Compositio Mathematica **136:** 217–235, 2003.

Existence of Moduli for Bi-Lipschitz Equivalence of Analytic Functions

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(Received: 11 July 2001; accepted in final form: 3 January 2002)

Abstract. We show that the bi-Lipschitz equivalence of analytic function germs $(\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ admits continuous moduli. More precisely, we propose an invariant of the bi-Lipschitz equivalence of such germs that varies continuously in many analytic families f_t: $(\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$. For a single germ f the invariant of f is given in terms of the leading coefficients of the asymptotic expansions of f along the branches of generic polar curve of f.

Mathematics Subject Classifications (2000). 14H15, 32S05, 32S15.

Key words. analytic function germs, bi-Lipschitz equivalence, moduli, polar curve, Newton polygon.

Consider a one parameter family of germs $f_t(x, y)$: $(\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$, $t \in \mathbb{C}$, given by

$$f_t(x, y) = f(x, y, t) = x^3 - 3t^2xy^4 + y^6. \tag{0.1}$$

We shall show that if t, t' are sufficiently generic then f_t and $f_{t'}$ are not bi-Lipschitz equivalent function germs, that is there is no germ of bi-Lipschitz homeomorphism $H\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that $f_t \circ H = f_{t'}$. This shows, in particular, that the bi-Lipschitz classification of function germs admits continuous moduli.

To our best knowledge this fact was not observed before. Since the bi-Lipschitz equivalence of complex analytic set germs does not admit moduli by [3] it is tempting to think that the same is true for function germs. In [6] Risler and Trotman asked explicitly whether the bi-Lipschitz equivalence of the zero sets $f_1^{-1}(0), f_2^{-1}(0)$ of complex analytic function germs f_1, f_2 with isolated singularities implies the bi-Lipschitz equivalence of f_1 and f_2. The above example shows that the answer is negative. We were asked whether the moduli exist also by our colleagues working on blow-analytic equivalence, S. Koike, T.-C. Kuo, and K. Kurdyka. The blow-analytic equivalence itself does not admit moduli, see [1], and the space of real analytic arcs seems to be the right object for study its properties. The action of a blow-analytic homeomorphism on this space is much easier to study if the homeomorphism does not change the order of contact between analytic arcs, that is if it is bi-Lipschitz. Unfortunately, by the above example, the blow-analytic bi-Lipschitz equivalence seems to be too strong for the equisingularity problem.

In order to distinguish bi-Lipschitz types of function germs of two complex variables we construct an invariant. The invariant for an arbitrary function germ of complex two variables is defined in Section 4. Its construction is based on the observation that the bi-Lipschitz homeomorphisms preserve (or rather do not move a lot) some regions around the relative polar curves. Consider, for instance, the family f_t of (0.1). We show in Section 1 that f_t cannot be trivialized by a Lipschitz vector field. For this we compare such a vector field, if it exists, on two branches of the relative polar curve $\partial f_t/\partial x = 0$. Then an elementary computation shows that such a vector field cannot be Lipschitz and tangent to the levels of f at the same time. Alternatively one can argue as follows. Though the flow of a Lipschitz vector field tangent to the levels of f does not necessarily preserve polar curves it can be proven that it cannot move them a lot. For instance, in the example (0.1), the image of $\partial f_t/\partial x = 0$ by the flow of such Lipschitz vector field has to stay in $U_\varepsilon = \{|\partial f_t/\partial x| \leqslant \varepsilon|\partial f_t/\partial y|\}$. The polar curve $\Gamma\colon \partial f_t/\partial x = 0$ has two branches (called polar arcs) that are tangent to the line $x = 0$. The asymptotic expansions of f on these polar arcs are different. The leading terms of these asymptotic expansions don't change if we replace Γ by U_ε. Fix a point p on one arc and take the closest point p' on the other. As we show $f(p)/f(p')$ tends to a precise value as $p \to 0$. This value is a bi-Lipschitz invariant. The example (0.1) is studied in details in Section 3.

Let $f(x, y)$ be an arbitrary function germ. We generalize the above idea and construct in Section 4 a set of bi-Lipschitz invariants coming from each set of mutually tangent polar arcs. In general a bi-Lipschitz homeomorphism does not come from the integration of a Lipschitz vector field and we do not know whether such a homeomorphism preserves (or rather does not move a lot) the sets of the form U_ε. Therefore in order to prove the invariance of the introduced numbers we shall study neighborhoods of polar curves maybe bigger than U_ε. These neighborhoods, denoted Y_δ, are introduced in Section 2 where we prove that they are preserved by bi-Lipschitz homeomorphisms. Moreover, in Section 5 we show that the asymptotic behavior of f on Y_δ is similar to that on the polar curves and hence will follow again that the leading coefficients of the expansions corresponding to the polar curves give a bi-Lipschitz invariant. Though the sets Y_δ play the crucial role in the proof of invariance we do not need them in order to calculate the invariant itself. For this it suffices to compute the leading coefficients of the expansion of f along the polar arcs of $\partial f/\partial x = 0$, with the only assumption that the x-axis is not tangent to the tangent cone of $f^{-1}(0)$ at the origin.

1. Trivializing Lipschitz Vector Fields

THEOREM 1.1. *There is no Lipschitz vector field*

$$\mathbf{v}(x, y, t) = \partial/\partial t + v_1(x, y, t)\partial/\partial x + v_2(x, y, t)\partial/\partial y,$$

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$v_1(0,0,t) \equiv v_2(0,0,t) \equiv 0$, *defined in a neighborhood of* $(0,0,t_0)$ *and tangent to the levels of* $f(x,y,t) = x^3 - 3t^2xy^4 + y^6$.

Proof. Suppose that, contrary to our claim, such $\mathbf{v}$ exists. Then

$$\partial f/\partial \mathbf{v} \equiv 0. \tag{1.1}$$

Let Γ denote the family of polar curves

$$\Gamma = \{(x,y,t) | \partial f/\partial x = 3(x^2 - t^2y^4) = 0\}.$$

Γ consists of two branches

$$x = \pm ty^2. \tag{1.2}$$

Develop (1.1) along each branch of Γ and substitute (1.2)

$$\begin{aligned} 0 = \partial f/\partial \mathbf{v} = \partial f/\partial t + v_2 \partial f/\partial y &= -6txy^4 + (-12t^2xy^3 + 6y^5)v_2 \\ &= \mp 6t^2y^6 + (\mp 12t^3y^5 + 6y^5)v_2. \end{aligned}$$

Hence

$$v_2(\pm ty^2, y, t) = \frac{\pm t^2 y}{1 \mp 2t^3} \tag{1.3}$$

We compare v_2 between two branches of Γ

$$v_2(ty^2, y, t) - v_2(-ty^2, y, t) = \frac{t^2y}{1-2t^3} - \frac{-t^2y}{1+2t^3} \sim y \tag{1.4}$$

at least near t supposed generic. On the other hand if v_2 is Lipschitz with Lipschitz constant L then

$$|v_2(ty^2, y, t) - v_2(-ty^2, y, t)| \leqslant 2L|ty^2|, \tag{1.5}$$

which contradicts (1.4). □

One may try to explain in a more geometric way why the existence of such a vector field is not possible. The flow of a Lipschitz vector field moves slowly the tangent spaces to the levels of f as follows from [5]. This means, in particular, that the polar curves Γ, though not necessarily preserved by the flow of $\mathbf{v}$, would stay in $U_\varepsilon := \{|\partial f/\partial x| < \varepsilon|\partial f/\partial y|\}$, for $\varepsilon > 0$ small. Suppose, for simplicity, that the flow of $\mathbf{v}$ preserves Γ. Then it cannot preserve f. Indeed, on each branch of Γ $f(\pm ty^2, y, t) = (1 \mp 2t^3)y^6$. If we compare the levels of f on these two branches, for instance by projecting onto the y-axis, we see that they are transverse. Consequently, the flow of $\mathbf{v}$ induces a significant movement of variable y on each branch of Γ. This is the meaning of (1.3). The movements on different branches of Γ are not compatible, they go in different directions which is algebraically expressed by (1.4). But this is not possible for a Lipschitz vector field because of the high contact between the branches of Γ, see (1.5).

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2. Neighborhoods of Γ Preserved by Bi-Lipschitz Homeomorphisms

Fix $g = f_t$ and $\tilde{g} = f_{t'}$. In order to show that g and $\tilde{g}$ are not bi-Lipschitz equivalent we would like to follow the geometric idea explained at the end of Section 1. The main problem is that we do not know whether the image of polar curve $\Gamma = \{(x, y) | \partial g/\partial x = 0\}$ by an arbitrary bi-Lipschitz homeomorphism H is contained in $\{|\partial\tilde{g}/\partial x| < \varepsilon|\partial\tilde{g}/\partial y|\}$. So our idea is to find maybe bigger neighborhoods of Γ which are preserved by such H.

Fix a single germ $g: (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ and a point $p_0 \in \mathbb{C}^2$ close to the origin. Fix another constant K sufficiently large which will be later related to the Lipschitz constant of a bi-Lipschitz homeomorphism. Let $c = g(p_0)$. Let $B(p_0, \rho)$ denote the open ball centered at p_0 and of radius ρ. Denote $X(p_0, \rho) := B(p_0, \rho) \cap g^{-1}(c)$. Suppose that $p, q \in X(p_0, \rho)$ belong to the same connected component of $X(p_0, K\rho)$, as p_0. Then one can join p and q by a piecewise C^1 curve in $X(p_0, K\rho)$. Let $\mathrm{dist}_{p_0,\rho,K}(p, q)$ denote the infimum of lengths of such curves that is the intrinsic distance of p and q in $X(p_0, K\rho)$. Define

$$\varphi(p_0, \rho, K) := \sup \frac{\mathrm{dist}_{p_0,\rho,K}(p, q)}{|p - q|}, \tag{2.1}$$

where the supremum is taken over all pairs of points p, q of $X(p_0, \rho)$ from the connected component of $X(p_0, K\rho)$ containing p_0. Clearly if p_0 is a nonsingular point of $g^{-1}(c)$ then $\varphi(p_0, \rho, K) \to 1$ as $\rho \to 0$ but φ is not necessarily an increasing function of ρ so we define

$$\psi(p_0, \rho, K) := \sup_{\rho' \leqslant \rho} \varphi(p_0, \rho', K) \tag{2.2}$$

Finally we define

$$Y(\rho, K, A) := \{p | \psi(p, \rho, K) \geqslant A\}. \tag{2.3}$$

Intuitively speaking $Y = Y(\rho, K, A)$, for A large, is the set of points where the curvature of levels $g^{-1}(c)$ is very large. We shall show that such Y are preserved by bi-Lipschitz homeomorphisms.

Let $H: (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be the germ of a bi-Lipschitz homeomorphism such that H sends the levels of $\tilde{g}$ to the levels of g, where $g, \tilde{g}: (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ are the germs of analytic functions. Fix $L \geqslant 1$, a common Lipschitz constant of H and its inverse H^{-1}. For $p_0 \in \mathbb{C}^2$ we denote by $\tilde{p}_0 = H(p_0)$ and similarly we add the tilde to distinguish the corresponding objects in the domain and the target space of H, that is for instance

$$\tilde{Y}(\rho, K, A) := \{\tilde{p} | \psi(\tilde{p}, \rho, K) \geqslant A\}, \tag{2.4}$$

will be a subset of the target space of H.

LEMMA 2.1. *Suppose* $K \geqslant L^2$. *Then*

$$\tilde{Y}(L^{-1}\rho, K, AL^2) \subset H(Y(\rho, K, A)) \subset \tilde{Y}(L\rho, K, AL^{-2}). \tag{2.5}$$

Proof. Since H is bi-Lipschitz

$$\tilde{X}(\tilde{p}_0, L^{-1}\rho) \subset H(X(p_0, \rho)) \subset \tilde{X}(\tilde{p}_0, L\rho). \tag{2.6}$$

Hence, if $K \geqslant L^2$,

$$\tilde{X}(\tilde{p}_0, L^{-1}\rho) \subset H(X(p_0, \rho)) \subset \tilde{X}(\tilde{p}_0, L^{-1}K\rho) \subset H(X(p_0, K\rho)). \tag{2.7}$$

If $\tilde{p}_0$ and $\tilde{p} \in \tilde{X}(\tilde{p}_0, L^{-1}\rho)$ are in the same connected component of $\tilde{X}(\tilde{p}_0, L^{-1}K\rho)$ then p_0 and $p = H^{-1}(\tilde{p})$ are in the same connected component of $H(X(p_0, K\rho))$. H changes the distances at most by a constant L also for the intrinsic distances calculated along a subvariety. Hence (2.7) implies

$$\psi(p_0, \rho, K) \geqslant L^{-2}\psi(\tilde{p}_0, L^{-1}\rho, K). \tag{2.8}$$

Similarly

$$L^2\psi(\tilde{p}_0, L\rho, K) \geqslant \psi(p_0, \rho, K). \tag{2.9}$$

The last two formulae give the statement. □

A similar argument shows the following.

LEMMA 2.2. *Let* $\delta > 0$ *and denote* $Y(\delta, K, M, A) := \{p | \psi(p, M|p|^{1+\delta}, K) \geqslant A\}$. *If* $K \geqslant L^2$ *then* $\tilde{Y}(\delta, K, ML^{-2-\delta}, AL^2) \subset H(Y(\delta, K, M, A)) \subset \tilde{Y}(\delta, K, ML^{2+\delta}, AL^{-2})$.

3. Computation on Example

THEOREM 3.1. *Let* f_t *and* $f_{t'}$ *be two analytic function germs given by* (0.1). *Suppose* $t, t', 1 \pm 2t^3$, *and* $1 \pm 2t'^3$ *are nonzero. If there is the germ of a bi-Lipschitz homeomorphism* $H\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ *such that* $f_t \circ H = f_{t'}$ *then* $t^3 = \pm t'^3$.

Proof. Fix $g(x, y) = x^3 - 3t^2xy^4 + y^6$ such that $t \neq 0$ and $1 \pm 2t^3 \neq 0$. Let

$$U = \{(x, y) | |\partial g/\partial x| < |\partial g/\partial y|\}$$

LEMMA 3.2. *Suppose* $(x, y) \in U$. *Then*

$$x = \pm ty^2 + \mathrm{O}(y^3), \tag{3.1}$$
$$g(x, y) = (1 \mp 2t^3)y^6 + \mathrm{O}(y^8). \tag{3.2}$$

Proof. We use

$$\partial g/\partial x = 3(x^2 - t^2y^4), \qquad \partial g/\partial y = 6y^3(y^2 - 2t^2x).$$

Consider the following change of variables $x = sy^2$. Then

$$\partial g/\partial x = 3y^4(s^2 - t^2), \tag{3.3}$$
$$\partial g/\partial y = 6y^5(1 - 2t^2s). \tag{3.4}$$

This implies that $s = \pm t + \mathrm{O}(y)$ on U, that is the first formula of the statement. The second formula is a direct consequence of the first one. □

LEMMA 3.3. *Let* $\delta > 0$, $C > 0$. *Then on the set*

$$\{p = (x, y) \mid \exists p_0 = (x_0, y_0) \in U, g(p) = g(p_0), |y - y_0| \leqslant C|y_0|^{1+\delta}\}$$

we have

$$x = O(y^2), \tag{3.5}$$

$$g(x, y) = (1 \mp 2t^3)y^6 + O(y^{6+\delta'}), \tag{3.6}$$

where $\delta' = \min\{\delta, 2\}$.

Proof. Since $\delta > 0$, $y \simeq y_0$. Thus (3.6) follows from (3.2) for (x_0, y_0).

Write $x = sy^2$. Then

$$y^6(s^3 - 3t^2 s + 1) = g(x, y) = (1 \mp 2t^3)y^6 + O(y^{6+\delta'})$$

and, hence, s is bounded. This gives (3.5). □

COROLLARY 3.4. *Let* $Y = Y(\delta, K, M, A) := \{p \mid \psi(p, M|p|^{1+\delta}, K) \geqslant A\}$, *where* $\delta > 0$, $M > 0$ *and* A, K *are sufficiently large constants. Then formulae* (3.5) *and* (3.6) *hold for all* $(x, y) \in Y$.

Proof. Fix p_0 and suppose $X(p_0, KM|p_0|^{1+\delta})$ does not intersect U. Then the projection $X_c = f^{-1}(c) \to \mathbb{C}$, $(x, y) \to y$, is submersive and X_c near each point of $X(p_0, KM|p_0|^{1+\delta})$ is the graph of a function $x = x(y)$ with the derivative bounded by 1. Therefore, if we suppose K and A big enough ($K, A > \sqrt{2}$ suffice), then $p_0 \notin Y$ (see the proof of Lemma 5.6 below for a more detailed argument). Consequently, if $p_0 = (x_0, y_0) \in Y$, then there is a point $p = (x, y) \in U$ such that

$$|p - p_0| \leqslant KM|p_0|^{1+\delta}. \tag{3.7}$$

Then, the corollary follows from Lemma 3.3. □

PROPOSITION 3.5. *Let* $Y = Y(\delta, K, M, A)$, *where* $\delta > 0, M > 0$, *and* A, K *are sufficiently large constants. Suppose that* $p_1, p_2 \in Y$ *and there is a* $\delta_1 > 0$ *such that* $|p_1 - p_2| \leqslant |p_1|^{1+\delta_1}$. *Then, for* $\delta_2 < \min\{\delta, \delta_1, 2\}$ *and in a sufficiently small neighborhood of the origin*

$$\left|\frac{g(p_1)}{g(p_2)} - a\right| \leqslant |p_1|^{\delta_2}, \tag{3.8}$$

where a is one of the following values: 1, $(1 + 2t^3)/(1 - 2t^3)$, $(1 - 2t^3)/(1 + 2t^3)$.

Proof. Let $p_i = (x_i, y_i)$, $i = 1, 2$. Then by Corollary 3.4

$$g(p_i) = (1 \mp 2t^3)y_i^6 + O(y_i^{6+\delta'}), \tag{3.9}$$

and

$$y_1 - y_2 = O(|p_1|^{1+\delta_1}) = O(y_1^{1+\delta_1}). \tag{3.10}$$

and, hence, the result. □

LEMMA 3.6. *If K and A are sufficiently large and $1 > \delta > 0$ then $Y = Y(\delta, K, M, A)$ is nonempty and contains the polar curve Γ. Moreover, in this case all the limits of $g(p_1)/g(p_2)$ given by Proposition 3.5 can be obtained by taking p_1 and p_2 along the branches of Γ.*

Proof. Fix K and δ, as above.

Consider the projection $\pi_c\colon X_c = g^{-1}(c) \to \mathbb{C}$, $\pi_c(x, y) = y$. π_c is a triple covering branched at the points of $\Gamma \cap X_c$, that is the points of coordinates

$$x = \pm t y^2, \quad y = (1 \mp 2t^3)^{-1/6} c^{1/6}. \tag{3.11}$$

The distances between different points of ramification or, equivalently, between their projections onto y-axis, are of size comparable to $|y|$.

Fix a point $p_0 = (x_0, y_0)$ of ramification of π_c. Let $\mathcal{V} = \{y; |y - y_0| < \varepsilon |y_0|\}$, ε small. Then, if p_0 is sufficiently close to the origin, $\pi_c^{-1}(\mathcal{V})$ contains $X(p_0, KM|p_0|^{1+\delta})$. On the other hand suppose that $p = (x, y) \in g^{-1}(c)$ satisfies

$$|y - y_0| \leqslant |y_0|^{1+\delta}, \tag{3.12}$$

Then, by Lemma 3.3, $x = \mathrm{O}(y^2)$ and, hence,

$$|p - p_0| \leqslant 2|y_0|^{1+\delta} \tag{3.13}$$

Denote $\mathcal{V}_\delta = \{y; |y - y_0| < |y_0|^{1+\delta}\}$. By (3.13), $\pi_c^{-1}(\mathcal{V}_\delta) \subset X(p_0, KM|p_0|^{1+\delta})$. Since π_c is a branched triple covering, $\pi_c^{-1}(\mathcal{V}_\delta)$ consists of two connected components, one which contains the p_0 and the other which projects diffeomorphically onto $\mathcal{V}_\delta$. In what follows we shall restrict ourselves to the connected component containing p_0. We shall denote by $\tilde{\pi}_c$ the restriction of π_c to this component.

Fix $y \in \mathcal{V}_\delta$ such that $|y - y_0| = \frac{1}{2}|y_0|^{1+\delta}$ and denote the two points of $\tilde{\pi}_c^{-1}(y)$ by $p_1 = (x_1, y)$ and $p_2 = (x_2, y)$. Then p_1, p_2 are in $\tilde{\pi}_c^{-1}(\mathcal{V}_\delta)$ which is connected. Note that the projection $\tilde{\pi}_c$ of any curve joining p_1 and p_2 in $\tilde{\pi}_c^{-1}(\mathcal{V}_\delta)$ passes around y_0. Hence the distance of p_1 and p_2 calculated along $g^{-1}(c)$ satisfies

$$\mathrm{dist}_{p_0, C|p_0|^{1+\delta}, K}(p_1, p_2) \geqslant |y_0|^{1+\delta}. \tag{3.14}$$

By Lemma 3.3, x_1 and x_2 are $\mathrm{O}(|y_0|^2)$ and, consequently,

$$|p_1 - p_2| = \mathrm{O}(|y_0|^2) = \mathrm{O}(|p_0|^2). \tag{3.15}$$

Thus

$$\psi(p_0, M|p_0|^{1+\delta}, K) \to \infty \tag{3.16}$$

as $p_0 \to 0$, that is Γ is contained, as the germ at the origin, in $Y(\delta, K, M, A)$ as claimed.

The last claim of the lemma can be checked directly by considering the limits along the two branches of Γ. $\square$

To complete the proof of Theorem 3.1 we note that Lemma 2.2, Proposition 3.5, and Lemma 3.6 give that the following sets are equal

$$\left\{\frac{1+2t^3}{1-2t^3}, \frac{1-2t^3}{1+2t^3}\right\} = \left\{\frac{1+2t'^3}{1-2t'^3}, \frac{1-2t'^3}{1+2t'^3}\right\}.$$

This implies $t^3 = \pm t'^3$ as claimed. □

4. Invariants of Bi-Lipschitz Equivalence

In this section we construct invariants of bi-Lipschitz equivalence of analytic function germs $(\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$. Our construction generalizes the one presented in the previous section.

Let $f(x, y): (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a germ of analytic function with Taylor expansion:

$$f(x, y) = H_k(x, y) + H_{k+1}(x, y) + \cdots. \tag{4.1}$$

We shall assume $f(x, y)$ is *mini-regular in x of order k* in the sense that $H_k(1, 0) \neq 0$. We shall also assume, for simplicity, that f has no multiple roots.

By *an analytic arc* we mean a fractional (convergent) power a series of the form

$$\lambda : \ x = \lambda(y) := c_1 y^{n_1/N} + c_2 y^{n_2/N} + \cdots, \quad c_i \in \mathbb{C}, \tag{4.2}$$

where $N \leqslant n_1 < n_2 < \cdots$ are positive integers having no common divisor, such that $\lambda(t^N)$ has positive radius of convergence. We can identify λ with the analytic arc $\lambda : \ x = c_1 t^{n_1} + c_2 t^{n_2} + \cdots, y = t^N$, $|t|$ small, which is not tangent to the x-axis (since $n_1/N \geqslant 1$).

Given two analytic arcs $x = \lambda_i(y), i = 1, 2$, by *the order of contact of* λ_1 *and* λ_2 we mean $\operatorname{ord}(\lambda_1, \lambda_2) = \operatorname{ord}_0(\lambda_1(y) - \lambda_2(y))$.

A polar arc $x = \gamma(y)$ is a branch of the polar curve Γ: $\partial f/\partial x = 0$. Since f is mini-regular in x, $x = \gamma(y)$ is not tangent to the x-axis and it is an arc in our sense. Let γ be a polar arc given by a fractional power series as in (4.2) and let θ is an Nth root of unity. Then we should consider $\gamma(\theta y)$ as a polar arc different from γ if $\theta \neq 1$.

We divide the polar arcs into two classes: those that are tangent to the tangent cone $C_0(X) = \{H_k = 0\}$ to $X = f^{-1}(0)$ at the origin, and those that are not. We shall call the polar arcs of the first type *tangential*. The nontangential polar arcs can be also described as the moving ones since their tangent at 0 moves when we change the system of coordinates, see the next section for details.

EXAMPLE 4.1. Consider $f(x, y) = y(x^2 - y^3) = x^2 y - y^4$. ($f$ is not mini-regular with respect neither to the x-axis nor to the y-axis but the computation in a special system of coordinates is easier) The generic polar curve of f

$$\begin{aligned} 0 &= \partial f/\partial x + a\partial f/\partial y = 2xy + a(x^2 - 4y^3) \\ &= (x - 2ay^2 + \cdots)(2y + ax + 2a^2y^2 + \cdots) \end{aligned}$$

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has two components: one tangential tangent to the line $x = 0$ and one tangent to the line $2y + ax = 0$ that moves as a moves.

Let γ be a polar arc. We associate to γ two numbers: $h_0 = h_0(\gamma) \in \mathbb{Q}_+$ and $c_0 = c_0(\gamma) \in \mathbb{C}^*$ given by the expansion

$$f(\gamma(y), y) = c_0 y^{h_0} + \cdots, \quad c_0 \neq 0. \tag{4.3}$$

Note that γ is tangential if and only if $h_0(\gamma) > k$.

Fix a tangent line l to X at 0, that is $l \subset C_0(X)$. Let $\Gamma(l)$ denote the set of all polar arcs tangent to l at 0. $\Gamma(l)$ is nonempty if and only if

$$l \subset \mathrm{Sing}(C_0(X)) := \{\partial H_k/\partial x = \partial H_k/\partial y = 0\}.$$

Define I(l) as the set of formal expressions $\mathrm{I(l)} = \{\mathrm{c_0}(\gamma) y^{\mathrm{h_0}(\gamma)} | \gamma \in \Gamma(\mathrm{l})\}/\mathbb{C}^*$, divided the action of $\mathbb{C}^*$, where $c \in \mathbb{C}^*$ acts by multiplication on y:

$$\{c_{01} y^{h_{01}}, \ldots, c_{0k} y^{h_{0k}}\} \sim \{c_{01} c^{h_{01}} y^{h_{01}}, \ldots, c_{0k} c^{h_{0k}} y^{h_{0k}}\}.$$

DEFINITION 4.2. Let $f(x, y)$ be an analytic function germ. By *the invariant* $\mathrm{Inv}(f)$ of f we mean the set of all I(l), where l runs over all lines in $\mathrm{Sing}(C_0(X))$.

THEOREM 4.3. *Let f_1, f_2 be two analytic functions germs $(\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ mini-regular in x. If f_1 and f_2 are bi-Lipschitz equivalent then* $\mathrm{Inv}(f_1) = \mathrm{Inv}(f_2)$.

Remark 4.4. (i) Theorem 4.3 implies that Inv(f) is independent of the choice of coordinates. This, of course, can be checked directly, for instance, by the methods of the next section.

(ii) Let f_t be the function of example of Section 3. Then there are two polar arcs $\gamma_\pm(y) = \pm t y^2$. They are both tangent to the y-axis. Thus $\mathrm{Inv(f}_t) = \{(1 \mp 2t^3) y^3\}/\mathbb{C}^*$.

Proof of Theorem 4.3. Consider the sets

$$Y_\delta = Y(\delta, K, M, A) := \{p | \psi(p, M|p|^{1+\delta}, K) \geqslant A\},$$

where K, A are large constants and $\delta > 0$ is sufficiently small. In what follows we suppress the constants to avoid an awkward notation. The sets Y_δ satisfy two important properties. Firstly they are preserved by bi-Lipschitz homeomorphism, see Lemma 2.2. Secondly the asymptotic behavior of f on Y_δ gives exactly the invariant Inv(f) thanks to the following proposition which will be proved in the next section.

PROPOSITION 4.5. *Let $\tilde{\Gamma}$ be the union of tangential polar arcs. Suppose $\delta > 0$ and sufficiently small. Then* (i) $\tilde{\Gamma} \subset \mathrm{Y}_\delta$. (ii) *There is a constant $B > 0$, which depends on the*

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constants in the definition of Y_δ*, such that for each* $p_0 \in Y_\delta$ *there is* $p \in \tilde{\Gamma}$ *such that* $f(p) = f(p_0)$ *and*

$$|p - p_0| \leqslant B|p_0|^{1+\delta}. \tag{4.4}$$

We shall show how Proposition 4.5 implies Theorem 4.3. First we explain how to recover Inv(f) from the asymptotic behavior of f on Y_δ. By Proposition 4.5 the tangent cone to Y_δ at the origin equals exactly $\mathrm{Sing}(C_0(X))$ that is to say, as germ at the origin, Y_δ is included in a 'horn' neighborhood of $\mathrm{Sing}(C_0(X))$ that is in a set of the form $\{p \in \mathbb{C}^2; \mathrm{dist}(p, \mathrm{Sing}(C_0(X))) \leqslant |p|^{1+\xi}\}$ for a $\xi > 0$. Indeed, clearly the union $\tilde{\Gamma}$ of tangential arcs is included in such a 'horn' neighborhood for an exponent ξ. Taking ξ maybe even smaller, in particular smaller than δ, we may assure, by Proposition 4.5, that the 'horn' neighborhood contains Y_δ. For $l \subset \mathrm{Sing}(C_0(X))$ denote by $Y_\delta(l)$ the part of Y_δ tangent to l that is $Y \cap \{p \in \mathbb{C}^2; \mathrm{dist}(p, \mathrm{Sing}(C_0(X))) \leqslant |p|^{1+\xi}\}$. For $p_0 = (x_0, y_0) \in Y_\delta(l)$ and $C > 0$ we denote

$$V(p_0) = \{f(p) | p \in Y_\delta \text{ and } |p - p_0| \leqslant C|p_0|^{1+\delta}\}. \tag{4.5}$$

LEMMA 4.6. *There is* $\eta > 0$ *such that* $V(p_0) \subset \bigcup_{\gamma \in \Gamma(l)} \{\tau | |\tau - c_0(\gamma) y_0^{h_0(\gamma)}| \leqslant |y_0|^{h_0(\gamma)+\eta}\}$.

Proof. Let $p \in Y_\delta$ and let $|p - p_0| \leqslant B|p_0|^{1+\delta}$. By Proposition 4.5 we may suppose that p_0 and p belong to $\Gamma(l)$ without changing the values of f on them. Then the lemma follows. □

Let f_1, f_2 be two analytic functions germs such that $f_1 = f_2 \circ H$, where H is a bi-lipschitz homeomorphism. By Lemma 2.2, $H(Y_{\delta,f_1}) \subset Y_{\delta,f_2}$ (the constants in definition of Y_δ change). Fix $l_1 \subset \mathrm{Sing}(C_0(f_1^{-1}(0)))$ and a polar arc γ tangent to l_1. The image $H(\gamma)$ is in Y_{δ,f_2} and, hence, in one of $Y_{\delta,f_2}(l_2)$, where $l_2 \subset \mathrm{Sing}(C_0(f_2^{-1}(0)))$. The line l_2 does not depend on the choice of γ with tangent line l_1 since all such polar arcs γ are mutually tangent and H is Lipschitz. Finally, by Proposition 4.5, $H(Y_{\delta,f_1}(l_1)) \subset Y_{\delta,f_2}(l_2)$. Let $p_1 \in Y_{\delta,f_1}(l_1)$ and let $p_2 = (x_2, y_2) = H(p_1)$. By the above,

$$V(p_1) \subset V(H(p_1)) \tag{4.6}$$

(again, maybe for a different constant C in the definition (4.5)). Denote by $\Gamma_i(l_i)$ the union of polar arcs in $Y_{\delta,f_i}(l_i)$, for $i = 1, 2$ respectively. By Lemma 4.6 and (4.6) $V(p_1) \subset \bigcup_{\gamma' \in \Gamma_2(l_2)} \{\tau | |\tau - c_0(\gamma') y_2^{h_0(\gamma')}| \leqslant |y_2|^{h_0(\gamma')+\eta}\}$. Thus for each polar arc $x_1 = \gamma(y_1)$ in $\Gamma_1(l_1)$, $f_1(\gamma(y_1), y_1) \in V(p_1)$ and

$$f_1(\gamma(y_1), y_1) = c_0(\gamma) y_1^{h_0(\gamma)} + \mathrm{o}(y_1^{h_0(\gamma)+\eta}) \in \bigcup_{\gamma' \in \Gamma_2(l_2)} \{\tau | |\tau - c_0(\gamma') y_2^{h_0(\gamma')}| \leqslant |y_2|^{h_0(\gamma')+\eta}\}.$$

Recall that $|y_i| \sim |p_i|$ since the normal cones are away of the x-axis and $|p_1| \sim |H(p_1)| = |p_2|$ since H is bi-Lipschitz. Hence, $|y_1/y_2|$ is bounded and so is $|y_2/y_1|$. As a consequence for each γ in $\Gamma_1(l_1)$ there is a $\gamma' \in \Gamma_2(l_2)$ such that

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$$|c_0(\gamma)y_1^{h_0(\gamma)} - c_0(\gamma')y_2^{h_0(\gamma')}| \leqslant |y_2|^{h_0(\gamma')+\eta}.$$

In particular $h_0(\gamma) = h_0(\gamma')$. Given two polar curves $\gamma, \tilde{\gamma} \in \Gamma_1(l_1)$ and the corresponding $\gamma', \tilde{\gamma}' \in \Gamma_2(l_2)$. Then $H(\gamma(y_1), y_1) = (x_2, y_2)$ and $H(\tilde{\gamma}(y_1), y_1) = (\tilde{x}_2, \tilde{y}_2)$ satisfy $y_2 - \tilde{y}_2 = o(y_2)$. Indeed, it follows from the tangency of γ and $\tilde{\gamma}$ since H is Lipschitz. This shows that

$$c_0(\tilde{\gamma})y_1^{h_0(\tilde{\gamma})} = c_0(\tilde{\gamma}')\tilde{y}_2^{h_0(\tilde{\gamma})} + o(\tilde{y}_2^{h_0(\tilde{\gamma})}) = c_0(\tilde{\gamma})(y_2/y_1)^{h_0(\tilde{\gamma})}y_1^{h_0(\tilde{\gamma})} + o(y_1^{h_0(\tilde{\gamma})}).$$

Considering all polar curves in $\in \Gamma_1(l_1)$ and taking the limit as $y_2 \to 0$ this gives

$$\{c_0(\gamma)y^{h_0(\gamma)}|\gamma \in \Gamma_1(l_1)\}/\mathbb{C}^* \subset \{c_0(\gamma')y^{h_0(\gamma')}|\gamma' \in \Gamma_2(l_2)\}/\mathbb{C}^*$$

where y_2/y_1 plays the role of a constant of $\mathbb{C}^*$. By symmetry we obtain the opposite inclusion. Therefore $\mathrm{I}(\Gamma_1(\mathrm{l}_1)) = \mathrm{I}(\Gamma_2(\mathrm{l}_2))$ and $\mathrm{Inv}(\mathrm{f}_1) = \mathrm{Inv}(\mathrm{f}_2)$, as required. $\square$

Remark 4.7. Let $f_t(x) = f(x, t)$ be an analytic family of analytic function germs $(\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with parameter $t \in \mathbb{C}$. Denote

$$\operatorname{grad}_x f_t = \sum_i \frac{\partial f_t}{\partial x_i}\frac{\partial}{\partial x_i}.$$

If f_t is equimultiple and the tangent cone to $X_t = f_t^{-1}(0)$ has isolated singularity at the origin for all t, then f_t is bi-Lipschitz trivial. Indeed, consider the Kuo vector field

$$\vec{v}(x, t) = \partial/\partial t - \frac{\partial f/\partial t}{|\operatorname{grad}_x f_t|^2}\operatorname{grad}_x f_t$$

that trivializes f. On may show that the assumption on the tangent cone implies $|\operatorname{grad}_x f_t| \sim |x|^{k-1}$, where k denotes the multiplicity of f at the origin. Therefore the partial derivatives of $\vec{v}$ are bounded and hence it follows easily that it is Lipschitz.

5. Cuspidal Neighborhoods of Polar Curves

In this section we give a proof of Proposition 4.5. The proof will be based on a detailed analysis of neighborhoods of polar curve $\Gamma : \partial f/\partial x = 0$. Our main method is the Newton diagram associated to each polar arc. First we recall briefly its construction. For the details the reader may consult [2].

Let $f(x, y)\colon (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a germ of analytic function with Taylor expansion (4.1) mini-regular in x. Fix an analytic arc λ as (4.2). Write

$$F(X, Y) := f(X + \lambda(Y), Y) := \sum c_{ij}X^i Y^{j/N}. \tag{5.1}$$

For each $c_{ij} \neq 0$, let us plot a dot at $(i, j/N)$, called a *Newton dot*. The set of Newton dots is called *the Newton diagram of f relative to λ*. By *the Newton polygon of f relative to λ* we mean the Newton polygon of F in the usual sense that is the union of compact faces of the boundary of the convex hull of Newton diagram.

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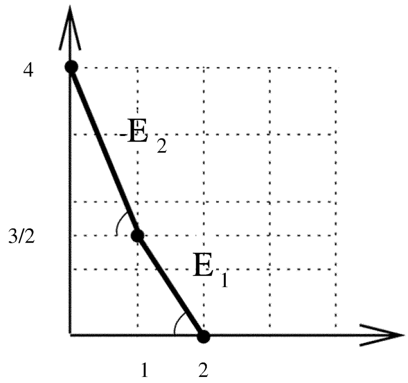
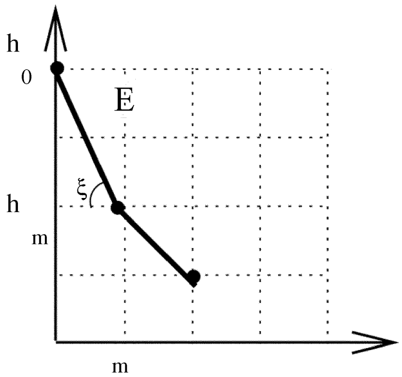
For instance, the Newton polygon of $f(x,y) = x^2 - y^3 + y^4$, relative to $\lambda : x = y^{3/2}$, is the following



The exponents of the development $f(\lambda(y), y) = F(0, Y)$ correspond to the Newton dots on the line $X = 0$. In particular, $\operatorname{ord} f(\lambda(y), y) = h_0$, where $(0, h_0)$ the lowest Newton dot on $X = 0$. Thus the arc λ is a root of f iff there is no Newton dots on $X = 0$.

Suppose that λ is not a root of f. Let $(0, h_0), (m, h_m)$ be the vertices of the highest Newton edge E as illustrated below.



Then $\xi = \xi(\lambda) : (h_0 - h_m)/m$ will be called *the Newton exponent associated to* λ. It satisfies $\xi \geqslant 1$ as follows from the assumptions on λ and f. Any arc $\tilde{\lambda}$ such that $\tilde{\lambda}(y) - \lambda(y) = o(y^{\xi})$, has the same associated Newton polygon with exactly the same Newton coefficients c_{ij} of (5.1) corresponding to the dots $(i, j/N)$ on the polygon. If $\tilde{\lambda}(y) - \lambda(y) = O(y^{\xi})$, then the coefficients corresponding to the dots on the last edge E may change. This may be calculated as follows. Collect the terms of $f(X + \lambda(Y), Y)$ corresponding to E:

$$F_{\mathrm{E}}(X, Y) := \sum c_{ij} X^i Y^{j/N}, \quad (i, j/N) \in \mathrm{E}.$$

F_{E} is weighted homogeneous of weights $(\xi, 1)$. Consider the associated polynomial of one complex variable $P_{\mathrm{E}}(z) := F_{\mathrm{E}}(z, 1)$. P_{E} is of degree m. Note also that $P_{\mathrm{E}}(0)$ gives the leading coefficient of $f(\lambda(y), y) = P_E(0)y^{h_0} + \cdots$. Suppose now that $\tilde{\lambda}(y) = \lambda(y) + ay^{\xi} + \cdots$. Then the corresponding polynomial for $\tilde{\lambda}$ equals $P_E(z+a)$.

The Newton diagram of $\partial f/\partial x$ is obtained from the Newton diagram of f by a shift by one unit to the left. All Newton dots of f on $X = 0$ disappear.

Let $x = \gamma(y)$ be a polar arc. Then γ is an analytic arc in the sense described above and we may consider the Newton polygon associated to γ. Note that an arc is a polar arc iff there is no Newton dot on the line $X = 1$ on the Newton diagram of f relative to this arc. In particular, the corresponding polynomial P_E satisfies $P_E'(0) = 0$. Hence we have the following.

PROPOSITION 5.1. *Let $\lambda(y)$ be an analytic arc and let ξ be the associated exponent. All polar arcs included in the cusp $\{|x - \lambda(y)| \leqslant M|y^{\xi})|\}$ are of the form $x = \gamma(y) = \lambda(y) + cy^{\xi} + o(y^{\xi})$, where c is a root of P_E'. The function f along such polar arc equals $f(\gamma(y), y) = P_E(c)y^{h_0} + o(y^{h_0})$.*

Conversely, for any root c of P_E' there is a polar arc γ of the form $\gamma(y) = \lambda(y) + cy^{\xi} + \cdots$. This can be proven using Newton's algorithm, see [2] for details.

Fix a polar arc γ. Denote $\xi = \xi(\gamma)$ and similarly $h_0 = h_0(\gamma)$. The exponent h_0 is finite provided γ is not a (multiple) root of f what we assume.

LEMMA 5.2. *$\xi(\gamma) > 1$ if and only if γ is tangential.*

Proof. Recall that γ is tangential if and only if it is tangent to $C_0(X)$ which is equivalent to $h_0(\gamma) > k$. This is equivalent to the slope ξ of the last edge being greater than 1. □

We consider the following family of (cuspidal) neighborhoods of γ: if $\xi > 1$

$$U(\gamma, \eta) = \{(x, y) | |x - \gamma(y)| < |y|^{\xi + \eta}\},$$

where $\eta > 0$, and for $\xi = 1$,

$$U_1(\gamma, a) = \{(x, y) | |x - \gamma(y)| < a|y|\},$$

where $a > 0$. For an $\varepsilon > 0$ denote $U_\varepsilon = \{|\partial f/\partial x| < \varepsilon|\partial f/\partial y|\}$.

THEOREM 5.3. *There are constants $\eta, C > 0$ and a positive integer s such that for sufficiently small $\varepsilon > 0$ $U_\varepsilon \subset \bigcup U(\gamma, \eta) \cup \bigcup U_1(\gamma, C\varepsilon^{1/s})$ where the first union is taken over all tangential polar arcs and the second one over the non-tangential ones.*

In particular the asymptotic behavior of f on U_ε is given by

$$f(x, y) = P_{E(\gamma)}(0)y^{h_0(\gamma)} + o(y^{h_0(\gamma)}),$$

on $U(\gamma, \eta)$ for γ tangential;

$$|f(x, y) - P_{E(\gamma)}(0)y^k| \leqslant C'\varepsilon^{1+1/s}|y|^k$$

on $U_1(\gamma, C\varepsilon^{1/s})$ for γ nontangential.

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Remark 5.4. In [4] T. Mostowski considered the families of generic absolute polar curves on complex analytic surface germs. In this case a phenomenon similar to the considered above happens. That is the homogeneous part of the surface, i.e. the tangent cone at the origin, gives rise to the families of polar curves with moving tangent at the origin when the projection (defining the polar curve) varies. On the other hand, the singular locus of the tangent cone gives rise to the families of polar curves with the tangent line fixed. The latter families appear in the construction of the Lipschitz stratification to the surface, see [4]. The technique of [4] can be certainly used to show Theorem 5.3 for 'generic' polar curves. In this paper we want to work with a more precise notion of 'genericity'. That is the only assumption on our system of coordinates we make is that the x-axis is not tangent to $f=0$ at the origin. We find the Newton polygon method particularly useful to handle this case.

Proof of Theorem 5.3. Let $x=\lambda(y)$ be an analytic arc and consider the Newton diagram relative to λ.

First we consider the following special but important case. We suppose that λ is not a root of f, that the highest Newton edge E of f is at least of length 2, i.e. $m\geqslant 2$, and that $\xi=\xi(\lambda)>1$. We suppose as well that $\operatorname{ord}_0\lambda(y)>1$. Recall that

$$\partial f/\partial x(\lambda(y),y)=P'_{\mathrm{E}}(0)y^{h_0-\xi}+\cdots \tag{5.2}$$

Differentiating the identity

$$f(\lambda(y),y)=P_{\mathrm{E}}(0)y^{h_0}+\mathrm{o}(y^{h_0}), \tag{5.3}$$

$P_{\mathrm{E}}(0)\neq 0$, we obtain

$$\begin{aligned} &P_{\mathrm{E}}(0)h_0y^{h_0-1}+\mathrm{o}(y^{h_0-1})\\ &\quad=\lambda'(y)\partial f/\partial x+\partial f/\partial y=\lambda'(y)(P'_{\mathrm{E}}(0)y^{h_0-\xi}+\mathrm{o}(y^{h_0-\xi}))+\partial f/\partial y. \end{aligned} \tag{5.4}$$

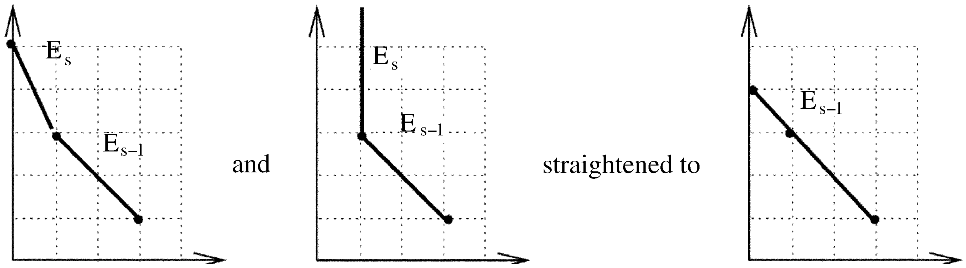
This allows us to compute $\partial f/\partial y$ along λ. If $x=\lambda(y)$ is included in U_ε then $\partial f/\partial y$ dominates $\lambda'(y)\partial f/\partial x$ and hence $\partial f/\partial y\sim y^{h_0-1}$. This means that $P'_{\mathrm{E}}(0)=0$ otherwise $\partial f/\partial x\sim P'_{\mathrm{E}}(0)y^{h_0-\xi}\gg\partial f/\partial y$. Hence, there is a polar arc γ such that the order of contact of λ and γ is bigger than ξ. Moreover, $\xi=\xi(\gamma)$.

We have a similar picture if we allow $\operatorname{ord}_0\lambda(y)=1$ that is $\lambda(y)=b_0y+\cdots, b_0\neq 0$. If $P'_{\mathrm{E}}(0)\neq 0$ then, by (5.4),

$$b_0\partial f/\partial x(\lambda(y),y)\simeq-\partial f/\partial y(\lambda(y),y)\sim y^{h_0-\xi}.$$

Thus the above cannot happen if λ is included in U_ε for $\varepsilon<|b_0|^{-1}$.

If $m=1$ or λ is a root of f then we may use the *straighten Newton polygon* of f, see [2]. It is defined as follows



Let ξ, resp. P_{E}, denote the associated exponent, resp. the highest edge polynomial, of the straighten polygon. Then there is a Newton dot $(1, h_1)$ on P_{E}, otherwise λ were multiple root of f which, as we have assumed, cannot happen. This means that $P'_{\mathrm{E}}(0) \neq 0$. Then the same argument as above shows that λ is not in U_ε for $\varepsilon < |b_0|^{-1}$, or not in any U_ε if $b_0 = 0$.

Finally we consider the case $\xi(\lambda) = 1$. Then $h_0 = k$ and this is the homogeneous part H_k of f which is in the play. Then $\lambda(y) = b_0 y + o(y)$, $b_0 \in \mathbb{C}$, that is the associated polynomial equals

$$P_{\mathrm{E}}(z) = H_k(z + b_0, 1). \tag{5.5}$$

By the assumption $\xi = 1$ we have that P_{E} cannot have a multiple root at 0. By (5.4) we may write

$$\partial f/\partial y(\lambda(y), y) = (kP_{\mathrm{E}}(0) - b_0 P'_{\mathrm{E}}(0))y^{k-1} + o(y^{k-1}).$$

Recall that $\partial f/\partial x(\lambda(y), y) = P'_{\mathrm{E}}(0)y^{k-1} + o(y^{k-1})$. Thus, if λ is contained U_ε then

$$|P'_{\mathrm{E}}(0)| \leqslant \varepsilon |kP_{\mathrm{E}}(0) - b_0 P'_{\mathrm{E}}(0)| \tag{5.6}$$

or equivalently in terms of $P(z) = H_k(z, 1)$

$$|P'(b_0)| \leqslant \varepsilon |kP(b_0) - b_0 P'(b_0)| \tag{5.7}$$

Since P is of degree k, the left hand side of (5.7) is of magnitude b_0^{k-1} for b_0 large and the right hand side is bounded by Cb_0^{k-1}. Hence (5.7) is impossible for $\varepsilon > 0$ small and $|b_0|$ large. Consequently, we may assume that $|b_0|$ is bounded. (5.7) implies also $(1 - |b_0|\varepsilon)|P'(b_0)| \leqslant \varepsilon |kP(b_0)|$ that gives, for $\varepsilon > 0$ sufficiently small,

$$|P'(b_0)| \leqslant 2\varepsilon k |P(b_0)|. \tag{5.8}$$

Note that P and P' cannot both vanish at b_0, otherwise 0 would be a multiple root of P_{E}. Thus we may suppose that b_0 is close to a root of P', say z_0, that is to say that b_0 is in a neighborhood of z_0 on which $P'(z) \sim (z - z_0)^s$, $s \geqslant 1$, and hence (5.8) gives $|b_0 - z_0| \leqslant 1/2C\varepsilon^{1/s}$ for a constant C sufficiently large. By Newton algorithm there is a polar arc γ such that $\gamma(y) = z_0 y + \cdots$ that is $\lambda \in U_1(\gamma, C\varepsilon^{1/s})$.

Note that we checked the statement of the first part of the theorem on any arc λ. Thus this part follows by the curve selection lemma and the Lojasiewicz Inequality (the existence of exponent η). Similarly follows the second part since it holds on any analytic arc. $\square$

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Fix $\varepsilon > 0$ small satisfying the statement of Theorem 5.3. For each line $l \subset \mathrm{Sing}(C_0(X))$ define $U_\varepsilon(l) = \bigcup_{\gamma\in\Gamma(l)} U_\varepsilon \cap U(\gamma,\eta)$. Then U_ε is a disjoint union of such $U_\varepsilon(l)$ and a part whose tangent cone is away of $C_0(X)$.

COROLLARY 5.5. *Let γ be a tangential polar arc. There is $\delta > 0$ such that for each $p \in U(\gamma,\eta)$ there is $p' \in \gamma$ such that $f(p) = f(p')$ and*

$$|p - p'| \leqslant |p|^{1+\delta}. \tag{5.9}$$

Proof. Let $p = (x_0, y_0)$, $|x_0 - \gamma(y_0)| < |y_0|^{\xi+\eta}$. Then by Theorem 5.3 there is an exponent $\alpha > 0$ such that

$$|f(x_0,y_0) - f(\gamma(y_0),y_0)| = \mathrm{o}(y_0^{h_0(\gamma)+\alpha}),$$

$$f(\gamma(y_0),y_0) = c_0(\gamma)y_0^{h_0(\gamma)} + \mathrm{o}(y_0^{h_0(\gamma)+\alpha}).$$

Therefore for $|y - y_0| \leqslant |y_0|^{\alpha+1}$

$$f(x_0,y_0) - f(\gamma(y),y) = C(y-y_0)y_0^{h_0(\gamma)-1} + \mathrm{o}(y_0^{h_0(\gamma)+\alpha}).$$

By Rouché's theorem, there is y', $|y' - y_0| \leqslant |y_0|^{\alpha+1}$ such that $f(\gamma(y'),y') = f(x_0,y_0)$. Set $p' = (x',y') = (\gamma(y'),y')$. Then $|x_0 - x'| \leqslant C'(|y_0|^{\xi+\eta} + |y_0|^{\alpha+1})$ and we may take $0 < \delta < \min\{\xi+\eta-1, \alpha\}$. □

LEMMA 5.6. *Given $\varepsilon, \delta > 0$ and the constants K, M, A of the definition Y_δ. We suppose that A and K are sufficiently large (that is $A \geqslant A(\varepsilon)$, $K \geqslant K(\varepsilon)$ where $A(\varepsilon)$, $K(\varepsilon)$ depend only on ε). Then there exists a constant C, depending on K, M, ε, such that for every $p_0 \in Y_\delta$ there is $p \in U_\varepsilon$ such that $f(p) = f(p_0)$ and $|p - p_0| < C|p_0|^{\delta+1}$.*

Proof. Suppose this is not the case; that is any point p satisfying $f(p) = f(p_0)$ and $|p - p_0| < C|p_0|^{\delta+1}$ is not in U_ε, the constant C to be specified later.

Denote $X_c = f^{-1}(c)$, where $c = f(p_0)$, and consider the projection $\pi_c\colon X_c \to \mathbb{C}$ onto the y-axis. At p_0 and at any point of X_c which is not in U_ε, π_c is submersive and X_c is the graph of a function $x = x(y)$ with bounded derivative (by a constant D depending only on ε). Denote, as before, by $X(p_0,\rho)$ the intersection of X_c with the ball $B(p_0,\rho)$ and consider $\mathcal{V}_N = \{y \mid |y - y_0| < N|y_0|^{\delta+1}\}$. There is a constant $N = N(C,\varepsilon)$ such that $\pi_c^{-1}(\mathcal{V}_N) \cap X(p_0, C|p_0|^{\delta+1})$ is the union of graphs of analytic functions defined on $\mathcal{V}_N$ and with derivative bounded by D and such that and π_c restricted to $\pi_c^{-1}(\mathcal{V}_N) \cap X(p_0, C|p_0|^{\delta+1})$ is a finite covering over $\mathcal{V}_N$. We denote this restriction by π'_c.

If $C' = C'(\varepsilon, M, \delta)$ is sufficiently large then for $N' = N(C',\varepsilon)$, $X(p_0, M|p_0|^{1+\delta}|) \subset \pi_c'^{-1}(\mathcal{V}_{N'})$ and if K was sufficiently large then $\pi_c'^{-1}(\mathcal{V}_{N'}) \subset X(p_0, KM|p_0|^{1+\delta}|)$. Finally we choose C such that for $N = N(C,\varepsilon)$

$$X(p_0, M|p_0|^{1+\delta}|) \subset \pi_c'^{-1}(\mathcal{V}_{N'}) \subset X(p_0, KM|p_0|^{1+\delta}|) \subset \pi_c'^{-1}(\mathcal{V}_N).$$

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This allows us to show that if $p \in X(p_0, M|p_0|^{1+\delta}|)$ is in the connected component of $X(p_0, KM|p_0|^{1+\delta}|)$ containing p_0 then it is in the same connected component of $\pi_c'^{-1}(\mathcal{V}_{N'})$ as p_0. Thus to conclude we note that on this connected component the distance along X_c is comparable to the euclidean one (again by a constant depending on ε). This is not possible if A were chosen sufficiently big. This ends the proof of lemma. □

LEMMA 5.7. *Let $x = \gamma(y)$ be a tangential polar arc. Then for any $\delta > 0$ sufficiently small and any set of constants $K \geqslant 1, C > 0, A > 0$, γ is contained in Y_δ.*

Proof. Let $\xi = \xi(\gamma)$ and let P_{E} be the polynomial of one complex variable z associated to the last edge E of the Newton polygon relative to γ. Fix any $0 < \delta < \xi - 1$, a constant $N > 0$, and another constant $\varepsilon' > 0$. We require ε' to be small so that $P_{\mathrm{E}}(z) - P_{\mathrm{E}}(0)$ has no other roots in $|z| \leqslant 3\varepsilon'$ but $z = 0$.

Let $p_0 = (x_0, y_0), x_0 = \gamma(y_0)$. We suppose p_0 sufficiently close to the origin, how close we shall determine later. Let $X_c = f^{-1}(c)$ where $c = f(p_0)$. Let X' denote the connected component of $\tilde{X} = \{(x, y) \in X_c | |y - y_0| \leqslant N|y_0|^{1+\delta}\}$ that contains p_0. We show that $X' \subset \{|x - \gamma(y)| \leqslant \varepsilon'|y|^{\xi}\}$.

For this we consider a continuous function $s \colon X' \to \mathbb{C}$ given by $s(x, y) = (x - \gamma(y))/y^{\xi}$. Let $p = (x, y) \in X'$ be such that $\{|x - \gamma(y)| \leqslant 2\varepsilon'|y|^{\xi}\}$ and write $x = \gamma(y) + sy^{\xi}$, $|s| \leqslant 2\varepsilon'$. Then

$$f(x, y) = P_{\mathrm{E}}(s)y^{h_0} + \mathrm{o}(y^{h_0}), \tag{5.10}$$

$h_0 = h_0(\gamma)$. On the other hand,

$$f(x, y) = f(x_0, y_0) = P_{\mathrm{E}}(0)y_0^{h_0} + \mathrm{o}(y_0^{h_0}) = P_{\mathrm{E}}(0)y^{h_0} + \mathrm{o}(y^{h_0}).$$

Hence

$$P_{\mathrm{E}}(s) - P_{\mathrm{E}}(0) = \mathrm{o}(1) \tag{5.11}$$

that is to say, it can be arbitrarily small if we have chosen y_0 sufficiently close to 0. Thus, s is close to a root of $P_{\mathrm{E}} - P_{\mathrm{E}}(0)$. But, by assumption on ε', this root has to be 0. Thus, we have shown that if $|s| \leqslant 2\varepsilon'$ then it is as close to 0 as we wish (if p_0 is close to the origin) that is, for instance, $|s| \leqslant \varepsilon'$. Thus a continuous function s defined on connected X' does not take values in $\varepsilon' < |s| \leqslant 2\varepsilon'$. This shows $X' \subset \{|x - \gamma(y)| \leqslant \varepsilon'|y|^{\xi}\}$.

Consider

$$\pi_c' \colon X' \to \mathcal{V}_N = \{y | |y - y_0| < N|y_0|^{1+\delta}\}.$$

π_c' is a finite covering branched at the points of polar arcs of the form $x = \gamma(y) + \mathrm{o}(y^{\xi})$. Since p_0 is a branching point this covering is at least of degree 2. Fix y_1 such that $|y_1 - y_0| = (N/2)|y_0|^{1+\delta}$ and $p_1 = (x_1, y_1), p_2 = (x_2, y_1)$ two distinct points in $(\pi_c')^{-1}(y_1)$. Then

https://doi.org/10.1023/A:1022726806349 Published online by Cambridge University Press

$$\operatorname{dist}(p_1, p_2) = |x_1 - x_2| \leqslant 2|y_1|^{\xi} \leqslant 3|y_0|^{\xi}.$$

On the other hand the projection by π_c of any curve joining p_1 and p_2 in X_c is at least of length $N|y_0|^{\delta+1}$ that is much longer than $|y_0|^{\xi}$. This shows the lemma. □

LEMMA 5.8. *For any $\delta > 0$ and A, K sufficiently large the tangent cone to Y_δ equals the singular part of $C_0(X)$.*

Proof. Fix $\delta > 0$ and write $Y_\delta(A, K)$ to emphasize the dependence of Y_δ on A and K. By Lemma 5.6 the tangent cone to Y_δ is contained in the tangent cone to U_ε. In particular, by Theorem 5.3, the x-axis, and its conical neighborhood, is not tangent to Y_δ. Note that here we just use the fact that the x-axis is not in $C_0(X)$, that is f is mini-regular in x. Of course the same argument works for any line l which is not in $C_0(X)$, that is for any l there is $A(l)$ and a conical neighborhood of l which is not tangent to $Y_\delta(A, K)$ for A, K sufficiently large. This does not suffice to conclude since the size of A and K may depend on l and the complement of the projectivized cone $\mathbb{P}C_0(X)$ in $\mathbb{P}^1$ is not compact.

But, again by Theorem 5.3, it suffices to show that $U_1(\gamma, C\varepsilon^{1/s})$, for each nontangential polar arc γ, is not tangent to Y_δ. And for ε small enough the projectivised tangent cone to $U_1(\gamma, C\varepsilon^{1/s})$, that is closed, is away of $\mathbb{P}C_0(X)$. Thus by the above argument Y_δ is tangent to the union of $U(\gamma, \eta)$ over tangential polar arcs γ. And the tangent cone to this union is the singular part of $C_0(X)$. □

Proof of Proposition 4.5. Fix ε given by Theorem 5.3. Choose $\delta > 0$ satisfying Corollary 5.5 and Lemma 5.7, any constant $C > 0$, and A, K sufficiently large so that the statements of Lemmas 5.6, 5.7, and 5.8 hold.

The first part (i) of Proposition 4.5 follows directly from Lemma 5.7. For the second part we use U_ε as an intermediary set. Firstly, a point p satisfying the statement of the proposition exists in U_ε by Lemma 5.6. Divide U_ε as in Theorem 5.3. Then, the point p in U_ε close to Y_δ cannot be in $U_1(\gamma, C\varepsilon^{1/s})$ for γ non-tangential by Lemma 5.8. Hence, it is in $p \in U(\gamma, \eta)$, for γ tangential. Finally, we may find such a p in $\tilde{\Gamma}$ by Corollary 5.5. This ends the proof. □

Acknowledgements

It is a pleasure to thank Krzysztof Kurdyka and Tadeusz Mostowski for very valuable and interesting discussions.

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