GENERALISED QUASIPERFECT NUMBERS

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The main concern of this thesis is an investigation of the equation
$\sigma(n) = 2n + k^2$, where n and k are relatively prime positive integers,
k is odd and $\sigma(n)$ is the sum of the positive divisors of n . We have
called any n which satisfies this equation for some k a generalised
quasiperfect number, or, more specifically, a k-quasiperfect number, since
such n , for $k = 1$, are known as quasiperfect numbers. No quasiperfect
number has yet been found, but neither has a proof that none exist. The
same is true of generalised quasiperfect numbers: we know of no example,
for any k , nor can we prove for any k that none exist.

Nonetheless, we have been able to establish a great number of
necessary conditions for the existence of a generalised quasiperfect
number, and many of these are direct generalisations of previously known
results about quasiperfect numbers. For example, the first necessary
property of a quasiperfect number to be established was that such a number
must be an odd perfect square; the same is true of a k-quasiperfect
number, for every k .

The following definitions are pertinent to the ensuing summary. For a
real number $\alpha \geq 1$, we say that a positive integer n is α-perfect if
$\sigma(n) = \alpha n$, and primitive α-abundant if $\sigma(n) > \alpha n$ with $\sigma(m) < \alpha m$ for
any proper divisor m of n . When $\alpha = 2$, the prefix is omitted.

Some of the main results of this thesis are as follows. Suppose there
is a k-quasiperfect number n . Then n must have at least four distinct
prime factors, and must exceed 10^{20} . If n has only four or five

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distinct prime factors, then k must exceed 10^{10} . If $k \leq 44366047$,
then n must be primitive abundant. If n is primitive abundant, then n
must have at least six distinct prime factors. Much of what was known for
the special case of quasiperfect numbers, prior to 1979, is included in
these results.

The thesis includes also a more general investigation of primitive
α-abundant numbers. We show that, for positive integers x, y, z , if
$x\sigma(n) = yn + z$, n is not primitive (y/x)-abundant and n/s is not
(y/x)-perfect (where s^d is the largest prime power in n), then

$$n < \max\{2(z-1)^{5/2}/y, \ 4(z+\tfrac{1}{2})^3/27y\} \ .$$

We table all solutions of the equation $\sigma(n) = yn + z$ $(y \geq 2, z \leq 210)$
for which n is not primitive y-abundant. These extend previous results
in the literature. Some new properties of a primitive α-abundant number
n are obtained: $\sigma(n)/n < \alpha + \min\{\tfrac{1}{2}, \ 3\alpha e^{-5\alpha/9}/2\}$ and
$\sigma(n)/n < \alpha + 1.6 \ \alpha/\log n$. We list all primitive abundant numbers n with
$\sigma(n)/n \geq 2.05$ and, in so doing, show that $\sigma(n)/n < 2.16104$.

The first result of the preceding paragraph has a useful application
to generalised quasiperfect numbers. We use it in showing that, if n is
k-quasiperfect and not primitive abundant, then $n < 2(1+10^{-15})k^6/27$.
This makes use of a result developed in connection with the existence of
odd perfect numbers and, to our knowledge, is the first application of that
theory outside the field itself.

It was conjectured in 1975 that the equation $\sigma(n) = 2n + k^2$, with
k odd, has no solutions at all. We show this is not true, and exhibit
seven solutions. (The smallest n for which $\sigma(n) = 2n + k^2$ for some odd
k is $n = 2^2 7^2 53^2$.) The final chapter of the thesis considers this and
other aspects of the equation $\sigma(n) = 2n + k^2$ when some of our original
restrictions on n and k are relaxed.

In the Appendix, some new results are proved concerning odd perfect
numbers. Suppose $N = p^a q_1^{2b_i} \ldots q_u^{2b_u}$ (where $p \equiv a \equiv 1 \ (\text{mod } 4)$ and

$p, q_1, \ldots, q_u$ are distinct odd primes) is an odd perfect number. We prove the following, which are extensions of earlier results in the literature. If $b_1 = \ldots = b_u = b$, then $b \neq 6, 8, 11, 14$ or 18 (a joint result with R.J. Williams). If $b_2 = \ldots = b_u = 1$, then $b_1 \neq 5$ or 6. If x is the number of powers $q_i^{2b_i}$ in which both $q_i \equiv 1 \pmod 4$ and $b_i \equiv 1 \pmod 2$, then $p - a \equiv 4x \pmod 8$. If $b_1 = \ldots = b_t = 1$ and $b_{t+1} = \ldots = b_u = 2$, then $1916 \leq u - t \leq \sqrt{a} + 2t$ and $N > 10^{83282}$. If $b_2 = 2$ and $b_3 = \ldots = b_u = 1$, then $b_1 \neq 3$.

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