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Sampling and interpolation for the discrete Hilbert and Kak–Hilbert transforms

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Abstract. The goal of the paper is to obtain analogs of the sampling theorems and of the Riesz–Boas interpolation formulas which are relevant to the discrete Hilbert and Kak–Hilbert transforms in l^2.

1 Introduction

The objective of the paper is to establish some analogs of the classical (Shannon) sampling theorems and Riesz–Boas interpolation formulas which are associated with the discrete Hilbert transform and Kak–Hilbert transform in l^2. The basic idea is to utilize their one-parameter uniformly bounded groups of operators in the space l^2 to reduce questions about sampling and interpolation to the classical ones. Such an approach to sampling and interpolation for general one-parameter uniformly bounded groups of class C_0 (i.e., continuous in strong topology) of operators in Banach spaces was developed in [13, 14]. The main part of the present paper is devoted to the discrete Hilbert transform. Here, we show in all the details how one can use one-parameter group of isometries generated by the discrete Hilbert transform in l^2 to obtain several relevant sampling and interpolation results.

If $\widetilde{H}$ is the discrete Hilbert transform in the space l^2 with the natural inner product $\langle\cdot,\cdot\rangle$, (see Section 2 for all the definitions) then the bounded operator $H = \pi\widetilde{H}$ generates a one-parameter group e^{tH}, $t \in \mathbb{R}$, of isometries of l^2. The fact that e^{tH}, $t \in \mathbb{R}$, is a group of isometries and the explicit formula for all e^{tH} were given in [7]. In our first sampling Theorem 3.2, we give an explicit formula for a function $\langle e^{tH}\mathbf{a}, \mathbf{a}^*\rangle$, $t \in \mathbb{R}$, for every $\mathbf{a}, \mathbf{a}^* \in l^2$, in terms of equally spaced "samples" $\langle e^{\gamma kH}\mathbf{a}, \mathbf{a}^*\rangle$, $k \in \mathbb{Z}$, for any $0 < \gamma < 1$. In two other sampling Theorems 3.5 and 3.7, we express the entire trajectory $e^{tH}\mathbf{a}$, $t \in \mathbb{R}$, $\mathbf{a} \in l^2$, in terms of the integer translations $e^{kH}\mathbf{a}$, $k \in \mathbb{Z}$. In Section 4, we have an analog of a sampling theorem with irregularly spaced "samples."

In Section 5, we present some analogs of the classical Riesz–Boas interpolation formulas. Namely, we give explicit formulas for $H^{2m-1}\mathbf{a}$, $m \in \mathbb{N}$, $\mathbf{a} \in l^2$, in terms of the vectors $e^{(k-1/2)H}\mathbf{a}$, $k \in \mathbb{Z}$, and for $H^{2m}\mathbf{a}$, $m \in \mathbb{N}$, in terms of $e^{kH}\mathbf{a}$, $k \in \mathbb{Z}$.

In Section 6, we briefly describe how similar results can be obtained in the case of the Kak–Hilbert transform.

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2 Some harmonic analysis associated with the discrete Hilbert transform

We will be interested in the operator $H = \pi\widetilde{H}$ where $\widetilde{H}$ is the discrete Hilbert transform operator

$$\widetilde{H} : l^2 \mapsto l^2,\quad \widetilde{H}\mathbf{a} = \mathbf{b},\quad \mathbf{a} = \{a_j\} \in l^2,\quad \mathbf{b} = \{b_m\} \in l^2,$$

which is defined by the formula

$$b_m = \frac{1}{\pi} \sum_{n\neq m,\ n\in\mathbb{Z}} \frac{a_n}{m-n},\quad m \in \mathbb{Z}. \tag{2.1}$$

Since H is a bounded operator, the following exponential series converges in l^2 for every $\mathbf{a} \in l^2$ and every $t \in \mathbb{R}$:

$$e^{tH}\mathbf{a} = \sum_{k=0}^{\infty} \frac{H^k\mathbf{a}}{k!} t^k. \tag{2.2}$$

In fact, H is a generator of a one-parameter group of operators e^{tH}, $t \in \mathbb{R}$, which means that [4, 11]

(1)

$$e^{t_1H}e^{t_2H} = e^{(t_1+t_2)H},\quad e^0 = I,$$

(2)

$$e^{-tH} = \left(e^{tH}\right)^{-1},$$

(3) for every $\mathbf{a} \in l^2$,

$$\lim_{t\to 0} \frac{e^{tH}\mathbf{a} - \mathbf{a}}{t} = H\mathbf{a}.$$

It is clear that for a general bounded operator A, the exponent can be extended to the entire complex plane $\mathbb{C}$, and one has the estimate

$$\|e^{zA}\| \leq \sum_{k=0}^{\infty} \frac{\|A\|^k|z|^k}{k!} = e^{\|A\||z|},\quad z \in \mathbb{C}. \tag{2.3}$$

In the nice paper by De Carli and Samad [7] about the group e^{tH}, the following results were obtained (among other interesting results):

(1) The explicit formulas for the operators e^{tH} were given.
(2) It was shown that every operator e^{tH} is an isometry in l^2.

The explicit formulas are given in the next statement.

Theorem 2.1 *The operator H generates in l^2 a one-parameter group of isometries $e^{tH}\mathbf{a} = \mathbf{b}$, $\mathbf{a} = (a_n) \in l^2$, $\mathbf{b} = (b_m) \in l^2$, which is given by the formulas*

$$b_m = \frac{\sin(\pi t)}{\pi} \sum_{n\in\mathbb{Z}} \frac{a_n}{m-n+t},$$

if $t \in \mathbb{R} \setminus \mathbb{Z}$*, and*

$$b_m = (-1)^t a_{m+t},$$

if $t \in \mathbb{Z}$*.*

As it was proved by Schur [18], the operator norm of $\widetilde{H} : l^2 \mapsto l^2$ is one and therefore the operator norm of H is π. It was shown in [8] that although the norm of the operator H is π, only a strong inequality $\|H\mathbf{a}\| < \pi\|\mathbf{a}\|$ can hold for every nontrivial $\mathbf{a} \in l^2$.

Let us remind that a Bernstein class [1, 12], which is denoted as $\mathbf{B}^p_\sigma(\mathbb{R}),\ \sigma \geq 0,\ 1 \leq p \leq \infty$, is a linear space of all functions $f : \mathbb{R} \mapsto \mathbb{C}$ which belong to $L^p(\mathbb{R})$ and admit extension to $\mathbb{C}$ as entire functions of exponential type σ. A function f belongs to $\mathbf{B}^p_\sigma(\mathbb{R})$ if and only if the following Bernstein inequality holds:

$$\left\| \left(\frac{d}{dx} \right)^k f \right\|_{L^p(\mathbb{R})} \leq \sigma^k \|f\|_{L^p(\mathbb{R})}, \quad x \in \mathbb{R},$$

for all natural k. Using the distributional Fourier transform

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\xi x} dx, \quad f \in L^p(\mathbb{R}),\ 1 \leq p \leq \infty,$$

one can show (Paley–Wiener theorem) that $f \in \mathbf{B}^p_\sigma(\mathbb{R}),\ 1 \leq p \leq \infty$, if and only if $f \in L^p(\mathbb{R}),\ 1 \leq p \leq \infty$, and the support of $\widehat{f}$ (in the sense of distributions) is in $[-\sigma, \sigma]$.

In what follows, the notation $\|\cdot\|$ will always mean $\|\cdot\|_{l^2}$. We note that since H is a bounded operator whose norm is π, one has, for all $\mathbf{a} \in l^2$, the following Bernstein-type inequality:

$$\|H^k \mathbf{a}\| \leq \pi^k \|\mathbf{a}\|. \tag{2.4}$$

Pick an $\mathbf{a}^* \in l^2$, and consider a scalar-valued function

$$\Phi(t) = \langle e^{tH}\mathbf{a}, \mathbf{a}^* \rangle, \qquad t \in \mathbb{R}.$$

The following lemma and the corollary after it can be considered as analogs of the Paley–Wiener theorem.

Lemma 2.2 *For every* $\mathbf{a} \in l^2$ *and every* $\mathbf{a}^* \in l^2$*, the function* Φ *belongs to the Bernstein class* $\mathbf{B}^\infty_\pi(\mathbb{R})$*.*

Proof We notice that

$$\left(\frac{d}{dt} \right)^k \Phi|_{t=0} = \langle H^k \mathbf{a}, \mathbf{a}^* \rangle.$$

Since the operator norm of H is π, we obtain that the Taylor series for Φ converges absolutely on $\mathbb{C}$

$$\left| \sum_{k=0} \left(\frac{d}{dt} \right)^k \Phi|_{t=0} \frac{z^k}{k!} \right| \leq \sum_{k=0} |\langle H^k \mathbf{a}, \mathbf{a}^* \rangle| \frac{|z|^k}{k!} = \|\mathbf{a}\| \|\mathbf{a}^*\| e^{\pi|z|} \tag{2.5}$$

and represents there a function of the exponential type π. In addition, the function Φ is bounded on the real line

$$|\Phi(t)| = \left|\langle e^{tH}\mathbf{a}, \mathbf{a}^*\rangle\right| \leq \|\mathbf{a}\|\|\mathbf{a}^*\|.$$

Lemma is proved. ∎

This lemma can also be reformulated as follows.

Corollary 2.1 *For a fixed* $\mathbf{a} \in l^2$, *the vector-valued function*

$$e^{tH}\boldsymbol{a} : \mathbb{R} \mapsto l^2 \tag{2.6}$$

has extension $e^{zH}\mathbf{a}$, $z \in \mathbb{C}$, *to the complex plane as an entire function of the exponential type* π *which is bounded on the real line.*

We already observed that the function Φ for any $\mathbf{a}$, $\mathbf{a}^* \in l^2$ belongs to $\mathbf{B}_{\pi}^{\infty}(\mathbb{R})$. Let us introduce a new function defined by the next formula if $t \neq 0$

$$\Psi(t) = \frac{\Phi(t) - \Phi(0)}{t} = \left\langle \frac{e^{tH}\mathbf{a} - \mathbf{a}}{t}, \mathbf{a}^* \right\rangle, \tag{2.7}$$

and in the case $t = 0$ by the formula

$$\Psi(0) = \frac{d}{dt}\Phi(t)|_{t=0} = \langle H\mathbf{a}, \mathbf{a}^*\rangle. \tag{2.8}$$

Lemma 2.3 *For every* $\mathbf{a} \in l^2$, $\mathbf{a}^* \in l^2$, *the function* Ψ *is in the Bernstein class* $\mathbf{B}_{\sigma}^{2}(\mathbb{R})$.

Proof The function Ψ is an entire function of the exponential type π. Indeed, the fact that $\Phi(t)$ is in $\mathbf{B}_{\pi}^{\infty}(\mathbb{R})$ means [12] that

$$\langle e^{tH}\mathbf{a}, \mathbf{a}^*\rangle = \langle \mathbf{a}, \mathbf{a}^*\rangle + \sum_{k=1}^{\infty} c_k t^k,$$

with $\overline{\lim}_{k\to\infty} \sqrt[k]{k!|c_k|} \leq \pi$, and then

$$\Psi(t) = \frac{\langle e^{tH}\mathbf{a}, \mathbf{a}^*\rangle - \langle \mathbf{a}, \mathbf{a}^*\rangle}{t} = \sum_{k=1}^{\infty} c_k t^{k-1},$$

where one obviously has $\overline{\lim}_{k\to\infty} \sqrt[k]{k!|c_{k+1}|} \leq \pi$. In addition, Ψ belongs to $L^2(\mathbb{R})$ since according to the Schwartz inequality,

$$|\Psi(t)|^2 = \left|\left\langle \frac{e^{tH}\mathbf{a} - \mathbf{a}}{t}, \mathbf{a}^*\right\rangle\right|^2 \leq \frac{(2\|\mathbf{a}\|\|\mathbf{a}^*\|)^2}{|t|^2}, \quad t \geq 1.$$

In other words, Ψ is in the Bernstein class $\mathbf{B}_{\sigma}^{2}(\mathbb{R})$. Lemma is proved. ∎

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The following so-called general Parseval formula can be found in [5]: For $f, g \in \mathbf{B}^2_\sigma$, $\sigma > 0$, one has

$$\int_{\mathbb{R}} f(t)\overline{g(t)}dt = \frac{\pi}{\sigma}\sum_{k\in\mathbb{Z}} f\left(\frac{k\pi}{\sigma}\right)\overline{g\left(\frac{k\pi}{\sigma}\right)}.$$

In our situation, the general Parseval formula gives the following result.

Theorem 2.4 *For every* $\mathbf{a}, \mathbf{a}^*, \mathbf{b}, \mathbf{b}^* \in l^2$, *the next equality holds*

$$\int_{\mathbb{R}} \left\langle \frac{e^{tH}\mathbf{a}-\mathbf{a}}{t}, \mathbf{a}^*\right\rangle \left\langle \frac{e^{tH}\mathbf{b}-\mathbf{b}}{t}, \mathbf{b}^*\right\rangle dt =$$

$$\langle H\mathbf{a}, \mathbf{a}^*\rangle \langle H\mathbf{b}, \mathbf{b}^*\rangle + \sum_{k\neq 0}\left\langle \frac{e^{kH}\mathbf{a}-\mathbf{a}}{k}, \mathbf{a}^*\right\rangle \left\langle \frac{e^{kH}\mathbf{b}-\mathbf{b}}{k}, \mathbf{b}^*\right\rangle.$$

3 Sampling theorems with regularly spaced samples for orbits $e^{tH}\mathbf{a}$

Below, we are going to use the following known fact (see [6]).

Theorem 3.1 *If* $h \in \mathbf{B}^\infty_\sigma(\mathbb{R})$, *then for any* $0 < \gamma < 1$, *the following formula holds:*

(3.1)
$$h(z) = \sum_{k\in\mathbb{Z}} h\left(\gamma\frac{k\pi}{\sigma}\right) \operatorname{sinc}\left(\gamma^{-1}\frac{\sigma}{\pi}z - k\right), \quad z\in\mathbb{C},$$

where the series converges uniformly on compact subsets of $\mathbb{C}$.

By using Theorem 3.1 and Lemma 2.2, we obtain our First Sampling Theorem.

Theorem 3.2 *For every* $\mathbf{a}, \mathbf{a}^* \in l^2$, *every* $0 < \gamma < 1$, *and every* $z \in \mathbb{C}$, *one has*

(3.2)
$$\left\langle e^{zH}\boldsymbol{a}, \boldsymbol{a}^*\right\rangle = \sum_{k\in\mathbb{Z}} \left\langle e^{(\gamma k)H}\boldsymbol{a}, \boldsymbol{a}^*\right\rangle \operatorname{sinc}\left(\gamma^{-1}z - k\right),$$

where the series converges uniformly on compact subsets of $\mathbb{R}$.

Explicitly, the formula (3.2) means that if z in (3.2) is a real $z = t$ which is not an integer, then (3.2) takes the form

(3.3)
$$\frac{\sin \pi t}{\pi}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}\ n\neq m}\frac{a_n b_m}{m-n+t} = S_1 + S_2,$$

where

$$S_1 = \sum_{k\in\mathbb{Z},\ \gamma k\in\mathbb{R}\setminus\mathbb{Z}} \frac{\sin \pi\gamma k}{\pi}\left(\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}\ n\neq m}\frac{a_n b_m}{m-n+\gamma k}\right)\operatorname{sinc}\left(\gamma^{-1}t-k\right)$$

and

$$S_2 = \sum_{k\in\mathbb{Z};\ \gamma k\in\mathbb{Z}}\left(\sum_{m\in\mathbb{Z}}(-1)^{\gamma k}a_{m+\gamma k}b_m\right)\operatorname{sinc}\left(\gamma^{-1}t-k\right).$$

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Next, if $z = t$ in (3.2) is an integer, then the formula (3.2) is given by

$$\sum_{m\in\mathbb{Z}} (-1)^t a_{m+t} b_m = S_1 + S_2.$$

We note that if $t = \gamma N,\ \ N \in \mathbb{Z}$, then (3.2) is evident since its both sides are (obviously) identical

$$\langle e^{tH}\mathbf{a}, \mathbf{a}^* \rangle = \langle e^{tH}\mathbf{a}, \mathbf{a}^* \rangle.$$

Remark 3.3 The situation with such a kind of "obvious interpolation" is very common for sampling formulas. Consider, for example, the following classical (Shannon) formula:

$$f(t) = \sum_{k\in\mathbb{Z}} f(k)\ \operatorname{sinc}(t-k), \tag{3.4}$$

for $f \in \mathbf{B}^2_\pi(\mathbb{R})$, where the series converges uniformly on compact subsets of $\mathbb{R}$ and also in $L^2(\mathbb{R})$. This formula is informative only when t is not integer. When $t = N \in \mathbb{Z}$, it clearly becomes a tautology $f(N) = f(N)$ because sinc z is zero for every $z \in \mathbb{Z} \setminus \{0\}$ and sinc $0 = 1$.

We are going to use the next known result (see [6]).

Theorem 3.4 *If $h \in \mathbf{B}^2_\sigma(\mathbb{R})$, then the following formula holds for $z \in \mathbb{C}$:*

$$h(t) = \sum_{k\in\mathbb{Z}} h\left(\frac{k\pi}{\sigma}\right)\ \operatorname{sinc}\left(\frac{\sigma}{\pi}t - k\right), \tag{3.5}$$

where the series converges uniformly on compact subsets of $\mathbb{R}$. The restriction of the series to the real line also converges in $L^2(\mathbb{R})$.

Theorem 3.5 *For every $\mathbf{a} \in l^2$,*

$$e^{tH}\boldsymbol{a} = \boldsymbol{a} + t\ \operatorname{sinc}(t) H\ \boldsymbol{a} + t \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{e^{kH}\boldsymbol{a} - \boldsymbol{a}}{k}\ \operatorname{sinc}(t-k), \tag{3.6}$$

where the series converges in the norm of l^2.

Proof Since Ψ is in $\mathbf{B}^2_\sigma(\mathbb{R})$, one can use Theorem 3.4 to obtain the following formula for every $\mathbf{a},\ \mathbf{a}^* \in l^2$, and every $t \in \mathbb{R}$:

$$\left\langle \frac{e^{tH}\mathbf{a} - \mathbf{a}}{t}, \mathbf{a}^* \right\rangle = \sum_{k\in\mathbb{Z}} \left\langle \frac{e^{kH}\mathbf{a} - \mathbf{a}}{k}, \mathbf{a}^* \right\rangle\ \operatorname{sinc}(t-k), \tag{3.7}$$

where the series converges uniformly on compact subsets of $\mathbb{R}$. Actually, this formula means that if $t \neq 0$, then

$$\left\langle \frac{e^{tH}\mathbf{a} - \mathbf{a}}{t}, \mathbf{a}^* \right\rangle = \langle H\mathbf{a}, \mathbf{a}^* \rangle\ \operatorname{sinc}(t) + \sum_{k\in\mathbb{Z}\setminus\{0\}} \left\langle \frac{e^{kH}\mathbf{a} - \mathbf{a}}{k}, \mathbf{a}^* \right\rangle\ \operatorname{sinc}(t-k), \tag{3.8}$$

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and for $t = 0$, it becomes just

$$\langle H\mathbf{a}, \mathbf{a}^* \rangle = \langle H\mathbf{a}, \mathbf{a}^* \rangle.$$

The formula (3.8) can be rewritten as

$$\left\langle e^{tH}\mathbf{a}, \mathbf{a}^* \right\rangle =$$

$$\langle \mathbf{a}, \mathbf{a}^* \rangle + t \langle H\mathbf{a}, \mathbf{a}^* \rangle \operatorname{sinc}(t) + t \sum_{k \in \mathbb{Z} \setminus \{0\}} \left\langle \frac{e^{kH}\mathbf{a} - \mathbf{a}}{k}, \mathbf{a}^* \right\rangle \operatorname{sinc}(t-k). \tag{3.9}$$

Next, we notice that the series

$$\sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{e^{kH}\mathbf{a} - \mathbf{a}}{k} \operatorname{sinc}(t-k)$$

converges in l^2 since for every fixed $t \in \mathbb{R}$,

$$\left\| \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{e^{kH}\mathbf{a} - \mathbf{a}}{k} \operatorname{sinc}(t-k) \right\| \le 2\|\mathbf{a}\| \sum_{k \ne 0, t} \frac{1}{|k||t-k|} < \infty.$$

It allows to rewrite (3.9) as

$$\left\langle e^{tH}\mathbf{a}, \mathbf{a}^* \right\rangle = \left\langle \mathbf{a} + tH\mathbf{a} \operatorname{sinc}(t) + t \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{e^{kH}\mathbf{a} - \mathbf{a}}{k} \operatorname{sinc}(t-k), \mathbf{a}^* \right\rangle.$$

Since this equality holds for all sequences $\mathbf{a}^* \in l^2$, we obtain (3.6). Theorem is proved. ∎

Next, we reformulate (3.6) in its "native" terms.

Proposition 3.6 *If $\mathbf{a} = (a_n) \in l^2$ and t is not integer, then the left-hand side of (3.6) is a sequence $e^{tH}\mathbf{a} = \mathbf{b} = (b_m) \in l^2$ with the entries*

$$b_m = \frac{\sin \pi t}{\pi} \sum_{n \in \mathbb{Z}} \frac{a_n}{m - n + t}, \tag{3.10}$$

and the right-hand side represents a sequence $\mathbf{c} = (c_m) \in l^2$ with the entries

$$c_m = a_m + t \operatorname{sinc}(t) \sum_{n \in \mathbb{Z}, n \ne m} \frac{a_n}{m - n} +$$

$$t \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{(-1)^k a_{m+k} - a_m}{k} \operatorname{sinc}(t-k). \tag{3.11}$$

If t in (3.6) is an integer $t = N$, then $b_m = (-1)^N a_{m+N}$ and

$$c_m = a_m + N \sum_{n \in \mathbb{Z}, n \ne m} \frac{a_n}{m - n} \operatorname{sinc}(N) +$$

$$N \sum_{k \ne 0} \frac{(-1)^k a_{m+k} - a_m}{k} \operatorname{sinc}(N-k) = (-1)^N a_{m+N}.$$

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Thus, in the case when $t = N$ is an integer, we obtain just a tautology

$$b_m = (-1)^N a_{m+N} = c_m.$$

The next theorem is a generalization of what is known as the Valiron–Tschakaloff sampling/interpolation formula [5].

Theorem 3.7 *For every $\mathbf{a} \in l^2$, one has*

$$e^{tH}\boldsymbol{a} = \operatorname{sinc}(t)\boldsymbol{a} + t\operatorname{sinc}(t)H\boldsymbol{a} + \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{t}{k}\operatorname{sinc}(t-k)e^{kH}\boldsymbol{a}, \tag{3.12}$$

where the series converges in the norm of l^2.

Proof If $h \in \mathbf{B}_\sigma^\infty(\mathbb{R})$, $\sigma > 0$, then for all $z \in \mathbb{C}$, the following Valiron–Tschakaloff sampling/interpolation formula holds [5]:

$$h(t) = \operatorname{sinc}\left(\frac{\sigma t}{\pi}\right) f(0) +$$

$$t\operatorname{sinc}\left(\frac{\sigma t}{\pi}\right) f'(0) + \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{\sigma t}{k\pi}\operatorname{sinc}\left(\frac{\sigma t}{\pi} - k\right) h\left(\frac{k\pi}{\sigma}\right), \tag{3.13}$$

the convergence being absolute and uniform on compact subsets of $\mathbb{C}$. If $\mathbf{a}, \mathbf{a}^* \in l^2$, then $\left\langle e^{tH}\mathbf{a},\ \mathbf{a}^*\right\rangle \in \mathbf{B}_\pi^\infty(\mathbb{R})$ and according to (3.13) with $\sigma = \pi$, we have

$$\left\langle e^{tH}\mathbf{a},\ \mathbf{a}^*\right\rangle = \operatorname{sinc}(t)\langle \mathbf{a},\ \mathbf{a}^*\rangle +$$

$$t\operatorname{sinc}(t)\langle H\mathbf{a},\ \mathbf{a}^*\rangle + \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{t}{k}\operatorname{sinc}(t-k)\left\langle e^{kH}\mathbf{a},\ \mathbf{a}^*\right\rangle.$$

Because the series

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{t}{k}\operatorname{sinc}(t-k)e^{kH}\mathbf{a}$$

converges in l^2, we obtain the formula (3.12). Theorem is proved. ∎

The following proposition formulates (3.12) in the specific language of l^2.

Proposition 3.8 *If $\mathbf{a} = (a_n) \in l^2$ and t is not an integer, then the left-hand side of (3.12) is a sequence $e^{tH}\mathbf{a} = \mathbf{b} = (b_m) \in l^2$ with entries*

$$b_m = \frac{\sin \pi t}{\pi}\sum_{n\in\mathbb{Z}} \frac{a_n}{m-n+t}, \tag{3.14}$$

and the right-hand side of (3.12) represents a sequence $\mathbf{c} = (c_m) \in l^2$ with entries

$$c_m = a_m \operatorname{sinc}(t) +$$

$$t\operatorname{sinc}(t)\sum_{n\in\mathbb{Z}, n\neq m} \frac{a_n}{m-n} + t\sum_{k\in\mathbb{Z}\setminus\{0\}} (-1)^k \frac{\operatorname{sinc}(t-k)}{k} a_{m+k}. \tag{3.15}$$

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When t in (3.12) is an integer $t = N$, then (3.12) is the tautology $b_m = (-1)^N a_{m+N} = c_m$.

4 An irregular sampling theorem

The following fact was proved in [9].

Theorem 4.1 *Let $\{t_k\}_{k\in\mathbb{Z}}$ be a sequence of real numbers such that*

$$\sup_{k\in\mathbb{Z}} |t_k - k| < 1/4. \tag{4.1}$$

Define the entire function

$$G(t) = (t - t_0) \prod_{k\in\mathbb{Z}} \left(1 - \frac{z}{t_k}\right)\left(1 - \frac{z}{t_{-k}}\right). \tag{4.2}$$

Then, for all $f \in \mathbf{B}^2_\pi(\mathbb{R})$, we have

$$f(t) = \sum_{k\in\mathbb{Z}} f(t_k) \frac{G(t)}{G'(t_k)(t - t_k)}$$

uniformly on every compact subset of $\mathbb{R}$.

As it was already noticed, for any $\mathbf{a}, \mathbf{a}^* \in l^2$, the function $\Psi(t)$ defined for all $t \neq 0$ as

$$\left\langle \frac{e^{tH}\mathbf{a} - \mathbf{a}}{t}, \mathbf{a}^* \right\rangle$$

and for $t = 0$ as $\Psi(0) = \langle H\mathbf{a}, \mathbf{a}^* \rangle$ belongs to $\mathbf{B}^2_\pi(\mathbb{R})$. Applying Theorem 4.1, we obtain the following theorem.

Theorem 4.2 *If $\mathbf{a}, \mathbf{a}^* \in l^2$ and a sequence $\{t_k\}$ satisfies (4.1), then*

$$\Psi(t) = \sum_{k\in\mathbb{Z}} \Psi(t_k) \frac{G(t)}{G'(t_k)(t - t_k)},$$

uniformly on every compact subset of $\mathbb{R}$.

5 Riesz–Boas interpolation formulas for the discrete Hilbert transform

Consider a trigonometric polynomial $P(t)$ of one variable t. The famous Riesz interpolation formula [12, 15, 16] can be written in the form

$$\left(\frac{d}{dt}\right) P(t) = \frac{1}{4\pi} \sum_{k=1}^{2n} (-1)^{k+1} \frac{1}{\sin^2 \frac{t_k}{2}} U_{t_k} P(t), \quad t \in \mathbb{T}, \quad t_k = \frac{2k-1}{2n}\pi, \tag{5.1}$$

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where $U_{t_k}P(t) = P(t_k + t)$. This formula was extended by Boas [2, 3], (see also [1, 12, 17]) to functions in $\mathbf{B}_\sigma^\infty(\mathbb{R})$ in the following form:

$$\left(\frac{d}{dt}\right) f(t) = \frac{n}{\pi^2}\sum_{k\in\mathbb{Z}} \frac{(-1)^{k-1}}{(k-1/2)^2} U_{\frac{\pi}{n}(k-1/2)} f(t), \quad t\in\mathbb{R}, \tag{5.2}$$

where $U_{\frac{\pi}{n}(k-1/2)} f(t) = f\left(\frac{\pi}{n}(k-1/2)+t\right)$. In turn, the formula (5.2) was extended in [6] to higher powers $(d/dt)^m$. In this section, we present some natural analogs of such formulas (which we call Riesz–Boas interpolation formulas) associated with the discrete Hilbert transform. Our objective is to obtain similar formulas where the operator d/dt is replaced by the discrete Hilbert transform H and the group of regular translations U_t is replaced by the group e^{tH}.

Let us introduce bounded operators

$$\mathcal{R}_H^{(2s-1)}\mathbf{a} = \sum_{k\in\mathbb{Z}} (-1)^{k+1} A_{s,k} e^{(k-1/2)H}\mathbf{a}, \quad \mathbf{a}\in l^2, \ s\in\mathbb{N}, \tag{5.3}$$

and

$$\mathcal{R}_H^{(2s)}\mathbf{a} = \sum_{k\in\mathbb{Z}} (-1)^{k+1} B_{s,k} e^{kH}\mathbf{a}, \quad \mathbf{a}\in l^2, \ s\in\mathbb{N}, \tag{5.4}$$

where $A_{s,k}$ and $B_{s,k}$ are defined as

$$A_{s,k} = (-1)^{k+1}\operatorname{sinc}^{(2s-1)}\left(\frac{1}{2}-k\right)$$

$$= \frac{(2s-1)!}{\pi(k-\frac{1}{2})^{2s}} \sum_{j=0}^{s-1} \frac{(-1)^j}{(2j)!}\left(\pi\left(k-\frac{1}{2}\right)\right)^{2j}, \quad s\in\mathbb{N}, \tag{5.5}$$

for $k\in\mathbb{Z}$,

$$B_{s,k} = (-1)^{k+1}\operatorname{sinc}^{(2s)}(-k) = \frac{(2s)!}{\pi k^{2s+1}} \sum_{j=0}^{s-1} \frac{(-1)^j(\pi k)^{2j+1}}{(2j+1)!}, \quad s\in\mathbb{N}, \tag{5.6}$$

for $k\in\mathbb{Z}\setminus\{0\}$, and

$$B_{s,0} = (-1)^{s+1}\frac{\pi^{2s}}{2s+1}, \quad s\in\mathbb{N}. \tag{5.7}$$

Both series converge in l^2 due to the following formulas (see [6]):

$$\sum_{k\in\mathbb{Z}} |A_{s,k}| = \pi^{2s-1}, \qquad \sum_{k\in\mathbb{Z}} |B_{s,k}| = \pi^{2s}, s\in\mathbb{N}. \tag{5.8}$$

Since $\|e^{tH}f\| = \|f\|$, it implies that

$$\|\mathcal{R}_H^{(2s-1)}\mathbf{a}\| \le \pi^{2s-1}\|\mathbf{a}\|, \quad \|\mathcal{R}_H^{(2s)}\mathbf{a}\| \le \pi^{2s}\|\mathbf{a}\|, \quad \mathbf{a}\in l^2, \ s\in\mathbb{N}. \tag{5.9}$$

Theorem 5.1 *For $\mathbf{a}\in l^2$, the following Riesz–Boas-type interpolation formulas hold true for $r\in\mathbb{N}$:*

$$H^r\boldsymbol{a} = \mathcal{R}_H^{(r)}\boldsymbol{a}, \quad \boldsymbol{a}\in l^2. \tag{5.10}$$

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More explicitly, if $r = 2s - 1,\ s \in \mathbb{N}$, *then*

$$H^{2s-1}\boldsymbol{a} = \sum_{k\in\mathbb{Z}} (-1)^{k+1} A_{s,k} e^{(k-1/2)H}\boldsymbol{a}, \tag{5.11}$$

and when $r = 2s,\ s \in \mathbb{N}$, *then*

$$H^{2s}\boldsymbol{a} = \sum_{k\in\mathbb{Z}} (-1)^{k+1} B_{s,k} e^{kH}\boldsymbol{a}. \tag{5.12}$$

Proof As we know, for any $\mathbf{a},\ \mathbf{a}^* \in l^2$, the function $\Phi(t) = \left\langle e^{tH}\mathbf{a}, \mathbf{a}^*\right\rangle$ belongs to $\mathbf{B}_\pi^\infty(\mathbb{R})$. Thus, by [6], we have

$$\Phi^{(2m-1)}(t) = \sum_{k\in\mathbb{Z}} (-1)^{k+1} A_{m,k} \Phi\left(t + (k - 1/2)\right), \quad m \in \mathbb{N},$$

$$\Phi^{(2m)}(t) = \sum_{k\in\mathbb{Z}} (-1)^{k+1} B_{m,k} \Phi\left(t + k\right), \quad m \in \mathbb{N}.$$

Together with

$$\left(\frac{d}{dt}\right)^k \Phi(t) = \left\langle H^k e^{tH}\mathbf{a}, \mathbf{a}^*\right\rangle,$$

it shows

$$\left\langle e^{tH} H^{2m-1}\mathbf{a}, \mathbf{a}^*\right\rangle = \sum_{k\in\mathbb{Z}} (-1)^{k+1} A_{m,k} \left\langle e^{(t+(k-1/2))H}\mathbf{a},\ \mathbf{a}^*\right\rangle, \quad m \in \mathbb{N},$$

and also

$$\left\langle e^{tH} H^{2m}\mathbf{a}, \mathbf{a}^*\right\rangle = \sum_{k\in\mathbb{Z}} (-1)^{k+1} B_{m,k} \left\langle e^{(t+k)H}\mathbf{a},\ \mathbf{a}^*\right\rangle, \quad m \in \mathbb{N}.$$

Since both series (5.3) and (5.4) converge in l^2 and the last two equalities hold for any $\mathbf{a}^* \in l^2$, we obtain the next two formulas

$$e^{tH} H^{2m-1}\mathbf{a} = \sum_{k\in\mathbb{Z}} (-1)^{k+1} A_{m,k} e^{(t+(k-1/2))H}\mathbf{a}, \quad m \in \mathbb{N}, \tag{5.13}$$

$$e^{tH} H^{2m}\mathbf{a} = \sum_{k\in\mathbb{Z}} (-1)^{k+1} B_{m,k} e^{(t+k)H}\mathbf{a}, \quad m \in \mathbb{N}. \tag{5.14}$$

In turn, when $t = 0$, these formulas become formula (5.10). Theorem is proved. ■

Let us introduce the notation

$$\mathcal{R}_H = \mathcal{R}_H^{(1)}.$$

One has the following "power" formula, which easily follows from the fact that operators $\mathcal{R}_H$ and H commute.

Corollary 5.1 *For any* $r \in \mathbb{N}$ *and any* $\mathbf{a} \in l^2$,

$$H^r\boldsymbol{a} = \mathcal{R}_H^{(r)}\boldsymbol{a} = \mathcal{R}_H^r\boldsymbol{a}, \tag{5.15}$$

where $\mathcal{R}_H^r\mathbf{a} = \mathcal{R}_H\left(\ldots(\mathcal{R}_H\mathbf{a})\right)$.

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Let us express (5.10) in terms of H and e^{tH}. Our starting sequence is $\mathbf{a} = (a_n)$, and then we use the notation $H^k\mathbf{a} = \left(a_m^{(k)}\right)$, $k \in \mathbb{Z}$. One has

$$H\mathbf{a} = \frac{1}{\pi} \sum_{(n,\ n\neq n_1)} \frac{a_n}{n_1 - n} = \left(a_{n_1}^{(1)}\right),$$

$$H^2\mathbf{a} = \frac{1}{\pi^2} \sum_{(n_1,\ n_1\neq n_2)} \sum_{(n,\ n\neq n_1)} \frac{a_{n_1}^{(1)}}{(n_2 - n_1)(n_1 - n)} = \left(a_{n_2}^{(2)}\right),$$

and so on up to an $r \in \mathbb{N}$

$$H^r\mathbf{a} = \frac{1}{\pi^r} \sum_{(n_{r-1},\ n_{r-1}\neq n_r)} \sum_{(n_{r-2},\ n_{r-2}\neq n_{r-1})} \cdots$$

(5.16)

$$\cdots \sum_{(n_1,\ n_1\neq n_2)} \sum_{(n,\ n\neq n_1)} \frac{a_{n_{r-1}}^{(r-1)}}{(n_r - n_{r-1})(n_r - n_{r-1})\dots(n_2 - n_1)(n_1 - n)} = \left(a_{n_r}^{(r)}\right).$$

Theorem 5.2 *For any $\mathbf{a} = (a_n) \in l^2$, and $r = 2s - 1$, we have in (5.11) the equality of two sequences where on the left-hand side we have a sequence whose general term is $a_m^{(2s-1)}$, and on the right-hand side we have a sequence whose general term is $c_{m,s}$ where*

$$c_{m,s} = \sum_{k\in\mathbb{Z}} (-1)^{k+1} \frac{\sin(\pi(k - 1/2))}{\pi} A_{s,k} \sum_{n\neq m} \frac{a_n}{m - n + (k - 1/2)}.$$

The equality (5.11) tells that $a_m^{(2s-1)} = c_{m,s}$.

For the case $r = 2s$, a sequence on the left-hand side of (5.12) has a general term $a_m^{(2s)}$, and a sequence on the right-hand side has a general term $d_{m,s}$ of the form

$$d_{m,s} = -\sum_{k\in\mathbb{Z}} B_{s,k} a_{m+k},$$

and (5.12) means that $a_m^{(2s)} = d_{m,s}$.

Let us introduce the following notations:

$$\mathcal{R}_H^{(2s-1)}(N)\mathbf{a} = \sum_{|k|\leq N} (-1)^{k+1} A_{s,k} e^{(k-1/2)H}\mathbf{a},$$

$$\mathcal{R}_H^{(2s)}(N)\mathbf{a} = \sum_{|k|\leq N} (-1)^{k+1} B_{s,k} e^{kH}\mathbf{a}.$$

One obviously has the following set of approximate Riesz–Boas-type formulas.

Theorem 5.3 *If $\mathbf{a} \in l^2$ and $r \in \mathbb{N}$, then*

(5.17) $$H^r\boldsymbol{a} = \mathcal{R}_H^{(r)}(N)\boldsymbol{a} + O(N^{-2}).$$

The next theorem contains another Riesz–Boas-type formula.

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Theorem 5.4 *If* $\mathbf{a} \in l^2$*, then the following sampling formula holds for* $\mathbf{a} \in \mathbb{R}$ *and* $n \in \mathbb{N}$*:*

$$H^n e^{tH}\boldsymbol{a} = \left(n \operatorname{sinc}^{(n-1)}(t) + t \operatorname{sinc}^{(n)}(t)\right) H\boldsymbol{a}$$

$$+ \sum_{k \neq 0} \left(n \operatorname{sinc}^{(n-1)}(t-k) + t \operatorname{sinc}^{(n)}(t-k)\right) \frac{e^{kH}\boldsymbol{a} - \boldsymbol{a}}{k}, \tag{5.18}$$

where the series converges in the norm of l^2*. In particular, for* $n \in \mathbb{N}$*, one has*

$$H^n \boldsymbol{a} = \mathcal{Q}_H^n \boldsymbol{a}, \tag{5.19}$$

where the bounded operator $\mathcal{Q}_H^{(n)}$ *is given by the formula*

$$\mathcal{Q}_H^{(n)} f = \left(\operatorname{sinc}^{(n-1)}(0) + \operatorname{sinc}^{(n)}(0)\right) n\, H\boldsymbol{a}$$

$$+ n \sum_{k \neq 0} \left(\operatorname{sinc}^{(n-1)}(-k) + \operatorname{sinc}^{(n)}(-k)\right) \frac{e^{kH}\boldsymbol{a} - \boldsymbol{a}}{k}. \tag{5.20}$$

Remark 5.5 We note that $(\operatorname{sinc} t)^{(m)}(0) = (-1)^m/(m+1)!$ if m is even, and $(\operatorname{sinc} t)^{(m)}(0) = 0$ if m is odd.

Proof For any $\mathbf{a}, \mathbf{a}^* \in l^2$, the function $\Phi(t) = \left\langle e^{tH}\mathbf{a}, \mathbf{a}^*\right\rangle$ belongs to $B_\pi^\infty(\mathbb{R})$. We consider Ψ which was introduced previously in (2.7) and (2.8). We have

$$\Psi(t) = \sum_{k \in \mathbb{Z}} \Psi(k) \operatorname{sinc}(t-k),$$

where the series converges in l^2. From here, we obtain the next formula

$$\left(\frac{d}{dt}\right)^n \Psi(t) = \sum_{k \in \mathbb{Z}} \Psi(k) \operatorname{sinc}^{(n)}(t-k),$$

and since

$$\left(\frac{d}{dt}\right)^n \Phi(t) = n \left(\frac{d}{dt}\right)^{n-1} \Psi(t) + t \left(\frac{d}{dt}\right)^n \Psi(t),$$

we obtain

$$\left(\frac{d}{dt}\right)^n \Phi(t) = n \sum_{k \in \mathbb{Z}} \Psi(k) \operatorname{sinc}^{(n-1)}(t-k) + t \sum_{k \in \mathbb{Z}} \Psi(k) \operatorname{sinc}^{(n)}(t-k).$$

Since $\left(\frac{d}{dt}\right)^n \Phi(t) = \left\langle H^n e^{tH}\mathbf{a}, \mathbf{a}^*\right\rangle$, and

$$\Psi(k) = \left\langle \frac{e^{kH}\mathbf{a} - \mathbf{a}}{k}, \mathbf{a}^* \right\rangle,$$

we obtain that the formulas (5.18)–(5.20) hold. Theorem is proved. ■

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6 The case of the Kak–Hilbert transform

We also briefly show how our methods can be applied to the Kak–Hilbert transform to obtain similar sampling and interpolation formulas. Kak–Hilbert transform also generates a one-parameter group of operators in l^2, but it is not a group of isometries like in the case of the discrete Hilbert transform. However, this group of operators is uniformly bounded. This uniform boundness is explored to include the case of Kak–Hilbert transform into our scheme.

The Kak–Hilbert transform

$$K\mathbf{a} = \mathbf{b}, \quad \mathbf{a} = (a_n) \in l^2, \quad \mathbf{b} = (b_n) \in l^2,$$

is defined by the formula

$$b_m = \frac{2}{\pi} \sum_{n \text{ even}} \frac{a_n}{m-n},$$

if m is odd, and by the formula

$$b_m = \frac{2}{\pi} \sum_{n \text{ odd}} \frac{a_n}{m-n},$$

if m is even.

It is known that K is an isometry in l^2 (see [10]). As a bounded operator, K generates a one-parameter group e^{tK} of bounded operators in l^2. One can verify the property $K^2 = -I$ which implies the explicit formula for e^{tK} (see [7]):

$$e^{tK} = (\cos t)\, I + (\sin t)\, K,$$

which gives the uniform bound $\|e^{tK}\| \leq 2$.

Pick an $\mathbf{a}^* \in l^2$, and consider a scalar-valued function

$$F(t) = \langle e^{tK}\mathbf{a}, \mathbf{a}^* \rangle, \qquad t \in \mathbb{R}.$$

Note that since K is an isometry, the analog of the Bernstein inequality takes the form

$$\|K^n \mathbf{a}\| = \|\mathbf{a}\|, \quad n \in \mathbb{N},$$

(compare to (2.4)). Using this inequality, one can easily prove the following analog of Lemma 2.2.

Lemma 6.1 *For every $\mathbf{a} \in l^2$ and every $\mathbf{a}^* \in l^2$, the function F belongs to the Bernstein class $\mathbf{B}_1^{\infty}(\mathbb{R})$.*

We also have the following corollary similar to (2.1).

Corollary 6.1 *For a fixed $\mathbf{a} \in l^2$, the vector-valued function*

$$e^{tK}\mathbf{a} : \mathbb{R} \mapsto l^2$$

has extension $e^{zK}\mathbf{a}$, $z \in \mathbb{C}$, to the complex plain as an entire function of the exponential type 1 which is bounded on the real line.

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Similarly to the case of the discrete Hilbert transform, one could prove the following statements.

Theorem 6.2 *For every* $\mathbf{a}$, $\mathbf{a}^* \in l^2$ *and every* $0 < \gamma < 1$, *the following formula holds true:*

$$\langle e^{tK}\mathbf{a}, \mathbf{a}^* \rangle = \sum_{n\in\mathbb{Z}} \langle e^{(\gamma n\pi)K}\mathbf{a}, \mathbf{a}^* \rangle \operatorname{sinc}\left(\frac{t}{\gamma\pi} - n\right),$$

where the series converges uniformly on compact subsets of $\mathbb{R}$.

The following formulas also hold true:

$$e^{tK}\mathbf{a} = \mathbf{a} + t \operatorname{sinc}(t/\pi)\, K\mathbf{a} + t \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{e^{n\pi K}\mathbf{a} - \mathbf{a}}{n\pi} \operatorname{sinc}\left(\frac{t}{\pi} - n\right),$$

where the series converges in the norm of l^2, *and*

$$e^{tK}\mathbf{a} = \operatorname{sinc}\left(\frac{t}{\pi}\right)\mathbf{a} + t \operatorname{sinc}\left(\frac{t}{\pi}\right) K\mathbf{a} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{t}{n\pi} \operatorname{sinc}\left(\frac{t}{\pi} - n\right) e^{n\pi K}\mathbf{a},$$

where the series converges in l^2.

One could also reformulate for the Kak–Hilbert transform all other results which were obtained for the discrete Hilbert transform. In particular, one could introduce bounded operators

$$\mathcal{T}_K^{(2m-1)}\mathbf{a} = \sum_{n\in\mathbb{Z}} (-1)^{n+1} A_{m,n} e^{(n-1/2)\pi K}\mathbf{a}, \quad \mathbf{a} \in l^2, \; m \in \mathbb{N},$$

and

$$\mathcal{T}_K^{(2m)}\mathbf{a} = \sum_{n\in\mathbb{Z}} (-1)^{n+1} B_{m,n} e^{n\pi K}\mathbf{a}, \quad \mathbf{a} \in l^2, \; m \in \mathbb{N},$$

and to prove relevant Riesz–Boas-type interpolation formulas

$$K^r\mathbf{a} = \mathcal{T}_K^{(r)}\mathbf{a}, \; \mathbf{a} \in l^2, \; r \in \mathbb{N}.$$

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