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ON A CONVEXITY THEOREM OF RUSKAI AND WERNER AND RELATED RESULTS

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Abstract. We show that the function

$$V_q(x) = \frac{2e^{x^2}}{\Gamma(q+1)} \int_x^\infty e^{-t^2}(t^2 - x^2)^q \, dt \quad (-1 < q \in \mathbf{R};\ 0 < x \in \mathbf{R}),$$

which has applications in the study of atoms in magnetic fields, satisfies certain monotonicity and convexity properties as well as inequalities. In particular, we prove that $1/V_q$ is convex on $(0, \infty)$ if and only if $q \geq 0$. This extends a recent result of M. B. Ruskai and E. Werner, who established the convexity for all integers $q \geq 0$.

2000 *Mathematics Subject Classification.* 33E20, 26D15.

1. Introduction. In an interesting paper published in 2000, M. B. Ruskai and E. Werner [**8**] discuss in detail the function

$$V_q(x) = \frac{2e^{x^2}}{\Gamma(q+1)} \int_x^\infty e^{-t^2}(t^2 - x^2)^q \, dt \quad (-1 < q \in \mathbf{R};\ 0 < x \in \mathbf{R}) \tag{1.1}$$

and its extensions. $V_q(x)$ is also defined for $x = 0$, if $q > -1/2$. The authors point out that their work was motivated by the fact that for an integer q this function 'arises naturally' [**8**, p. 436] in the study of atoms in magnetic fields. Indeed, V_q can be regarded as one-dimensional regularization of the Coulomb potential. See [**3**], [**4**], and [**8**] for details and references.

The special case $q = 0$ leads to Mills's ratio

$$\frac{1}{\sqrt{2}} V_0\left(\frac{x}{\sqrt{2}}\right) = e^{x^2/2} \int_x^\infty e^{-t^2/2} \, dt,$$

which has applications in statistics. Inequalities for this and related functions are given in [**7**, Section 2.26].

A remarkable number theoretic property of

$$\frac{V_q(0)}{\sqrt{\pi}} = 2^{-2q}\binom{2q}{q} = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2q-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2q)} \quad (0 \leq q \in \mathbf{Z}),$$

known as normalized binomial mid-coefficient, can be found in [**2**].

A central role in [**8**] is the study of convexity properties of $V_q(x)$. The authors show that the arithmetic mean of $V_0(x), \ldots, V_{n-1}(x)$ $(n \geq 1)$ is convex on $(0, \infty)$ with respect to x. In particular, $x \mapsto V_q(x)$ is convex for $q = 0$. But, this is not true, if $q > 1/2$. In

1993, M. Wirth [10] established that $1/V_0$ is convex on $(0,\infty)$. Ruskai and Werner provide a substantial extension of this theorem. They prove that for all integers $q \ge 0$ the function $x \mapsto 1/V_q(x)$ is convex on $(0,\infty)$. An application of this result reveals that $1/V_q$ is subadditive, that is,

$$\frac{1}{V_q(x+y)} \le \frac{1}{V_q(x)} + \frac{1}{V_q(y)} \quad (x, y > 0;\ 0 \le q \in \mathbf{Z}). \tag{1.2}$$

The ratio $V_{q+1}(x)/V_q(x)$ $(1 \le q \in \mathbf{Z})$ is of importance in the proof of the convexity of $1/V_q$. This ratio has an interesting monotonicity property: it is increasing with respect to x. The authors also study $V_q(x)$ as function of q. They establish that $q \mapsto V_q(x)$ and $q \mapsto -qV_q(x)$ $(x > 0)$ are decreasing.

It is our aim to continue the work of Ruskai and Werner. In Section 3, we determine all real parameters p and q such that $x \mapsto V_p(x)$ and $x \mapsto 1/V_q(x)$ are convex on $(0,\infty)$. Moreover, we give an answer to the question: for which q is $x \mapsto V_q(x)$ completely monotonic on $(0,\infty)$? And, we prove that for every $x > 0$ the function $q \mapsto V_q(x)$ is convex on $(-1,\infty)$. In Section 4, we extend and complement inequality (1.2). Further, we provide all parameters q such that $x \mapsto V_q(x)$ is supermultiplicative on $(0,\infty)$, and we present a differential inequality involving $(V_q^{(k)}(x))^n$ and $(V_q^{(n)}(x))^k$. Finally, we study the monotonicity behaviour of the functions $x \mapsto V_p(x)/V_q(x)$ and $x \mapsto V_p(x) - V_q(x)$.

2. Lemmas. In this section, we collect some lemmas, which we need to prove our theorems. First, we present integral representations for $V_q(x)$ and its first and second derivatives with respect to x.

LEMMA 1. *For all $q > -1$ and $x > 0$ we have*

$$V_q(x) = \frac{1}{\Gamma(q+1)} \int_0^\infty e^{-(sx+s^2/4)}(sx + s^2/4)^q \, ds, \tag{2.1}$$

$$V_q(x) = \frac{x^{q+1/2}}{\Gamma(q+1)} \int_0^\infty e^{-xs} \frac{s^q}{(x+s)^{1/2}} \, ds, \tag{2.2}$$

$$V_q'(x) = -\frac{x^{q+1/2}}{\Gamma(q+1)} \int_0^\infty e^{-xs} \frac{s^q}{(x+s)^{3/2}} \, ds, \tag{2.3}$$

$$V_q''(x) = \frac{x^{q-1/2}}{\Gamma(q+1)} \int_0^\infty e^{-xs} \frac{(2x-s)s^q}{(x+s)^{5/2}} \, ds. \tag{2.4}$$

Proof. We substitute in (1.1) $t = x + s/2$ and $t = \sqrt{x^2 + xs}$, respectively, and obtain (2.1) and (2.2), respectively. Next, we set $s = u/x$ in (2.2). This yields

$$V_q(x) = \frac{1}{\Gamma(q+1)} \int_0^\infty e^{-u} \frac{u^q}{(x^2+u)^{1/2}} \, du. \tag{2.5}$$

Further, if we differentiate (2.5) once and twice, respectively, and substitute $u = xs$, then we get (2.3) and (2.4), respectively. □

Proofs for the next two lemmas are given in [8].

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LEMMA 2. *Let $q > -1$ and $x > 0$. The function $a \mapsto aV_q(ax)$ is strictly increasing on $(0, \infty)$.*

LEMMA 3. *Let $q > -1$. Then we have the asymptotic formula*

$$V_q(x) = \frac{1}{x} - \frac{q+1}{2x^3} + \frac{3(q+1)(q+2)}{8x^5} + O\left(\frac{1}{x^7}\right). \tag{2.6}$$

The following integral inequality was first proved by P. L. Tchebyschef. References for this and related results can be found in [**7**, Section 2.5].

LEMMA 4. *Let $f, g : [a, b] \to \mathbf{R}$ be both increasing or both decreasing and let $p : [a, b] \to [0, \infty)$ be integrable. Then*

$$\int_a^b p(x)f(x)\,dx \int_a^b p(x)g(x)\,dx \le \int_a^b p(x)f(x)g(x)\,dx \int_a^b p(x)\,dx.$$

Moreover, we need an inequality for convex functions due to M. Petrović [**7**, pp. 22–23].

LEMMA 5. *Let $f : [0, a] \to \mathbf{R}$ be convex. If $x_j \in [0, a]$ $(j = 1, \ldots, n)$ and $x_1 + \ldots + x_n \in [0, a]$, then*

$$f(x_1) + \ldots + f(x_n) \le f(x_1 + \ldots + x_n) + (n-1)f(0).$$

A function $f : (0, \infty) \to \mathbf{R}$ is called *completely monotonic*, if f has derivatives of all orders and satisfies $(-1)^n f^{(n)}(x) \ge 0$ for all $x > 0$ and $n = 0, 1, 2, \ldots$. In particular, completely monotonic functions are decreasing and convex. These functions have numerous applications in probability theory, physics, and other branches. We refer to [**1**], where details and references can be found. The basic properties of completely monotonic functions are collected in [**9**, Chapter IV].

LEMMA 6. *If f is completely monotonic on $(0, \infty)$, then we have for all real numbers $x > 0$ and integers n, k with $n \ge k \ge 0$:*

$$(-1)^{nk}\left(f^{(k)}(x)\right)^n \le (-1)^{nk}\left(f^{(n)}(x)\right)^k (f(x))^{n-k}.$$

A proof of Lemma 5 is given in [**5**].

3. Complete monotonicity and convexity. Our first theorem provides all parameters q such that V_q is completely monotonic on $(0, \infty)$.

THEOREM 1. *Let $q > -1$ be a real number. The function $x \mapsto V_q(x)$ is completely monotonic on $(0, \infty)$ if and only if $q \in (-1, 0]$.*

Proof. Let $q \in (-1, 0]$ and $x > 0$ be real numbers. Further, let $n \ge 0$ be an integer. Using the Leibniz rule for the n-th derivative of a product we conclude from (2.1):

$$(-1)^n V_q^{(n)}(x) = \frac{1}{\Gamma(q+1)} \int_0^\infty e^{-s(x+s/4)} \sum_{\nu=0}^{n} \binom{n}{\nu} s^{q+n-\nu} \left(x + \frac{s}{4}\right)^{q-\nu} \prod_{j=0}^{\nu-1} (j-q)\,ds > 0,$$

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which implies that V_q is completely monotonic on $(0, \infty)$. Next, we show: if $q > 0$, then V_q is not convex and, thus not completely monotonic on $(0, \infty)$. We consider three cases.

Case 1: $0 < q < 1/2$.
Let

$$\lambda_q(x, s) = \left(\frac{s}{x+s}\right)^q \quad \text{and} \quad \mu_q(x, s) = \frac{s - 2xe^x}{(x+s)^{5/2-q}},$$

where $s \in [0, 1]$ and $x > 0$ is sufficiently small. The function $s \mapsto \lambda_q(x, s)$ is increasing on $[0, 1]$, so that we get

$$\lambda_q(x, s)\mu_q(x, s) \geq \lambda_q(x, 2xe^x)\mu_q(x, s).$$

Hence,

$$\begin{aligned}\int_0^1 \lambda_q(x, s)\mu_q(x, s)\, ds &\geq \left(\frac{2e^x}{1+2e^x}\right)^q \int_0^1 \frac{s - 2xe^x}{(x+s)^{5/2-q}}\, ds \\ &= \left(\frac{2e^x}{1+2e^x}\right)^q \frac{1}{(3/2-q)(1/2-q)} \big[x^{q-1/2}(1 + e^x(2q-1)) \\ &\quad - (x+1)^{q-3/2}(x + xe^x(2q-1) + 3/2 - q)\big].\end{aligned}$$

Since $0 < q < 1/2$, we conclude that the expression on the right-hand side tends to ∞, if x tends to 0. This yields

$$0 < \int_0^1 \frac{s^q}{(x+s)^{5/2}}(s - 2xe^x)\, ds. \tag{3.1}$$

Using (3.1) we obtain

$$\begin{aligned}2x \int_0^1 e^{-xs} \frac{s^q}{(x+s)^{5/2}}\, ds &< 2x \int_0^1 \frac{s^q}{(x+s)^{5/2}}\, ds < e^{-x} \int_0^1 \frac{s^{q+1}}{(x+s)^{5/2}}\, ds \\ &< \int_0^1 e^{-xs} \frac{s^{q+1}}{(x+s)^{5/2}}\, ds.\end{aligned}$$

Thus,

$$\Gamma(q+1)x^{1/2-q} V_q''(x) < \int_0^1 e^{-xs} \frac{(2x-s)s^q}{(x+s)^{5/2}}\, ds < 0$$

for all sufficiently small x.

Case 2: $q = 1/2$.
We define

$$W(x^2) = \frac{\sqrt{\pi}}{2} V_{1/2}''(x) = \int_0^\infty e^{-t} t^{1/2} \frac{2x^2 - t}{(x^2+t)^{5/2}}\, dt.$$

Let $y \in (0, 1/2)$. We get

$$\begin{aligned}W(y) &\leq 2y \int_0^1 e^{-t} \frac{t^{1/2}}{(y+t)^{5/2}}\, dt - \int_0^1 e^{-t} \frac{t^{3/2}}{(y+t)^{5/2}}\, dt \\ &\leq 2y \int_0^1 \frac{t^{1/2}}{(y+t)^{5/2}}\, dt - \frac{1}{e} \int_0^1 \frac{t^{3/2}}{(y+t)^{5/2}}\, dt.\end{aligned}$$

Since

$$\lim_{y\to 0} y\int_0^1 \frac{t^{1/2}}{(y+t)^{5/2}}\,dt = \lim_{y\to 0}\frac{2}{3(y+1)^{3/2}} = \frac{2}{3} \quad\text{and}\quad \lim_{y\to 0}\int_0^1 \frac{t^{3/2}}{(y+t)^{5/2}}\,dt = \infty,$$

we conclude that $W(y)$ is negative for sufficiently small y.

Case 3: $q > 1/2$.
The substitution $s = t/x$ in (2.3) leads to

$$V_q'(x) = -\frac{x}{\Gamma(q+1)}\int_0^\infty e^{-t}\frac{t^q}{(x^2+t)^{3/2}}\,dt$$

This implies $\lim_{x\to 0} V_q'(x) = 0$. Since $V_q'(x) < 0$ for $x > 0$, we conclude that V_q' is not increasing on $(0,\infty)$. □

REMARK 1. In particular, we have proved: the function $x \mapsto V_q(x)$ is convex on $(0,\infty)$ if and only if $q \in (-1, 0]$.

REMARK 2. Since a completely monotonic function is log-convex, and a log-convex function is convex, we obtain: the function $x \mapsto V_q(x)$ is log-convex on $(0,\infty)$ if and only if $q \in (-1, 0]$.

Next, we study the convexity of $1/V_q$. Ruskai and Werner [**8**] conjecture that for all real numbers $q > -1$ the function $x \mapsto 1/V_q(x)$ is convex on $(0,\infty)$. The following theorem reveals that this is true for all $q \ge 0$, but false for all $q \in (-1, 0)$.

THEOREM 2. *Let $q > -1$ be a real number. The function $x \mapsto 1/V_q(x)$ is convex on* $(0,\infty)$ *if and only if $q \ge 0$. Moreover, if $q \ge 0$, then $1/V_q$ is strictly convex on* $(0,\infty)$.

Proof. Let $q \ge 0$ and $x > 0$. Differentiation with respect to x yields

$$(V_q(x))^3\left(\frac{1}{V_q(x)}\right)'' = 2(V_q'(x))^2 - V_q(x)V_q''(x). \tag{3.2}$$

Using (2.2)–(2.4) and the convolution theorem we get

$$\frac{(\Gamma(q+1))^2}{2x^{2q+1}}[2(V_q'(x))^2 - V_q(x)V_q''(x)] = \int_0^\infty e^{-xt}\Lambda_q(x,t)\,dt, \tag{3.3}$$

where

$$\Lambda_q(x,t) = \int_0^t \frac{[s(t-s)]^q}{[(x+s)(x+t-s)]^{5/2}}[(x+s)(x+t-s) - (1 - s/(2x))(x+t-s)^2]\,ds.$$

Let $t > 0$. We define

$$\Theta_q(x,t) = 8x\left(\frac{2}{t}\right)^{2q+2}\Lambda_q(x,t). \tag{3.4}$$

Next, we substitute $s = t(1+y)/2$. This leads to

$$\Theta_q(x,t) = \int_{-1}^1 \Psi_q(x,t,y)[\alpha(t)y^3 + \beta(x,t)y^2 + \gamma(x,t)y + \delta(x,t)]\,dy,$$

where

$$\Psi_q(x,t,y) = (1-y^2)^q\left[\left(x+\frac{t}{2}\right)^2 - \left(\frac{ty}{2}\right)^2\right]^{-5/2},$$

$$\alpha(t) = t^2,\quad \beta(x,t) = -12xt - t^2,\quad \gamma(x,t) = 20x^2 + 8xt - t^2,\quad \delta(x,t) = (2x+t)^2.$$

Since $y \mapsto \Psi_q(x, t, y)$ is even, we obtain

$$\Theta_q(x, t) = 2 \int_0^1 \Psi_q(x, t, y)[\beta(x, t)y^2 + \delta(x, t)]\, dy.$$

We put $t = 2a$ $(a > 0)$ and $x = ra$ $(r > 0)$. Then we get

$$\left(\frac{a}{2}\right)^3 \Theta_q(ra, 2a) = \int_0^1 (1 - y^2)^q \frac{(r+1)^2 - (6r+1)y^2}{[(r+1)^2 - y^2]^{5/2}}\, dy.$$

Applying Lemma 4 with $f(y) = (1 - y^2)^q$, $g(y) = (r+1)^2 - (6r+1)y^2$, and $p(y) = [(r+1)^2 - y^2]^{-5/2}$ yields

$$\left(\frac{a}{2}\right)^3 \Theta_q(ra, 2a) \geq \frac{\int_0^1 p(y)f(y)\, dy \int_0^1 p(y)g(y)\, dy}{\int_0^1 p(y)\, dy}.$$

Since

$$\int_0^1 p(y)\, dy > 0, \quad \int_0^1 p(y)f(y)\, dy > 0, \quad \text{and} \quad \int_0^1 p(y)g(y)\, dy = \frac{r^{1/2}}{(r+1)^2(r+2)^{3/2}} > 0,$$

we conclude that $\Theta_q(ra, 2a)$ is positive. Thus, (3.2)–(3.4) imply that $(1/V_q(x))'' > 0$ for $x > 0$.

It remains to show that if $-1 < q < 0$, then $1/V_q$ is not convex on $(0, \infty)$. First, let $-1/2 < q < 0$. We have for $x > 0$:

$$\int_0^\infty \frac{e^{-sx}s^q}{(x+s)^{3/2}}\, ds \geq \int_0^x \frac{e^{-sx}s^q}{(x+s)^{3/2}}\, ds \geq \int_0^x \frac{e^{-x^2}x^q}{(x+s)^{3/2}}\, ds = x^{q-1/2}e^{-x^2}(2 - \sqrt{2}),$$

so that (2.3) yields $\lim_{x\to 0}(-V_q'(x)) = \infty$. Since $V_q(0) = \Gamma(q + 1/2)/\Gamma(q+1)$, we get

$$\lim_{x\to 0}\left(\frac{1}{V_q(x)}\right)' = \lim_{x\to 0}\frac{-V_q'(x)}{(V_q(x))^2} = \infty.$$

This implies that $(1/V_q)'$ is not increasing on $(0, \infty)$.

Next, let $-1 < q \leq -1/2$. We assume that $1/V_q$ is convex on $(0, \infty)$. Then we have

$$\frac{1}{V_q((x+y)/2)} \leq \frac{1}{2}\left(\frac{1}{V_q(x)} + \frac{1}{V_q(y)}\right) \quad (x, y > 0). \tag{3.5}$$

Since $\lim_{y\to 0} V_q(y) = \infty$, we obtain from (3.5):

$$\frac{1}{V_q(x/2)} \leq \frac{1}{2V_q(x)} \quad (x > 0).$$

This contradicts Lemma 2. □

It is natural also to study properties of $V_q(x)$ as function of q, where $x > 0$ is a fixed number. We now give an affirmative answer to a question posed by Ruskai and Werner [**8**]: is $V_q(x)$ convex with respect to q?

THEOREM 3. *Let $x > 0$ be a real number. The function $q \mapsto V_q(x)$ is strictly convex on $(-1, \infty)$.*

Proof. Let $x > 0$. Since $V_q(x)$ is continuous with respect to q, it suffices to show that

$$V_{(a+b)/2}(x) < \frac{1}{2}(V_a(x) + V_b(x)) \tag{3.6}$$

for all real numbers a, b with $b > a > -1$. Using (2.2), the integral formula

$$\frac{1}{x^r} = \frac{1}{\Gamma(r)} \int_0^\infty e^{-xt} t^{r-1}\, dt \quad (r > 0;\ x > 0),$$

and the convolution theorem we get

$$\begin{aligned}
&\Gamma(a+1)\Gamma(b+1)x^{-(a+b+3/2)}[V_a(x) + V_b(x) - 2V_{(a+b)/2}(x)] \\
&= \int_0^\infty e^{-xs} s^b\, ds \int_0^\infty e^{-xs} \frac{s^a}{(x+s)^{1/2}}\, ds + \int_0^\infty e^{-xs} s^a\, ds \int_0^\infty e^{-xs} \frac{s^b}{(x+s)^{1/2}}\, ds \\
&\quad - \frac{2\Gamma(a+1)\Gamma(b+1)}{(\Gamma((a+b)/2+1))^2} \int_0^\infty e^{-xs} s^{(a+b)/2}\, ds \int_0^\infty e^{-xs} \frac{s^{(a+b)/2}}{(x+s)^{1/2}}\, ds \\
&= \int_0^\infty e^{-xt} \sigma_{a,b}(x, t)\, dt,
\end{aligned} \tag{3.7}$$

where

$$\sigma_{a,b}(x, t) = \int_0^t \frac{1}{(x+s)^{1/2}} \left[(t-s)^b s^a + (t-s)^a s^b - \frac{2\Gamma(a+1)\Gamma(b+1)}{(\Gamma((a+b)/2+1))^2} ((t-s)s)^{(a+b)/2} \right] ds.$$

Let $t > 0$. We substitute $s = t(1+y)/2$ and obtain

$$\sigma_{a,b}(x, t) = \left(\frac{t}{2}\right)^{a+b+1} \int_0^1 P(x, t, y) Q_{a,b}(y)\, dy,$$

with

$$P(x, t, y) = (x + t(1+y)/2)^{-1/2} + (x + t(1-y)/2)^{-1/2}$$

and

$$Q_{a,b}(y) = (1-y)^a(1+y)^b + (1-y)^b(1+y)^a - \frac{2\Gamma(a+1)\Gamma(b+1)}{(\Gamma((a+b)/2+1))^2}(1-y^2)^{(a+b)/2}.$$

Next, we define for $y \in (0, 1)$:

$$\begin{aligned}
R_{a,b}(y) = (1-y^2)^{-(a+b)/2} Q_{a,b}(y) &= \left(\frac{1+y}{1-y}\right)^{(b-a)/2} + \left(\frac{1-y}{1+y}\right)^{(b-a)/2} \\
&\quad - \frac{2\Gamma(a+1)\Gamma(b+1)}{(\Gamma((a+b)/2+1))^2}.
\end{aligned}$$

Differentiation with respect to y gives

$$R'_{a,b}(y) = \frac{b-a}{1-y^2}\left[\left(\frac{1+y}{1-y}\right)^{(b-a)/2} - \left(\frac{1-y}{1+y}\right)^{(b-a)/2}\right] > 0,$$

which implies that $y \mapsto R_{a,b}(y)$ is strictly increasing on $(0, 1)$. The gamma function is strictly log-convex on $(0, \infty)$, so that we obtain

$$R_{a,b}(0) = 2\left(1 - \frac{\Gamma(a+1)\Gamma(b+1)}{\left(\Gamma((a+b)/2+1)\right)^2}\right) < 0.$$

Further, we have $\lim_{y\to 1} R_{a,b}(y) = \infty$. Thus, there exists a number $y_0 \in (0, 1)$ such that $R_{a,b}(y) < 0$ for $y \in (0, y_0)$ and $R_{a,b}(y) > 0$ for $y \in (y_0, 1)$. Since $y \mapsto P(x, t, y)$ is strictly increasing on $[0, 1]$, we get:

$$\text{if} \quad y \in (0,1),\ y \neq y_0, \quad \text{then} \quad P(x,t,y)Q_{a,b}(y) > P(x,t,y_0)Q_{a,b}(y).$$

This leads to

$$\sigma_{a,b}(x,t) > \left(\frac{t}{2}\right)^{a+b+1} P(x,t,y_0)\int_0^1 Q_{a,b}(y)\,dy. \tag{3.8}$$

Using

$$\int_0^1 [(1-y)^a(1+y)^b + (1-y)^b(1+y)^a]\,dy = 2^{a+b+1}\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)},$$
$$\int_0^1 (1-y^2)^{(a+b)/2}\,dy = \frac{1}{2}\sqrt{\pi}\,\frac{\Gamma((a+b)/2+1)}{\Gamma((a+b+1)/2+1)},$$

and the duplication formula

$$\Gamma(2x) = \frac{1}{\sqrt{\pi}}2^{2x-1}\Gamma(x)\Gamma(x+1/2) \quad (x>0)$$

we obtain

$$\int_0^1 Q_{a,b}(y)\,dy = 0. \tag{3.9}$$

From (3.7)–(3.9) we conclude that (3.6) holds. □

4. Inequalities and monotonicity. Applying Theorem 2, Lemma 2, and Lemma 5 we are able to extend and to complement inequality (1.2).

THEOREM 4. *Let $q \geq 0$ be a real number. Then we have for all $x, y \geq 0$:*

$$0 < \frac{1}{V_q(x)} + \frac{1}{V_q(y)} - \frac{1}{V_q(x+y)} \leq \frac{\Gamma(q+1)}{\Gamma(q+1/2)}. \tag{4.1}$$

Both bounds are best possible.

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Proof. Let $q, x, y \geq 0$. As remarked in [**8**], the convexity of $1/V_q$ and the inequality $V_q(x/2) < 2V_q(x)$ lead to

$$\frac{1}{V_q(x+y)} < \frac{2}{V_q((x+y)/2)} \leq \frac{1}{V_q(x)} + \frac{1}{V_q(y)}.$$

Also, Lemma 5 yields

$$\frac{1}{V_q(x)} + \frac{1}{V_q(y)} \leq \frac{1}{V_q(x+y)} + \frac{1}{V_q(0)} = \frac{1}{V_q(x+y)} + \frac{\Gamma(q+1)}{\Gamma(q+1/2)}.$$

Let

$$w_q = V_q(x) - \frac{1}{x}.$$

Then we get

$$\frac{1}{V_q(x)} - \frac{1}{V_q(x+y)} = \frac{x(x+y)[w_q(x+y) - w_q(x)] - y}{[1 + xw_q(x)][1 + (x+y)w_q(x+y)]} \quad \text{and}$$

$$\frac{1}{V_q(y)} - y = -\frac{y^2 w_q(y)}{1 + yw_q(y)}.$$

Using (2.6) gives

$$\lim_{x\to\infty} x^3 w_q(x) = -\frac{q+1}{2}.$$

This implies

$$\lim_{y\to\infty} \lim_{x\to\infty} \left(\frac{1}{V_q(x)} + \frac{1}{V_q(y)} - \frac{1}{V_q(x+y)} \right) = 0.$$

If we set $x = y = 0$, then equality holds on the right-hand side of (4.1). Thus, the bounds given in (4.1) are sharp. □

Since $x \mapsto V_q(x)$ is positive and strictly decreasing on $(0, \infty)$, we obtain

$$V_q(x+y) < V_q(x) + V_q(y) \quad (x, y > 0). \tag{4.2}$$

This means that for all $q > -1$ the function V_q is strictly subadditive on $(0, \infty)$. However, there is no parameter $q > -1$ such that V_q is submultiplicative on $(0, \infty)$. Otherwise, from

$$V_q(xy) \leq V_q(x)V_q(y) \quad (x, y > 0)$$

we get $V_q(1) \leq (V_q(1))^2$ or $1 \leq V_q(1)$, which contradicts

$$V_q(x) = \frac{1}{\Gamma(q+1)} \int_0^\infty e^{-u} \frac{u^q}{(x^2+u)^{1/2}} du < \frac{1}{\Gamma(q+1)} \int_0^\infty e^{-u} \frac{u^q}{x} du = \frac{1}{x} \quad (x > 0). \tag{4.3}$$

This leads to the question: do there exist parameters q such that V_q is supermultiplicative on $(0, \infty)$? The following theorem gives an answer.

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THEOREM 5. *Let $q > -1$ be a real number. The function $x \mapsto V_q(x)$ is strictly supermultiplicative on* $(0, \infty)$*, that is,*

$$V_q(x)V_q(y) < V_q(xy) \quad \textit{for all} \quad x, y > 0 \tag{4.4}$$

if and only if $q \geq q_0$, where $q_0 = 0.72117...$ is the only solution of $\Gamma(t+1) = \Gamma(t+1/2)$ on $(-1/2, \infty)$.

Proof. Let $q \geq q_0$. We consider two cases. First, let $0 < y \leq 1$. Then we obtain

$$V_q(xy) \geq V_q(x) \quad \text{and} \quad V_q(y) < V_q(0) = \frac{\Gamma(q+1/2)}{\Gamma(q+1)}.$$

This implies

$$V_q(xy) - V_q(x)V_q(y) \geq V_q(x)\big(1 - V_q(y)\big) > V_q(x)(1 - V_q(0)). \tag{4.5}$$

The function $q \mapsto 1 - V_q(0)$ is strictly increasing on $(-1/2, \infty)$. Hence, we get

$$1 - V_q(0) \geq 1 - V_{q_0}(0) = 0. \tag{4.6}$$

Combining (4.5) and (4.6) we obtain $V_q(xy) > V_q(x)V_q(y)$.

Next, let $y > 1$. Applying Lemma 2 and (4.3) we get

$$V_q(xy) > V_q(x)\frac{1}{y} > V_q(x)V_q(y).$$

It remains to show: if (4.4) holds, then $q \geq q_0$. Again, we consider two cases. Let $q > -1/2$. We set $x = y$ in (4.4) and let x tend to 0. This leads to $V_q(0) \leq 1 = V_{q_0}(0)$. Thus, $q \geq q_0$.

Now, we assume that $-1 < q \leq -1/2$. We prove that the inequality

$$V_q(x/2) < V_q(1/2)V_q(x) \tag{4.7}$$

is valid for all sufficiently small x. Using (1.1) we conclude that (4.7) is equivalent to

$$0 < \int_x^\infty (t^2 - x^2)^q \big[V_q(1/2)e^{3x^2/4 - t^2} - 2^{-2q-1}e^{-t^2/4}\big]\, dt = I_q(x), \quad \text{say.}$$

We define

$$z(q) = -\frac{\log(4^q V_q(1/2))}{\log(4)}.$$

From (2.5) we obtain

$$2\Gamma(q+1)[4^q V_q(1/2) - 1/2] = \int_0^\infty e^{-s/4}\frac{s^q}{(1+s)^{1/2}}\, ds - \int_0^\infty e^{-s}s^q\, ds.$$

Since $e^{-s/4}/(1+s)^{1/2} > e^{-s}$ for $s > 0$, we get

$$4^q V_q(1/2) > 1/2 \quad \text{for} \quad q > -1. \tag{4.8}$$

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This implies $z(q) < 1/2$. Let $\omega = \omega(q)$ be a real number such that

$$z(q) < \omega < 1/2. \tag{4.9}$$

We have

$$I_q(x) = \int_x^\infty A_q(x, t) B_q(x, t)\, dt,$$

where

$$A_q(x, t) = (t^2 - x^2)^{q+\omega} \quad \text{and} \quad B_q(x, t) = (t^2 - x^2)^{-\omega}\big[V_q(1/2) e^{3x^2/4 - t^2} - 2^{-2q-1} e^{-t^2/4}\big].$$

Since $q + \omega \leq -1/2 + \omega < 0$, we conclude that $t \mapsto A_q(x, t)$ is strictly decreasing on (x, ∞). Moreover, the function

$$t \mapsto b_q(x, t) = (t^2 - x^2)^\omega e^{t^2} B_q(x, t)$$

is strictly decreasing on $[x, \infty)$ with $\lim_{t\to\infty} b_q(x, t) = -\infty$. Applying (4.8) yields $b_q(x, x) > 0$. Thus, there exists a number $t_0 > x$ such that $b_q(x, t)$ is positive for $t \in (x, t_0)$ and negative for $t \in (t_0, \infty)$. This implies

$$A_q(x, t) B_q(x, t) > A_q(x, t_0) B_q(x, t) \quad \text{for} \quad x < t \neq t_0.$$

Hence, we obtain

$$I_q(x) > A_q(x, t_0) \int_x^\infty (t^2 - x^2)^{-\omega}\big[V_q(1/2) e^{3x^2/4 - t^2} - 2^{-2q-1} e^{-t^2/4}\big]\, dt = A_q(x, t_0) J_q(x), \quad \text{say.} \tag{4.10}$$

We have

$$2J_q(0) = \Gamma(1/2 - \omega)\big[V_q(1/2) - 4^{-(q+\omega)}\big]. \tag{4.11}$$

Since $V_q(1/2) > 4^{-(q+\omega)}$ is equivalent to $\omega > z(q)$, we conclude from (4.9) and (4.11) that $J_q(0) > 0$. This implies that there is a number $\epsilon > 0$ such that $J_q(x)$ is positive for $x \in (0, \epsilon)$. From (4.10) we get $I_q(x) > 0$ for $x \in (0, \epsilon)$. The proof of Theorem 5 is complete. □

Remark 3. Comments on the relevance of sub- and supermultiplicative functions in various fields as well as references on this subject can be found in [**6**].

Remark 4. Inequality (4.2) can be improved. In fact, from

$$\frac{2}{((x+y)^2 + u)^{1/2}} < \frac{1}{(x^2 + u)^{1/2}} + \frac{1}{(y^2 + u)^{1/2}} \quad (x, y, u > 0)$$

and (2.5) we obtain for all $q > -1$:

$$2 < \frac{V_q(x) + V_q(y)}{V_q(x + y)} \quad (x, y > 0). \tag{4.12}$$

Let $q > -1/2$ and $x = y$. If we let x tend to 0, then the ratio on the right-hand side of (4.12) converges to 2. Thus, (at least) for $q > -1/2$ the lower bound 2 cannot be replaced by a larger term, which is independent of x and y.

We now present an inequality which reveals a connection between $(V_q^{(k)}(x))^n$ and $(V_q^{(n)}(x))^k$.

THEOREM 6. *The inequality*

$$(-1)^{nk}\left(\frac{V_q^{(k)}(x)}{V_q(x)}\right)^n \le (-1)^{nk}\left(\frac{V_q^{(n)}(x)}{V_q(x)}\right)^k \tag{4.13}$$

holds for all real numbers $x > 0$ and integers n, k with $n \ge k \ge 0$ if and only if $q \in (-1, 0]$.

Proof. Let $q \in (-1, 0]$, $x > 0$, and $n \ge k \ge 0$. Applying Theorem 1 and Lemma 6 we conclude that (4.13) is valid. Conversely, we assume that (4.13) holds for all $x > 0$ and n, k with $n \ge k \ge 0$. We set $n = 2$ and $k = 1$ and obtain

$$(V_q'(x))^2 \le V_q(x)V_q''(x).$$

This means that V_q is log-convex on $(0, \infty)$, so that Remark 2 implies $q \in (-1, 0]$. □

Finally, we study the monotonicity behaviour of the ratio V_p/V_q and the difference $V_p - V_q$.

THEOREM 7. *Let $p, q > -1$ be real numbers.*

(i) *The function $x \mapsto V_p(x)/V_q(x)$ is increasing on $(0, \infty)$ if and only if $p \ge q$.*

(ii) *The function $x \mapsto V_p(x) - V_q(x)$ is increasing on $(0, \infty)$ if and only if $p \ge q$.*

If $p > q > -1$, then V_p/V_q and $V_p - V_q$ are strictly increasing on $(0, \infty)$.

Proof. Since the proofs of (i) and (ii) are similar, we only establish part (i). First, we assume that $p > q > -1$. Applying (2.2), (2.3), and the convolution theorem we get for $x > 0$:

$$\Gamma(p+1)\Gamma(q+1)x^{-(p+q+1)}(V_q(x))^2\left(\frac{V_p(x)}{V_q(x)}\right)' = V_p'(x)V_q(x) - V_p(x)V_q'(x)$$

$$= \int_0^\infty e^{-xs}\frac{s^p}{(x+s)^{1/2}}\,ds \int_0^\infty e^{-xs}\frac{s^q}{(x+s)^{3/2}}\,ds$$

$$- \int_0^\infty e^{-xs}\frac{s^q}{(x+s)^{1/2}}\,ds \int_0^\infty e^{-xs}\frac{s^p}{(x+s)^{3/2}}\,ds = \int_0^\infty e^{-xt}\Delta_{p,q}(x,t)\,dt, \tag{4.14}$$

where

$$\Delta_{p,q}(x,t) = \int_0^t \frac{s^q(t-s)^q}{(x+s)^{3/2}(x+t-s)^{1/2}}[(t-s)^{p-q} - s^{p-q}]\,ds.$$

The substitution $s = t(1+y)/2$ leads to

$$\Delta_{p,q}(x,t) = \left(\frac{t}{2}\right)^{p+q+1}\int_{-1}^1 \phi_q(x,t,y)(x+t(1-y)/2)[(1-y)^{p-q} - (1+y)^{p-q}]\,dy$$

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with

$$\phi_q(x,t,y) = \frac{(1-y^2)^q}{[(x+t(1-y)/2)(x+t(1+y)/2)]^{3/2}}.$$

Since $y \mapsto \phi_q(x,t,y)$ is even and $y \mapsto (1-y)^{p-q} - (1+y)^{p-q}$ is odd, we obtain

$$\Delta_{p,q}(x,t) = -2\left(\frac{t}{2}\right)^{p+q+2} \int_0^1 \phi_q(x,t,y)y[(1-y)^{p-q} - (1+y)^{p-q}]\,dy > 0. \tag{4.15}$$

From (4.14) and (4.15) we conclude that $(V_p(x)/V_q(x))' > 0$ for $x > 0$.

We define

$$h_q(x) = V_q(x) - \frac{1}{x} + \frac{q+1}{2x^3}.$$

Then, (2.6) gives

$$h_q(x) = O\left(\frac{1}{x^5}\right). \tag{4.16}$$

If $x \mapsto V_p(x)/V_q(x)$ is increasing on $(0,\infty)$, then we get for all $x > 0$:

$$\begin{aligned} 0 &\le [V_p(2x)V_q(x) - V_p(x)V_q(2x)]x^4 \\ &= [h_p(2x)h_q(x) - h_p(x)h_q(2x)]x^4 + [-h_p(x)/2 + h_p(2x) + h_q(x)/2 - h_q(2x)]x^3 \\ &\quad + [(q+1)h_p(x)/8 - (q+1)h_p(2x) - (p+1)h_q(x)/8 \\ &\quad + (p+1)h_q(2x)]x/2 + 3(p-q)/16. \end{aligned}$$

Applying (4.16) we obtain that the expression on the right-hand side converges to $3(p-q)/16$, if x tends to ∞. Thus, $p \ge q$. □

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REFERENCES

1. H. Alzer and C. Berg, Some classes of completely monotonic functions, *Ann. Acad. Scient. Fennicae* **27** (2002), 445–460.

2. T. Bang and B. Fuglede, No two quotients of normalized binomial mid-coefficients are equal, *J. Number Th.* **35** (1990), 345–349.

3. R. Brummelhuis and M. B. Ruskai, A simple one-dimensional model for atoms in strong magnetic fields, *Contemporary Math.* **217** (1998), 109–119.

4. R. Brummelhuis, M. B. Ruskai and E. Werner, One dimensional regularizations of the Coulomb potential with application to atoms in strong magnetic fields, *Studies Adv. Math.* **16** (2000), 67–75.

5. A. M. Fink, Kolmogorov-Landau inequalities for monotone functions, *J. Math. Anal. Appl.* **90** (1982), 251–258.

6. C. E. Finol and M. Wójtowicz, Multiplicative properties of real functions with applications to classical functions, *Aequat. Math.* **59** (2000), 134–149.

7. D. S. Mitrinović, *Analytic inequalities* (Springer-Verlag, 1970).

8. M. B. Ruskai and E. Werner, Study of a class of regularizations of $1/|x|$ using Gaussian integrals, *SIAM J. Math. Anal.* **32** (2000), 435–463.

9. D. V. Widder, *The Laplace transform* (Princeton Univ. Press, Princeton, 1941).

10. M. Wirth, On considère la fonction de **R** dans **R** défine par $f(x) = e^{-x^2/2}$; démontrer que la fonction g de **R** dans **R** défine par $g(x) = f(x)/\int_x^\infty f(t)\,dt$ est convexe, *Rev. Math. Spéciales* **104** (1993), 187–188.