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NON-SIMPLICITY OF LOCALLY FINITE BARELY TRANSITIVE GROUPS

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We answer the following questions negatively: Does there exist a simple locally finite barely transitive group (LFBT-group)? More precisely we have: There exists no simple LFBT -group. We also deal with the question, whether there exists a LFBT-group G acting on an infinite set Ω so that G is a group of finitary permutations on Ω. Along this direction we prove: If there exists a finitary LFBT-group G, then G is a minimal non-FC p-group. Moreover we prove that: If a stabilizer of a point in a LFBT-group G is abelian, then G is metabelian. Furthermore G is a p-group for some prime p, $G/G' \cong C_{p\infty}$, and G' is an abelian group of finite exponent.

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Let Ω be an infinite set. Then a transitive subgroup G of $\mathrm{Sym}(\Omega)$ is said to be barely transitive if every orbit of every proper subgroup of G is finite. More generally, we say that a group G is barely transitive if it can be represented as a barely transitive subgroup of $\mathrm{Sym}(\Omega)$ for some infinite set Ω. This is easily seen to be equivalent to the condition that G has a subgroup H of infinite index such that $\bigcap_{g\in G} H^g = \{1\}$ and such that $|K : K \cap H|$ is finite for every proper subgroup K of G. Throughout this article, if G is a barely transitive group, then H will denote a fixed subgroup of G with the above properties.

In this article, we shall study locally finite barely transitive groups, which we shall call LFBT-groups. Metabelian LFBT-groups were constructed by B. Hartley in [**4**] and [**5**]. It is unknown whether perfect LFBT-groups exists. We shall prove that there are no simple LFBT-groups; and, as a consequence, improve on some of the results in [**8**].

Theorem 1. *There exists no simple LFBT-group.*

It is also natural to ask whether there exists a LFBT-group G acting on an infinite set Ω so that G is a group of finitary permutations on Ω.

Theorem 2. *If there exists a finitary LFBT-group G, then G is a minimal non-FC, p-group.*

* The Society is saddened by the death of Professor Brian Hartley.

Theorem 3. *If G is a finitary permutation group on Δ and $G = \langle g_i | g_i^p = 1, i = 1, 2, 3 \ldots \rangle$, then G is not a LFBT-group on Δ.*

In [7], it was asked how restrictions on H affect the structure of a LFBT-group. We shall prove the following result.

Proposition 1. *Let G be a LFBT-group. If H is abelian, then G is metabelian. Furthermore*

(i) *G is a p-group, p prime.*

(ii) *G/G' is isomorphic to $C_{p\infty}$.*

(iii) *G' is an abelian group of finite exponent.*

It should be pointed out that in each of the LFBT-groups constructed in [4] and [5], the subgroup H is abelian.

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Proofs of the results

We will begin by collecting together some of the basic properties of LFBT-groups. Complete proofs of these results can be found in [8].

$$G \text{ has no proper subgroup of finite index.} \tag{1}$$

Suppose

$$H = H_0 < H_1 < H_2 < \ldots < H_n \ldots \tag{2}$$

is a chain of subgroups of G above H. Since $|H_n : H|$ is finite, there is a finite subgroup L_n of H_n with $H_n = HL_n$. Let $F_n = \langle L_1, \ldots, L_n \rangle$.

We have

$$F_1 < F_2 < \ldots < F_n < \ldots \tag{3}$$

and

$$H_n = HF_n \tag{4}$$

$$\text{Evidently } G = \bigcup_n H_n \tag{5}$$

follows from the fact that $K \leq G$ and $|K : K \cap H| = \infty$ implies $K = G$. Again by the same reason we have

$$G = \bigcup_n F_n \tag{6}$$

$$\text{Further if } X < G, \text{then } X \leq H_n \text{ for some } n \tag{7}$$

Now suppose that there is no simple LFBT-group. Since it is clear that any simple homomorphic image of G would have to be a LFBT-group, it follows that G has no maximal normal subgroup. Hence G is a union of proper normal subgroups. In particular

$$F_n^G < G \tag{8}$$

Proposition 2. *Let G be a LFBT-group. Then either G is a p-group for some prime p or there are infinitely many primes dividing the order of the elements of G.*

Proof. By (3) and (6) we have $F_1 \leq F_2 \leq \dots$ a sequence of finite subgroups of G such that $G = \cup_{i=1}^{\infty} F_i$. Assume that G is not a p-group and there are only finitely many primes, say $p_1, \dots, p_k$, dividing the order of the elements of G. Let S_{i1} be a Sylow p_i-subgroup of F_1 and let S_{i2} be a Sylow p_i-subgroup of F_2 containing S_{i1}, etc. Then $S_i = \cup_{j=1}^{\infty} S_{ij}$ is a maximal p_i-subgroup of G. We shall show that $G = \langle S_1, \dots, S_k \rangle$. The group $F_j \cap \langle S_1, \dots, S_k \rangle$ contains the groups $S_{1j}, \dots, S_{kj}$, hence equals to F_j, for all j. This implies that G is generated by a finite number of proper maximal p_i-subgroups which is impossible by [**8**, Lemma 2.10]. Thus infinitely many primes must divide the order of the elements of G.

Proof of Theorem 1. Assume that there exists a simple LFBT-group G. By (6) G is countable and by [**8**, Lemma 2.10], G cannot be generated by two proper subgroups. Then by [**1**, Corollary 1.9] such a group can be embedded in a finitary linear group $FGL(V)$ on a vector space V over a field of characteristic p.

By [**2**, Theorem B], for an infinite simple periodic group G of finitary transformations on a space over a field of characteristic p the following are valid:

(1) If $p = 0$, then for each finite subgroup K of G, there exists a finite quasisimple subgroup H that contains K and is such that $K \cap Z(H) = \{1\}$.

(2) If $p > 0$, then for each finite subgroup K of G, there exists a finite subgroup H that contains K and is such that $H = H', H/O_p(H)$ is a quasisimple group and $K \cap S(H) = \{1\}$ where $S(H)$ is the maximal soluble normal subgroup of H.

In the first case, G has a sequence of finite subgroups $G_1 < G_2 < \dots$ where $G = \cup_{i=1}^{\infty} G_i$ and $G_i \cap Z(G_{i+1}) = \{1\}$ (i.e. A Kegel sequence $(G_i, Z(G_i))$ $i = 1, 2, 3, \dots$). By

[**8**, Lemma 4.2], G cannot be a barely transitive group. (For details about Kegel sequences and reductions on Kegel sequences, see [**6**].)

For the second case, let $G = \cup_{i=1}^{\infty} G_i$, where $G_i/O_p(G_i)$ are finite quasisimple groups. We shall show that there exists an element x in G such that $C_G(x)$ involves an infinite non-linear locally finite simple group; then we shall get a contradiction. Let $\bar{G}_i = G_i/O_p(G_i)$.

By using the classification of finite simple groups and reduction on Kegel sequences we may assume that

(i) each $\bar{G}_i/Z(\bar{G}_i)$ is an alternating group *or*

(ii) each $\bar{G}_i/Z(\bar{G}_i)$ is a classical group of fixed Lie type over a field of characteristic p_i.

For (i), the centralizer $C_G(x)$ of any element x of order prime to p involves an infinite non-linear locally finite simple group. See [**6**, Lemma 2.5].

For (ii), let $\{p_i : i \in N\}$ be the set of primes that appear as characteristic of the fields. If one of the primes, say p_j, in this set appears infinitely often, then we choose an element of prime order relatively prime to p and p_j. Existence of this element is guaranteed by Proposition 2.

If none of the primes appears as a characteristic of the fields infinitely many times, then we may assume that each prime appears as a characteristic only once. Here we may need to pass, if necessary, to a subsequence and delete some of the terms in the Kegel sequence. Again passing to a subsequence, if necessary, we may assume that there exists a prime, say n, which does not appear as a characteristic in the list and is different from p. Let x be an element of order n so that x becomes a semisimple element in all the classical simple groups $\bar{G}_i/Z(\bar{G}_i)$. Then by [**6**, Theorem C (iv)] we get $C_{\bar{G}_i}(x) \in T_{n+[\frac{4}{n}]}$. Here T_n denotes the class of locally finite groups having a series of finite length in which there are at most n non-abelian simple factors and the rest are locally soluble. (For details see [**6**, Section 2].) But by coprime action $C_{\bar{G}_i}(x)$ equals $C_{G_i}(x)O_p(G_i)/O_p(G_i)$. This implies by [**6**, Lemma 2.1] that $C_{G_i}(x)$ is in $T_{n+[\frac{4}{n}]}$. Then by [**6**, Lemma 2.3] it follows that $C_G(x)$ is in $T_{n+[\frac{4}{n}]}$ and involves an infinite non-linear finite locally simple group.

In any case the centralizer of one of the elements x involves an infinite non-linear locally finite simple group and this is impossible by [**8**, Lemma 4.1]. Therefore there exists no simple LFBT-group.

Proposition 3. *Let G be a LFBT-group. If H is almost locally p-soluble, then G is almost locally p-soluble. In particular G is a p-group and every proper normal subgroup is nilpotent of finite exponent.*

Proof. Let p be a prime. If K is any locally finite group, let K_p be the product of all normal locally p-soluble subgroups of K. Then K_p is locally p-soluble and K/K_p has no non-trivial locally p-soluble normal subgroup.

Suppose H is almost locally p-soluble. If K is a proper normal subgroup of G, then

$|K : H_p \cap K|$ must be finite, so that K/K_p is finite. By (1) $[K, G] \leq K_p$, and so K must equal K_p. By (3) and (8), G must equal G_p. i.e. G is locally p-soluble. Now the rest of the theorem follows from [**8**, Theorem 1.1].

Therefore the restriction of local p-solubility on G of [**8**, Theorem 1.1] is reduced to the restriction of almost local p-solubility of H.

Corollary 1. *Let G be a LFBT-group. If H is nilpotent, then G is a p-group and each proper subgroup of G is nilpotent.*

Proof. By Proposition 3 and (8) G is a p-group and a union of nilpotent proper normal subgroups. Let X be any proper subgroup of G. Then $|X : X \cap H| < \infty$. Let Y be a normal subgroup of X of finite index and contained in $X \cap H$. Then $X = F^X Y$ for some finite subgroup F of X. Hence X is nilpotent.

Proof of Proposition 1. Assume that H is abelian. By Theorem 1, G is not simple. By (8) G is a union of proper normal subgroups. Let N be a proper normal subgroup of G. Let A be a normal subgroup of N of finite index and contained in H. Let B be the FC-radical of N. Then $B/Z(B)$ is finite, so $N/Z(B)$ is as well. $(G/Z(B))/C_{(G/Z(B))}(N/Z(B)) \leq Aut(N/Z(B))$ which is finite. By (1) again we have $[N, G]$ abelian. So G' is a proper subgroup. Now (i) and (ii) follows from the theorem in [**4**].

It remains to show that G' is abelian. Let $M = FC(G')$. We have $|G' : M|$ is finite. Then the commutator group G'/M is finite. This implies that G/M is an FC-group. It follows from (1) that G/M is abelian. Thus $M = G'$. But then, G' is an abelian by finite FC-group. Therefore G' is central by finite. However G' does not have a subgroup N of finite index. Then we get G' is abelian. Now (iii) follows from the theorem in [**4**].

Lemma 1. *If there exists a finitary LFBT-group on a set* Ω, *then* $G = G'$.

Proof. Assume if possible that G is a finitary LFBT-group on the set Ω, and $G \neq G'$. Let Δ be an orbit of G' containing $\alpha \in \Omega$. Then Δ is a finite G-block. Let $\Xi = \{\Delta g : g \in G\}$ be the set of distinct orbits of G' on Ω. Then G acts on Ξ transitively and there exists a homomorphism ρ, from G to finitary symmetric group on Ξ. By (1) $K = \operatorname{Ker} \rho \neq G$. Then G/K is an infinite abelian group acting on Ξ faithfully and transitively. Now let $gK \in G/K$ and $\Delta g_1.gK \neq \Delta g_1$. Then $\Delta g_1 g_2.gK \neq \Delta g_1 g_2$ for all g_2. Since G/K acts transitively on Ξ, gK moves every element of Ξ. Hence $|\, Supp\, g \,| > |\, Supp\, \rho(g) \,|$ which is infinite. But this is impossible as G is a finitary permutation group on Ω. Hence $G = G'$.

Proof of Theorem 2. By definition the orbits of each proper subgroup of G are finite. As G acts transitively on Ω by (1), Ω is a countable set. Let K be a proper subgroup of G and let $\{\Omega_i : i = 1, 2, 3 \ldots\}$ be the set of distinct orbits of K. Then each Ω_i is a finite K set and K acts on Ω_i transitively. Hence K can be embedded into restricted direct product of finite groups. It follows that K is an FC-group. This implies that G is a minimal non-FC-group. But by [9] a perfect locally finite minimal non-FC-group is a p-group.

Lemma 2. *If there exists a finitary LFBT-group on a set Δ, then G does not have a maximal G-block. Moreover $\Delta = \cup_{i=1}^{\infty}\Delta_i$, where Δ_i are finite G-blocks.*

Proof. By [8, Lemma 2.8] G is not a primitive permutation group. Hence we have non-trivial G-blocks

$$\Delta_1 < \Delta_2 < \dots \quad \text{and let} \quad \delta \in \Delta_1.$$

Assume if possible that Δ_n is a maximal G-block. Then we have an equivalence relation corresponding to Δ_n. Let ρ be the set of equivalence classes corresponding to the equivalence relation of Δ_n. Then G acts on ρ transitively and Δ_n is a maximal G-block of the permutational pair (G, Δ) so (G, ρ) is a primitive permutation group and the stabilizer of a point in ρ is a maximal subgroup of G but this is impossible by [8, Lemma 2.10]. Hence the existence of maximal G-block Δ_n is impossible. Therefore we have an infinite tower of G-blocks $\Delta_1 < \Delta_2 < \Delta_3 < \dots$ and $\cup_{i=1}^{\infty}\Delta_i = \Delta$.

The following lemma might have an independent interest in finitary permutation groups.

We use [3] as a reference for the properties of the wreath product.

Lemma 3. *Let $G = \langle g_i : g_i^p = 1, \quad i = 1, 2, 3, \dots \rangle$ be a transitive finitary permutation group on a set Δ and $\Delta = \cup_{i=1}^{\infty}\Delta_i$ where $\Delta_1 < \Delta_2 \dots$ and Δ_i are finite G-blocks. Then G has a subgroup isomorphic to $Wr^N C_p$.*

Proof. Let g be an element of G of order p. Then there exists a G-block Δ_{i_1} such that Supp $g \subseteq \Delta_{i_1}$. Since G is transitive not all g_i, $i = 1, 2, 3 \dots$, can stabilize Δ_{i_1}. So there exists g_{i_1} such that $g_{i_1}^p = 1$ and $\Delta_{i_1} g_{i_1} \neq \Delta_{i_1}$. Now consider $G_{i_1} = \langle g, g_{i_1} \rangle$. The elements g and $g^{g_{i_1}^n}$, $1 \leq n \leq p-1$ commute. Since $\langle g^{g_{i_1}^n} \rangle$ and $\langle g^{g_{i_1}^m} \rangle$, $1 \leq n, m \leq p-1$, $n \neq m$ moves distinct points of Δ, the intersection $\langle g^{g_{i_1}^n} \rangle \cap \langle g^{g_{i_1}^m} \rangle = 1$ for all $n \neq m$ and

$$\langle g, g^{g_{i_1}}, g^{g_{i_1}^2}, \dots, g^{g_{i_1}^{p-1}} \rangle = \langle g \rangle \times \langle g^{g_{i_1}} \rangle \times \langle g^{g_{i_1}^2} \rangle \times \dots \times \langle g^{g_{i_1}^{p-1}} \rangle$$

and

$$\langle g, g_{i_1} \rangle = \langle g \rangle \times \langle g \rangle^{g_{i_1}} \times \langle g \rangle^{g_{i_1}^2} \times \dots \times \langle g \rangle^{g_{i_1}^{p-1}} \rtimes \langle g_{i_1} \rangle.$$

Hence $\langle g, g_{i_1} \rangle \cong \langle g \rangle \wr < \langle g_{i_1} \rangle \cong C_p \wr C_p$. As *Supp* $xy \subseteq$ *Supp* $x \cup$ *Supp* y, again by (9) there exists Δ_{i_2} such that Supp $G_{i_1} \subseteq \Delta_{i_2}$ and $|\Delta_{i_2}| < \infty$ so there exists $g_{i_2} \in G$ such that $g_{i_2}^p = 1$ and $\Delta_{i_2} \cap \Delta_{i_2} g_{i_2} = \emptyset$. Then the elements of G_{i_1} and $\langle g_{i_2} \rangle$ do not commute but, for any $x \in G_{i_1}$, the element x and $x^{g^n_{i_2}}$, $\quad 1 \le n \le p-1$ commute. Then

$$\langle G_{i_1}, g_{i_2}\rangle = G_{i_1} \times G_{i_2}^{g_{i_2}} \times \ldots \times G_{i_1}^{(g_{i_2})^{p-1}} \rtimes \langle g_{i_2}\rangle$$

$$G_{i_2} = \langle g, g_{i_1} g_{i_2}\rangle \cong G_{i_1} \wr \langle g_{i_2}\rangle.$$

We can continue this process since G_{i_j} is a finite group and we have a tower of finite G-blocks. Then we have

$$G_{i_j} \cong G_{i_{j-1}} \wr \langle g_{i_j}\rangle \text{ and } G_{i_1} < G_{i_2} < \ldots \quad .$$

In order to simplify the notation let us suppress the i in the subscripts i.e. we have $G_i \cong W_i$ where $W_i = C_p \wr C_p \wr \ldots \wr C_p$ (i times). Suppose we have an isomorphism $\psi_j : G_j \to W_j$. We need to extend this isomorphism ψ_j to an isomorphism ψ_{j+1} between G_{j+1} and W_{j+1}. Let

$$\psi_{j+1} : \prod_{t=0}^{p-1} x_t^{(g^t_{j+1})} g^s_{j+1} \to \prod_{t=0}^{p-1} \psi_j(x_t)^{w^t_{j+1}} w^s_{j+1}, \quad x_t \in G_j, \quad (0 \le s \le p-1)$$

Clearly ψ_{j+1} is a well defined map from G_{j+1} to W_{j+1}. It follows that ψ_{j+1} is an isomorphism.

Then $\{G_j, \psi_j : j = 1, 2, 3, \ldots\}$ is a direct system and $\psi_{j+1}|_{G_j} = \psi_j$. Let $\Psi : (K =: \cup G_i) \to W$. If $g \in K$ there exists i such that $g \in G_i$, then $\psi(g) = \psi_i(g)$. Ψ is an isomorphism and $K \le G$.

Let $\Omega_i = \{\Delta_i, \Delta_i g_i, \ldots, \Delta_i g_i^{p-1}\}$. Then $\langle g_i\rangle$ acts on Ω_i transitively. Let

$$\Omega = Dr_{i \in N} \Omega_i$$

Now as in [3]; choose $(\Delta_i)_{i \in N}$ as the reference point. Then every g_i^n, $\quad i = 1, 2, 3, \ldots$, $1 \le n \le p$ gives a permutation of Ω so

$$\langle g_i \theta_i : g_i^p = 1, i \in N\rangle$$

acts on Ω as in the definition and hence

$$\langle g_i \theta_i : g_i^p = 1, i \in N\rangle = Wr^N C_p$$

where $\theta_i : \langle g_i\rangle \to Sym(\Omega)$. Hence K is the required subgroup of G.

Proof of Theorem 3. Assume to the contrary that G is a LFBT-group. By Lemma 2 we have

$$\Delta_1 < \Delta_2 < \Delta_3 < \dots \quad \text{and} \quad \bigcup_{i=1}^{\infty} \Delta_i = \Delta. \tag{9}$$

By Lemma 3 G has a subgroup K isomorphic to $Wr^N C_p$. If K is a proper subgroup of G, then bare transitivity of G implies that K is a residually finite group and hence K' is residually finite. But by [3, p. 173] K' is a perfect p-group hence this is impossible. If $K = G$, then G has proper subgroups isomorphic to K but this is impossible by the above paragraph. Hence the assumption that G is a LFBT-group lead us a contradiction.

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