2, respectively, and functions $f_1, f_2 \in C^m(\Omega)$ with $\operatorname{supp} f_j \subseteq U_j$ and $(Tf_j)(\gamma) \neq 0$ for $j = 1, 2$. But then $f_1 f_2 = 0$, whereas $(Tf_1)(Tf_2) \neq 0$, which contradicts the condition that T is disjointness preserving.

On the other hand, if one assumes that $\delta_\gamma \circ T$ has no support point, then for each $\omega \in \Omega$ there is an open neighborhood U_ω of ω such that $(Tf)(\gamma) = 0$, for all $f \in C^m(\Omega)$ with $\operatorname{supp} f \subseteq U_\omega$. Now, let $K \subseteq \Omega$ be a compact set outside of which T vanishes, and choose finitely many points $\omega_1, \ldots, \omega_r \in K$ such that $K \subseteq U_{\omega_1} \cup \cdots \cup U_{\omega_r}$. Then there exist $g_1, \ldots, g_r \in C^m(\Omega)$ such that $g_1 + \cdots + g_r \equiv 1$ on some open neighborhood W of K and $\operatorname{supp} g_j \subseteq U_j$, for $j = 1, \ldots, r$. For arbitrary $f \in C^m(\Omega)$, we obtain that $f \equiv fg_1 + \cdots + fg_r$ on W, and therefore $T(f) = T(fg_1 + \cdots + fg_r)$, by the choice of K. Moreover, from $\operatorname{supp}(fg_j) \subseteq U_j$ we infer that $T(fg_j)(\gamma) = 0$ for $j = 1, \ldots, r$ and therefore $(Tf)(\gamma) = 0$, for all $f \in C^m(\Omega)$, which means that $\delta_\gamma \circ T = 0$.

https://doi.org/10.1017/S0017089501020134 Published online by Cambridge University Press

Consequently, if we introduce the set

$$\Gamma_* := \{\gamma \in \Gamma : (Tf)(\gamma) \neq 0 \text{ for some } f \in C^m(\Omega)\},$$

then, for every $\gamma \in \Gamma_*$, there exists exactly one support point ω for $\delta_\gamma \circ T$. Let

$$\varphi : \Gamma_* \to \Omega$$

denote the mapping that assigns to every $\gamma \in \Gamma_*$ the corresponding support point $\omega \in \Omega$.

To establish the continuity of φ, we first note that the range of φ is contained in the compact set K. Hence it suffices to show that, for every net $(\gamma_\iota)_{\iota \in J}$ in Γ_* and every pair of points $\gamma \in \Gamma_*$ and $\omega \in \Omega$, the condition that $\gamma_\iota \to \gamma$ and $\varphi(\gamma_\iota) \to \omega$ ensures that $\varphi(\gamma) = \omega$. If we assume that $\varphi(\gamma) \neq \omega$, then we obtain disjoint open neighborhoods U of $\varphi(\gamma)$ and V of ω. By the definition of φ, there is a function $f \in C^m(\Omega)$ for which $\operatorname{supp} f \subseteq U$ and $(Tf)(\gamma) \neq 0$. Since the function Tf is continuous on Γ, the convergence $\gamma_\iota \to \gamma$ and $\varphi(\gamma_\iota) \to \omega$ implies that there exists some $\iota \in J$ for which $(Tf)(\gamma_\iota) \neq 0$ and $\varphi(\gamma_\iota) \in V$. Again by the definition of φ, we obtain a function $g \in C^m(\Omega)$ for which $\operatorname{supp} g \subseteq V$ and $(Tg)(\gamma_\iota) \neq 0$. But this is impossible, since $fg = 0$ and T is disjointness preserving. Thus $\varphi(\gamma) = \omega$, which shows that φ is continuous.

It is interesting to note that every disjointness preserving operator on $C^m(\Omega)$ behaves, modulo its support function, like a local operator, in the sense that

$$\varphi(\operatorname{coz} Tf) \subseteq \operatorname{supp} f \qquad \text{for all } f \in C^m(\Omega).$$

To see this, let $\gamma \in \operatorname{coz} Tf$ be given, and assume that $\varphi(\gamma) \notin \operatorname{supp} f$. Since $\Omega \setminus \operatorname{supp} f$ is an open neighborhood of the support point $\varphi(\gamma)$, there exists a function $g \in C^m(\Omega)$ for which $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$ and $(Tg)(\gamma) \neq 0$. Thus $fg = 0$ and $(Tf)(\gamma)(Tg)(\gamma) \neq 0$. But this is impossible, since T preserves disjointness.

If $T : C^m(\Omega) \to C(\Omega)$ is a disjointness preserving linear mapping that vanishes outside some compact subset of Ω, then it follows that the operator T is local precisely when its support function is the identity mapping on Ω_*. Indeed, by the result of the preceding paragraph and Lemma 1, the latter condition implies that T is local. For the converse, suppose that $\varphi(\omega) \neq \omega$, for some $\omega \in \Omega$, and choose disjoint open neighborhoods U and V of $\varphi(\omega)$ and ω, respectively. Then there exist functions $f \in C^m(\Omega)$ and $g \in C(\Omega)$ for which $\operatorname{supp} f \subseteq U$ and $(Tf)(\omega) \neq 0$, while $\operatorname{supp} g \subseteq V$ and $g(\omega) = 1$. Thus $fg = 0$ and $(Tf)g \neq 0$, so that T cannot be local.

In addition to the preceding properties of the support function, our representation of disjointness preserving operators requires the following result on differentiable functions.

LEMMA 7. *Let $\omega \in \Omega$, and let $f \in C^m(\Omega)$ be a function for which $(D^\alpha f)(\omega) = 0$, for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$. Then there exists a sequence $(g_k)_{k \in \mathbb{N}}$ of functions in $C^m(\Omega)$ such that each g_k is identically equal to one on some open neighborhood of ω, while $g_k f \to 0$ as $k \to \infty$, in the topology of $C^m(\Omega)$.*

Proof. For each $k \in \mathbb{N}$, let $V_k := \{\lambda \in \Omega : \|\lambda - \omega\| < 1/k\}$, where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$. By a classical result on differentiable functions, [**13**,

https://doi.org/10.1017/S0017089501020134 Published online by Cambridge University Press

Lemma 1.5.1] or [**15**, Lemma I.4.2], there exists a sequence of functions $g_k \in C^m(\Omega)$ for which $g_k \equiv 1$ on some open neighborhood of ω, $\operatorname{supp} g_k \subseteq V_k$, and $\|D^\alpha g_k\|_\infty \leq c\, k^{|\alpha|}$ for all $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$, where $c > 0$ is a constant that depends only on m and n, but not on k.

Moreover, given an arbitrary function $f \in C^m(\Omega)$ that satisfies $(D^\alpha f)(\omega) = 0$, for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$, a well-known application of the mean value theorem ensures that $|f(\lambda)|/\|\lambda - \omega\|^m \to 0$ as $\lambda \to \omega$ in Ω.

For each $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$, this implies that $|(D^\alpha f)(\lambda)|/\|\lambda - \omega\|^{m-|\alpha|} \to 0$ as $\lambda \to \omega$ in Ω. Consequently, for every $\varepsilon > 0$, there exists some $\delta(\varepsilon) > 0$ such that

$$\left|(D^\alpha f)(\lambda)\right| \leq \varepsilon \|\lambda - \omega\|^{m-|\alpha|},$$

for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$ and all $\lambda \in \Omega$ with $\|\lambda - \omega\| < \delta(\varepsilon)$.

Now, let $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$ be given, and consider an arbitrary integer $k \in \mathbb{N}$ for which $1/k < \delta(\varepsilon)$. For each point $\lambda \in \Omega$ with $\|\lambda - \omega\| > 1/k$, it follows from $\operatorname{supp} g_k \subseteq V_k$ that $(D^\alpha(g_k f))(\lambda) = 0$. On the other hand, if $\|\lambda - \omega\| \leq 1/k$, then the multivariate version of the Leibniz rule and the preceding estimates entail that

$$\left|(D^\alpha(g_k f))(\lambda)\right| \leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \left|(D^{\alpha-\beta} g_k)(\lambda)\right| \left|(D^\beta f)(\lambda)\right| \leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} c\, \varepsilon\, k^{|\alpha|-m},$$

where the summations are taken over all $\beta \in \mathbb{N}_0^n$ for which $\beta_j \leq \alpha_j$ for $j = 1, \ldots, n$. Thus $D^\alpha(g_k f) \to 0$ as $k \to \infty$, uniformly on Ω. The assertion follows. $\square$

Now we are in a position to establish the canonical representation of disjointness preserving operators from $C^m(\Omega)$ into $C_0(\Gamma)$ or $C(\Gamma)$. We begin with the case of the range space $C_0(\Gamma)$. The other case will then be reduced to this one.

THEOREM 8. *Given an arbitrary collection of functions $h_\alpha \in C_0(\Gamma)$, for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$, and a continuous mapping $\varphi : \Gamma \to \Omega$ for which the closure of the range is a compact subset of Ω, the definition*

$$(Tf)(\gamma) := \sum_{|\alpha| \leq m} h_\alpha(\gamma)(D^\alpha f)(\varphi(\gamma)) \qquad \text{for all } f \in C^m(\Omega) \text{ and } \gamma \in \Gamma$$

yields a disjointness preserving continuous linear operator $T : C^m(\Omega) \to C_0(\Gamma)$ whose support function coincides with φ on Γ_*.

Conversely, if $T : C^m(\Omega) \to C_0(\Gamma)$ denotes an arbitrary disjointness preserving continuous linear operator, then there exist functions $h_\alpha \in C_0(\Gamma_*)$, for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$ and a continuous mapping $\varphi : \Gamma_* \to \Omega$ whose range is contained in a compact subset of Ω such that

$$(Tf)(\gamma) = \sum_{|\alpha| \leq m} h_\alpha(\gamma)(D^\alpha f)(\varphi(\gamma)) \qquad \text{for all } f \in C^m(\Omega) \text{ and } \gamma \in \Gamma_*.$$

This representation of T is unique.

Proof. (i) To establish the first half of the theorem, we note that the conditions on the functions h_α and φ ensure that T indeed maps into $C_0(\Gamma)$. Moreover, for a

https://doi.org/10.1017/S0017089501020134 Published online by Cambridge University Press

suitable constant $c > 0$ we obtain the estimate $\|Tf\|_\infty \le c\,\sigma_{m,K}(f)$ for all $f \in C^m(\Omega)$, where K is any compact subset of Ω that contains the range of φ. In particular, it follows that T is continuous and vanishes outside K.

To show that T preserves disjointness, we observe that, for arbitrary $f, g \in C^m(\Omega)$ with $fg = 0$, it follows that f vanishes in a neighborhood of any point $\omega \in \Omega$ for which $g(\omega) \neq 0$. This entails that $(D^\alpha f)g = 0$ for every $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le m$. Hence, by the same argument, $(D^\alpha f)(D^\beta g) = 0$, for all $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha|, |\beta| \le m$, and therefore $(Tf)(Tg) = 0$.

To see that $\varphi \,|\, \Gamma_*$ is the support mapping of T, let a point $\gamma \in \Gamma_*$ be given, and let U be an open neighborhood of $\varphi(\gamma)$. Then there exist functions $g, e \in C^m(\Omega)$ such that $(Tg)(\gamma) \neq 0$, $\operatorname{supp} e \subseteq U$, and $e \equiv 1$ on some open neighborhood of $\varphi(\gamma)$. Evidently, the function $f := ge \in C^m(\Omega)$ satisfies $\operatorname{supp} f \subseteq \operatorname{supp} e \subseteq U$. Moreover, since f and g agree on an open neighborhood of $\varphi(\gamma)$, the inclusion $\varphi(\operatorname{coz} T(f-g)) \subseteq \operatorname{supp}(f-g)$ then guarantees that $(Tf)(\gamma) = (Tg)(\gamma) \neq 0$, as desired.

(ii) Let $T : C^m(\Omega) \to C_0(\Gamma)$ be a disjointness preserving continuous linear operator, and let $\gamma \in \Gamma_*$ be given. Since, by continuity, T vanishes outside some compact subset of Ω, we may consider the corresponding support point $\varphi(\gamma) \in \Omega$.

We first claim that $(Tf)(\gamma) = 0$, for every $f \in C^m(\Omega)$ with the property that $(D^\alpha f)(\varphi(\gamma)) = 0$, for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le m$. To see this, we apply Lemma 7 to obtain a sequence of functions $g_k \in C^m(\Omega)$ such that $g_k f \to 0$ in $C^m(\Omega)$ and $g_k \equiv 1$ on some open neighborhood U_k of $\varphi(\gamma)$. By the continuity of T, it follows that $T(g_k f) \to 0$ as $k \to \infty$, uniformly on Γ. On the other hand, for each $k \in \mathbb{N}$, we know that $\varphi(\gamma) \notin \operatorname{supp}(f - g_k f)$. Because $\varphi(\operatorname{coz} T(f - g_k f)) \subseteq \operatorname{supp}(f - g_k f)$, we conclude that $\gamma \notin \operatorname{coz} T(f - g_k f)$, and therefore $(Tf)(\gamma) = (T(g_k f))(\gamma)$. This shows that $(Tf)(\gamma) = 0$.

Now, for an arbitrary function $f \in C^m(\Omega)$, we may apply the result of the preceding paragraph to the function $f - p$, where p denotes the mth Taylor polynomial for f about $\varphi(\gamma)$ given by

$$p(\omega) := \sum_{|\alpha| \le m} \frac{(D^\alpha f)(\varphi(\gamma))}{\alpha!} (\omega - \varphi(\gamma))^\alpha \qquad \text{for all } \omega \in \Omega,$$

with the convention that $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$. Hence, if Z denotes the identity function on Ω, then we obtain

$$(Tf)(\gamma) = (Tp)(\gamma) = \sum_{|\alpha| \le m} \frac{(D^\alpha f)(\varphi(\gamma))}{\alpha!} T((Z - \varphi(\gamma))^\alpha)(\gamma).$$

This establishes the representation

$$(Tf)(\gamma) = \sum_{|\alpha| \le m} h_\alpha(\gamma)\,(D^\alpha f)(\varphi(\gamma)) \qquad \text{for all } f \in C^m(\Omega) \text{ and } \gamma \in \Gamma_*$$

with the choice

$$h_\alpha(\gamma) := \frac{1}{\alpha!} T((Z - \varphi(\gamma))^\alpha)(\gamma) = \sum_{\beta \le \alpha} \frac{(-\varphi(\gamma))^{a-\beta}}{\beta!\,(\alpha - \beta)!} T(Z^\beta)(\gamma)$$

https://doi.org/10.1017/S0017089501020134 Published online by Cambridge University Press

for all $\gamma \in \Gamma_*$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$, where the last identity follows from the multivariate version of the binomial theorem. Since the support mapping φ is continuous and bounded, and since T maps into $C_0(\Gamma)$, it is immediate that each of the functions h_α belongs to $C_0(\Gamma_*)$.

To prove the last assertion of the theorem, we note that φ coincides with the support mapping of T, while the uniqueness of the functions h_α follows by a simple inductive argument involving the functions $T(Z^\alpha)$ for $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$. □

THEOREM 9. *A disjointness preserving continuous linear operator $T : C^m(\Omega) \to C(\Gamma)$ has a unique representation of the form*

$$(Tf)(\gamma) = \sum_{|\alpha| \leq m} h_\alpha(\gamma)(D^\alpha f)(\varphi(\gamma)) \qquad \text{for all } f \in C^m(\Omega) \text{ and } \gamma \in \Gamma_*,$$

with continuous functions $\varphi : \Gamma_* \to \Omega$ and $h_\alpha : \Gamma_* \to \mathbb{C}$, for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$.

Conversely, for every choice of continuous functions $\varphi : \Gamma \to \Omega$ and $h_\alpha : \Gamma \to \mathbb{C}$ for $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$, the preceding formula, evaluated for all $f \in C^m(\Omega)$ and $\gamma \in \Gamma$, defines a disjointness preserving continuous linear operator $T : C^m(\Omega) \to C(\Gamma)$.

Proof. Given a disjointness preserving continuous linear operator $T : C^m(\Omega) \to C(\Gamma)$ and a compact subset K of Γ, we observe that the definition

$$T_K f := (Tf) \,|\, K \qquad \text{for all } f \in C^m(\Omega)$$

yields a disjointness preserving continuous linear operator $T_K : C^m(\Omega) \to C(K)$. Hence, by Theorem 8, we have a unique representation of the form

$$(T_K f)(\gamma) = \sum_{|\alpha| \leq m} h_{\alpha,K}(\gamma)(D^\alpha f)(\varphi_K(\gamma)) \qquad \text{for all } f \in C^m(\Omega) \text{ and } \gamma \in K_*,$$

with continuous functions $\varphi_K : K_* \to \Omega$ and $h_{\alpha,K} \in C_0(K_*)$, for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$. By uniqueness, for every pair of compact sets $K, L \subseteq \Gamma$, we obtain both $\varphi_K \equiv \varphi_L$ and $h_{\alpha,K} \equiv h_{\alpha,L}$ on $K_* \cap L_*$, for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$. Moreover, it is immediate that Γ_* is equal to the union of the sets K_* as K runs over all compact subsets of Γ. Since Γ is locally compact, we thus obtain continuous functions $\varphi : \Gamma_* \to \Omega$ and $h_\alpha : \Gamma_* \to \mathbb{C}$ such that $\varphi \equiv \varphi_K$ and $h_\alpha \equiv h_{\alpha,K}$ on K, for every compact set $K \subseteq \Gamma$ and all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$. It follows that T has a representation of the desired form. The remaining assertions are clear from Theorem 8. □

The following result due to Peetre [**19**] is a simple consequence of Theorems 3 and 9. Our approach is more elementary, since it avoids the theory of distributions.

COROLLARY 10. *The local linear operators from $C^m(\Omega)$ into $C(\Omega)$ are precisely the linear partial differential operators on $C^m(\Omega)$ with continuous coefficients.*

We conclude with a brief discussion of disjointness preserving operators on certain Banach algebras of differentiable functions of the type $C^m([0, 1])$.

Given a bounded open subset Ω of $\mathbb{R}^n$, let $C^m(\overline{\Omega})$ consist of the functions $f \in C^m(\Omega)$ which, together with all their partial derivatives up to the order m, have

https://doi.org/10.1017/S0017089501020134 Published online by Cambridge University Press

continuous extensions to the closure $\overline{\Omega}$, also denoted by $D^\alpha f$, for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$. Endowed with pointwise operations and the norm $\sigma_{m,\overline{\Omega}}$, the space $C^m(\overline{\Omega})$ is a Banach algebra.

To ensure reasonable Taylor expansions for all functions in $C^m(\overline{\Omega})$ even at the boundary points of $\overline{\Omega}$, some condition on Ω is needed. Following Whitney [**21**], we say that Ω is *locally regular* if, for every point $\omega \in \overline{\Omega}$, there exist an open neighborhood U of ω in $\mathbb{R}^n$ and a constant $c > 0$ such that any two points $u, v \in \Omega \cap U$ can be joined by a rectifiable curve in $\Omega \cap U$ of length not exceeding $c\|u - v\|$. Simple examples are provided by the convex bounded open sets.

If Ω is open, bounded, and locally regular, then every $f \in C^m(\overline{\Omega})$ has a Taylor expansion of order m at every point $\omega \in \overline{\Omega}$, in the sense that

$$\|\lambda - \omega\|^{-m} \left| f(\lambda) - \sum_{|\alpha| \leq m} \frac{(D^\alpha f)(\omega)}{\alpha!} (\lambda - \omega)^\alpha \right| \to 0 \qquad \text{as } \lambda \to \omega \text{ in } \Omega;$$

see [**21**, Lemma 3], and also [**13**, Proposition 5.5.4] for a short approach. Hence a glance at the proof of Lemma 7 reveals that this lemma remains valid for arbitrary points $\omega \in \overline{\Omega}$ and functions $f \in C^m(\overline{\Omega})$ provided that Ω is locally regular. One may therefore proceed exactly as in the proof of Theorem 8 to obtain the following result.

THEOREM 11. *Let $\Omega \subseteq \mathbb{R}^n$ be open, bounded, and locally regular. Then every disjointness preserving continuous linear operator $T : C^m(\overline{\Omega}) \to C_0(\Gamma)$ has a unique representation of the form*

$$(Tf)(\gamma) = \sum_{|\alpha| \leq m} h_\alpha(\gamma)(D^\alpha f)(\varphi(\gamma)) \qquad \textit{for all } f \in C^m(\overline{\Omega}) \textit{ and } \gamma \in \Gamma_*,$$

where $\varphi : \Gamma_* \to \overline{\Omega}$ is continuous and $h_\alpha \in C_0(\Gamma_*)$ for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$.

Conversely, given a continuous function $\varphi : \Gamma \to \overline{\Omega}$ and $h_\alpha \in C_0(\Gamma)$, for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$, the preceding formula defines a disjointness preserving continuous linear operator $T : C^m(\overline{\Omega}) \to C_0(\Gamma)$.

REFERENCES

1. Y. A. Abramovich, Multiplicative representation of disjointness preserving operators, *Indag. Math.* **45** (1983), 265–279.

2. E. Albrecht and M. M. Neumann, Automatic continuity of generalized local linear operators, *Manuscripta Math.* **32** (1980), 263–294.

3. E. Albrecht and M. M. Neumann, Local operators between spaces of ultradifferentiable functions and ultradistributions, *Manuscripta Math.* **38** (1982), 131–161.

4. E. Albrecht and M. M. Neumann, Automatic continuity for operators of local type, in *Radical Banach algebras and automatic continuity*, Lecture Notes in Math. No. 975 (Springer-Verlag, Berlin, 1983), 342–355.

5. J. Araujo and K. Jarosz, Separating maps on spaces of continuous functions, *Contemp. Math.* **232** (1999), 33–37.

6. W. Arendt, Spectral properties of Lamperti operators, *Indiana Univ. Math. J.* **32** (1983), 199–215.

7. E. Beckenstein, L. Narici, and A. R. Todd, Automatic continuity of linear maps on spaces of continuous functions, *Manuscripta Math.* **62** (1988), 257–275.

https://doi.org/10.1017/S0017089501020134 Published online by Cambridge University Press

8. H. G. Dales, A discontinuous homomorphism from $C(X)$, *Amer. J. Math.* **101** (1979), 647–734.

9. J. J. Font, Automatic continuity of certain isomorphisms between regular Banach function algebras, *Glasgow Math. J.* **39** (1997), 333–343.

10. J. J. Font, Disjointness preserving mappings between Fourier algebras, *Colloq. Math.* **77** (1998), 179–187.

11. J. J. Font and S. Hernández, On separating maps between locally compact spaces, *Arch. Math. (Basel)* **63** (1994), 158–165.

12. K. Jarosz, Automatic continuity of separating linear isomorphisms, *Canad. Math. Bull.* **33** (1990), 139–144.

13. K. B. Laursen and M. M. Neumann, *An introduction to local spectral theory* (Clarendon Press, Oxford, 2000).

14. W. A. J. Luxemburg, *Some aspects of the theory of Riesz spaces*, University of Arkansas Lecture Notes in Math. **4** (Fayetteville, Arkansas, 1979).

15. B. Malgrange, *Ideals of differentiable functions* (Oxford University Press, 1966).

16. P. Meyer-Nieberg, *Banach lattices* (Springer-Verlag, 1991).

17. L. Narici and E. Beckenstein, The separating map: a survey, *Rend. Circ. Mat. Palermo (2) Suppl.* **52** (1998), 637–648.

18. M. M. Neumann and V. Pták, Automatic continuity, local type and causality, *Studia Math.* **82** (1985), 61–90.

19. J. Peetre, Réctification à l'article "Une caractérisation abstraite des opérateurs différentiels", *Math. Scand.* **8** (1960), 116–120.

20. W. Rudin, *Real and complex analysis* (McGraw-Hill, New York, 1967).

21. H. Whitney, Functions differentiable on the boundaries of regions, *Ann. of Math.* **35** (1934), 482–485.

https://doi.org/10.1017/S0017089501020134 Published online by Cambridge University Press