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TWO MORE HEREDITARILY SEPARABLE NON-LINDELÖF SPACES

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0. Introduction. Our method using CH is a blend of two earlier constructions (Hajnal-Juhász [**2**] and Ostaszewski [**4**]) of hereditarily separable (HS), regular, non-Lindelöf, first countable spaces. [**4**] produces a much better space than ours in § 1; it has all of our properties except that it is not realcompact (which is probably more interesting), and it is countably compact as well; however, the construction works only under $\diamond$, which implies the continuum hypothesis (CH) but is not equivalent to it. The argument of [**2**], like ours, just needs CH, but it is much more complicated, and it is not immediate that the space produced is locally compact or perfectly normal (although, in fact, it is; see the remark at the end of § 1).

In § 2, we use a more complicated version of the technique in § 1 to construct a first countable, cardinality ω_1, HS, Dowker space. A Dowker space is a normal, Hausdorff space which is not countably paracompact. There is a known "real" Dowker space but all of its cardinal functions are large [**7**]. There is a known HS Dowker space but its construction depends on the existence of a Souslin line [**6**]. It was an old conjecture that the existence of a small cardinality (or small cardinal function) Dowker space depended on the existence of a Souslin line, and this conjecture is disproved by our construction. Using our technique and $\diamond$ (which implies both CH and the existence of a Souslin line) we can construct a first countable, cardinality ω_1, HS, Dowker space which is also locally compact and σ-countably compact; but we choose the weaker hypothesis over the stronger conclusion.

In § 2 we use Lusin sets in our construction. A subset L of the line is Lusin if L is uncountable and every nowhere dense subset of L is countable. If we assume CH, then there are Lusin sets in the line. However if we assume Martin's axiom and the negation of CH, then there are *no* Lusin sets in the line. If we assume Martin's axiom and the negation of CH, then there is *no* non-Lindelöf, first countable, regular topology on a subset of the line which refines the usual topology and has the property that the closure of a set in the two topologies differs by an at most countable set. Since our construction in § 1 yields just such a topology, both constructions are independent of the usual axioms for set theory.

1. The basic idea for obtaining this space is to start with the usual topology of the real numbers ($\mathbf{R}$), which has many of the properties we want; in particular

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it is HS. It is also hereditarily Lindelöf (HL), which we don't want, so we use a somewhat finer topology (i.e., add more open sets) which keeps the good properties and gets rid of the bad ones.

We shall in fact describe, under CH, a general machine for refining topologies. If one inputs a first countable T_2 space of cardinality ω_1, it outputs a finer topology on the same set which is first countable, T_3, locally compact, locally countable, and not Lindelöf. Additional properties of the input topology imply more about the output topology; in particular, starting with $\mathbf{R}$ yields the space described in the abstract.

Our machine is a modification of the following very simple procedure (see [**1**]) for constructing HS non-Lindelöf spaces. Let X be any topological space of cardinality ω_1. Say $\langle x_\xi : \xi < \omega_1 \rangle$ is a $1-1$ enumeration of X. For $\alpha \leqq \omega_1$, let $X_\alpha = \{x_\xi : \xi < \alpha\}$ (so $X = X_{\omega_1}$). If each X_α were open (so X would be right separated), then the X_α would form an open cover with no countable subcover. It is also easy to check that the topology on X generated by the original open sets plus the X_α is HS if the original topology is. Unfortunately, this new topology is not usually T_3. Our procedure does yield a T_3 topology, but it works only under CH and when X is T_2 and first countable, all of which we now assume.

To avoid confusion between the various topologies, we use ρ for the original topology on X and ρ_α for the subspace topology on X_α inherited from ρ, whereas τ_α and τ denote the topologies we are about to construct on X_α and X. We adopt the convention that a topology *is* the set of open sets in a space, so, for example, when $\xi < \eta$, "$\tau_\xi = \tau_\eta \cap \mathscr{P}(X_\xi)$" means that X_ξ is open in τ_η and τ_ξ is the induced subspace topology.

By CH, fix an enumeration $\langle S_\mu : \mu < \omega_1 \rangle$ of all countable subsets of X so that each $S_\mu \subseteq X_\mu$.

We construct τ_η by induction on η so as to make the following hold for all $\xi < \eta \leqq \omega_1$:

1) $\tau_\xi = \tau_\eta \cap \mathscr{P}(X_\xi)$.
2) Each τ_η is first countable, locally compact, and T_2.
3) $\tau_\eta \supseteq \rho_\eta$.
4) For each $\mu < \xi$, if $x_\xi \in \mathrm{cl}_\rho(S_\mu)$ then $x_\xi \in \mathrm{cl}_{\tau_\eta}(S_\mu)$.

For $\beta \leqq \omega$, let τ_β be discrete. For $\omega < \beta \leqq \omega_1$, we assume that we have done the construction below β so that (1)–(4) hold for all $\xi < \eta < \beta$, and we show how to define τ_β so that (1)–(4) hold for all $\xi < \eta \leqq \beta$.

If β is a limit, condition (1) forces us to take $\tau_\beta = \{U \subseteq X_\beta : \forall \eta < \beta (U \cap X_\eta \in \tau_\eta)\}$, and it is easy to check that this definition preserves 1–4.

Before proceeding with the successor stage, observe that our conditions imply that each τ_α is also T_3 and zero-dimensional, and for $\alpha < \omega_1$, also metrizable.

Now, if $\beta = \alpha + 1$, then we have τ_α and we must define τ_β on $X_\alpha \cup \{x_\alpha\}$. Our main problem is to handle (4) for $\xi = \alpha$. If there are no $\mu < \alpha$ such that $x_\alpha \in \mathrm{cl}_\rho(S_\mu)$, let τ_β be the topology whose base is $\tau_\alpha \cup \{\{x_\alpha\}\}$, so that the point x_α

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becomes isolated. Otherwise, let $\langle \mu_n : n \in \omega \rangle$ enumerate $\{\mu < \alpha : x_\alpha \in \mathrm{cl}_\rho(S_\mu)\}$ with each such μ being listed ω times. Let $\{U_n : n \in \omega\}$ be a nested open base at X_α in the topology ρ, and pick $p_n \in S_{\mu_n} \cap U_n$. $\{p_n : n \in \omega\}$ is a discrete subset of X_α in ρ_α (since it converges in ρ to $x_\alpha \notin X_\alpha$), and thus also in τ_α, so we may find disjoint τ_α-clopen compact sets K_n ($n \in \omega$) with each $p_n \in K_n \subseteq U_n$ (since τ_α is metrizable, zero-dimensional, and locally compact). Define τ_β to have as a base the sets of the form $\{x_\alpha\} \cup \bigcup_{m>n} K_m$ for $n = 0, 1, \ldots,$ along with all the sets in τ_α. It is then easy to check (1)–(4) for all $\xi < \eta \leqq \beta$. Note that $\bigcup_n K_n$ is closed in (X_α, T_α) since $K_n \subseteq U_n$ and ρ is T_2.

Let $\tau = \tau_{\omega_1}$. Then τ is first countable, locally compact (and thus T_3), locally countable, and not Lindelöf. Further properties of τ may be deduced from further properties of ρ; all require at least that ρ is *HS*.

THEOREM. *If ρ is HS, then*

a) *τ is HS.*

b) *If all closed sets are G_δ in ρ, the same is true of τ.*

c) *If ρ is T_3 and HL, τ is normal.*

d) *If ρ is T_3 and Lindelöf, τ is realcompact.*

The main tool in the proof of the theorem is a lemma stating that τ is not too much finer than ρ, namely:

LEMMA. *If ρ is HS and $A \subseteq X$, then* $|\mathrm{cl}_\rho(A) - \mathrm{cl}_\tau(A)| \leqq \omega$.

Proof. Let $B \subseteq A$ be countable and ρ-dense in A, so $\mathrm{cl}_\rho(B) = \mathrm{cl}_\rho(A)$. $B = S_\mu$ for some μ. By Condition (4), whenever $\xi > \mu$, if $x_\xi \in \mathrm{cl}_\rho(B)$ then $x_\xi \in \mathrm{cl}_\tau(B)$; so $\mathrm{cl}_\rho(A) - \mathrm{cl}_\tau(A) \subseteq \{x_\xi : \xi \leqq \mu\}$.

For part (a) of the theorem, if $C \subseteq X$, there is a countable $A \subseteq C$ which is ρ-dense in C, so $A \cup (C - \mathrm{cl}_\tau(A))$ is τ-dense in C and countable. Similarly, for (b), if A is τ – closed, $\mathrm{cl}_\rho(A) - A$ is countable and $\mathrm{cl}_\rho(A)$ is a $\rho - G_\delta$ and hence a $\tau - G_\delta$, so A is a $\tau - G_\delta$.

For (c), we follow the usual proof that T_3 Lindelöf spaces are normal (even though τ isn't Lindelöf). Let H, K be τ-closed disjoint subsets of X. To show that they can be separated, it is sufficient to produce a countable cover of X by τ-open sets U such that $\mathrm{cl}_\tau(U)$ intersects at most one of H and K; call such U "nice." By the lemma, $\mathrm{cl}_\rho(H) \cap \mathrm{cl}_\rho(K)$ is countable, and around each of its points we may put a nice U. Since $X - (\mathrm{cl}_\rho(H) \cap \mathrm{cl}_\rho(K))$ is ρ-Lindelöf, we may cover it with a countable collection of nice U (which are in fact ρ-open and whose ρ-closures intersect at most one of H and K). These two collections together produce the desired cover.

(d) just uses the well-known fact [**8**] that any refinement of a first countable T_3 Lindelöf topology is realcompact. For a direct proof, let $\mathscr{U}$ be a countably complete τ-Z-ultrafilter. Then $\mathscr{V} = \{H \in \mathscr{U} : H \text{ is } \rho\text{-closed}\}$ has non-empty intersection since ρ is Lindelöf. Fix $p \in \bigcap \mathscr{V}$. Let $f : X \to [0, 1]$ be ρ-continuous with $f^{-1}\{0\} = \{p\}$. For each n, $f^{-1}[1/n, 1] \notin \mathscr{V}$, so $f^{-1}[1/n, 1] \notin \mathscr{U}$, so $f^{-1}[0, 1/n] \in \mathscr{U}$, so $f^{-1}[0, 1/n] \in \mathscr{V}$. Thus, $\{p\} \in \mathscr{V} \subseteq \mathscr{U}$, so $\mathscr{U}$ is fixed.

We conclude this section with three remarks:

1) Our space answers a question of Hodel [**3**]; namely, it is an example of a p-space with no uncountable discrete subsets and countable pseudoweight, but uncountable weight.

2) In retrospect, the construction of Hajnal-Juhász [**2**] may be viewed as running the Sorgenfrey line through our machine.

3) Our construction answers a question of Pfeffer ([**5**, p. 137]), namely, assuming CH, there is a first countable compact T_2 space which admits a non-regular Borel measure.

To see this, we first generalize the Alexandrov duplicate construction. If ρ is a compact T_2 topology on X and τ is a locally compact refinement of ρ, we may define a compact T_2 topology on $X \times 2$ as follows: $X \times \{0\}$ is open and has the τ-topology, and neighborhoods of a point $(x, 1)$ in $X \times \{1\}$ are in the form $U \times \{1\} \cup [(U - K) \times \{0\}]$, where $x \in U$ and K is τ-compact (and hence ρ-compact). Then $X \times \{1\}$ is closed and has the ρ-topology. If (X, ρ) is first countable, so is $X \times 2$, since it obviously has countable pseudocharacter. The Alexandrov duplicate is the special case where τ is discrete.

Now, if X is $[0, 1]$, ρ is the usual topology, and τ is locally countable, then in $X \times 2$, every G_δ containing $X \times \{1\}$ is co-countable. If there is a non-atomic Borel measure μ on (X, τ), one may extend μ to $X \times 2$ by identifying (X, τ) with $X \times \{0\}$ and declaring $X \times \{1\}$ to have measure 0. Every G_δ containing $X \times \{1\}$ will have measure 1, so the measure on $X \times 2$ will not be regular.

The existence of such a μ is trivial if there is a real-valued measurable cardinal $\leqq c$, since we may then take τ to be discrete. But also, under CH, we may let τ be as constructed by our machine. Then by our lemma above, the τ-Borel sets are just the ρ-Borel sets, so μ may be taken to be Lebesgue measure.

2. We again assume CH and let ρ be the usual topology on $\mathbf{R}$. Let $\mathscr{B}$ be a countable basis for $(\mathbf{R}, \rho)$ made up of open intervals.

Fact 1. Using CH, there is a family $\{L_i | i \in \omega\}$ of disjoint Lusin sets in $\mathbf{R}$ such that each L_i meets every member of $\mathscr{B}$ in an uncountable set.

Define $X = \bigcup_{i \in \omega} L_i$. Let Λ be the set of all limit ordinals in ω_1. We can index $X = \{x_\alpha | \alpha \in \omega_1\}$ in a one-to-one way such that for each $\lambda \in \Lambda$ and $i \in \omega$, $\{x_\alpha \in L_i | \lambda \leqq \alpha < \lambda + \omega\}$ is dense in $(\mathbf{R}, \rho)$.

Fact 2. Using CH, for each $i \in \omega$, we can index the set of all countable subsets of L_i as $\{S_{\alpha, i} | \alpha \in \omega_1\}$ in such a way that, for $\lambda \in \Lambda$, $S_{\lambda+\omega, i} = \{x_\alpha \in L_i | \lambda \leqq \alpha < \lambda + \omega\}$.

For each $\alpha \in \omega_1$, define $X_\alpha = \{x_\beta \in X | \beta < \alpha\}$ and let ρ_α be the subspace topology of X_α in $(\mathbf{R}, \rho)$.

Fact 3. Using CH, we can index the set of all countable subsets of X as $\{A_\lambda | \lambda \in \Lambda\}$ in such a way that, if $\lambda \in \Lambda$, there is $\lambda^* < \lambda$ such that $A_\lambda \subset X_{\lambda^*}$.

Our aim is to define a new topology τ on X and we do this by induction as in § 1.

Our induction hypothesis for $\beta \in \omega_1$ is that for all $\lambda \in \Lambda$ with $\lambda < \beta$, a subset Z_λ of X_{λ^*} has been defined and, for each $\alpha < \beta$ in ω_1 and $n \in \omega$, a subset $U_{\alpha,n}$ of $X_{\alpha+1}$ containing x_α has been defined in such a way that, if τ_β is the topology on X_β induced by using $\{U_{\alpha,n}|\alpha < \beta \text{ and } n \in \omega\}$ as a subbasis, then:

(1) (X_β, τ_β) is metric, and $\tau_\beta \subset \rho_\beta$.

(2) For all $\alpha < \beta$, $U_{\alpha,0} \supset U_{\alpha,1} \supset \ldots$.

(3) For all $\alpha < \beta$, $\{U_{\alpha,n}|n \in \omega\}$ is a clopen basis for x_α in (X_β, τ_β).

(4) For $\lambda \in \Lambda$ and $\lambda < \beta$, Z_λ is clopen in (X_β, τ_β).

(5) For $\alpha < \beta$, $x_\alpha \in L_i \cap B$ for some $B \in \mathscr{B}$, and $n \in \omega$, there is a finite subset G of $\{U_{\gamma,j}|\gamma < \alpha, j \in \omega\}$ and a $k \in \omega$ such that $U_{\alpha,n} - B = \bigcup G - B$ and $U_{\alpha,k} \subset B \cap (\bigcup_{j \leqq i} L_j)$.

(6) For $\lambda \in \Lambda$ and $\lambda < \beta$, if A_λ is closed in $(X_\lambda, \tau_\lambda)$, then $A_\lambda \subset Z_\lambda$.

(7) For $i \in \omega$, $a \in \omega$, $\lambda' \in \Lambda$, $\lambda = \lambda' + \omega$, $\beta = \alpha + 1$, $\alpha = \lambda + a$, $x_\alpha \in L_i$, and $j \leqq i$, then $x_\alpha \in \mathrm{cl}_{\tau_\beta} S_{\lambda,j}$.

(8) If $\gamma_1 < \gamma_2 < \ldots$ have λ as a limit in ω_1, $i \in \omega$, $a \in \omega$, $\alpha = \lambda + a$, $\beta = \alpha + 1$, $x \in L_i$, $j \leqq i$, and, for each n, $x_\alpha \in \mathrm{cl}_\rho S_{\gamma_n,j}$, then $x_\alpha \in \mathrm{cl}_{\tau_\beta} \bigcup_n S_{\gamma_n,j}$.

We use (6), (7), and (8) later and they are trivial to check inductively, but the fact that they hold inductively is not used in the construction; we also make no use at this time of the fact that each L_i is a Lusin set. Later we use this fact to insure that whenever we have disjoint closed sets to separate, one of them is countable; then we use (6) to guarantee the existence of a clopen set Z_λ which separates our countable closed set A_λ from a tail of the space. Observe that both A_λ and Z_λ are only defined for $\lambda \in \Lambda$. We use (7) and (8) to achieve hereditary separability.

If $\alpha \in \omega$, define $U_{\alpha,n} = \{x_\alpha\}$ for all $n \in \omega$; the induction hypothesis is then satisfied for all $\beta \in \omega$.

So assume that $\omega \leqq \beta \in \omega_1$ and that the induction hypothesis is satisfied for all $\beta' < \beta$. Then if $\beta \in \Lambda$ one can easily check that the induction hypothesis holds for β.

Now assume that $\beta = \alpha + 1$ for some $\alpha \in \omega_1$, and $\alpha = \lambda + a$ for some $\lambda \in \Lambda$ and $a \in \omega$. If $\alpha = \lambda$ consider A_λ. If A_λ is *not* closed in $(X_\lambda, \tau_\lambda)$, define $Z_\lambda = \emptyset$. If A_λ *is* closed in the zero-dimensional metric countable space $(X_\lambda, \tau_\lambda)$, since $X_\lambda - X_{\lambda^*}$ is closed and disjoint from A_λ, we can find a clopen set Z_λ in $(X_\lambda, \tau_\lambda)$ such that $A_\lambda \subset Z_\lambda$ and $Z_\lambda \subset X_{\lambda^*}$.

Choose $B_{\alpha,0} \supset B_{\alpha,1} \supset \ldots$ from $\mathscr{B}$ such that $x_\alpha = \bigcap_{n\in\omega} B_{\alpha,n}$. Choose $\lambda_0 < \lambda_1 < \ldots$ having λ as a limit in ω_1. Let $\{\lambda^n|n \in \omega\}$ be an indexing of $\{\lambda' \in \Lambda|\lambda' \leqq \lambda\}$. Define $\mathscr{S}_\alpha = \{S_{\gamma,j}|\gamma \leqq \alpha, j \in \omega\}$ and $\mathscr{B}_\alpha = \{B_{\alpha,n}|n \in \omega\}$.

Since $\mathscr{S}_\alpha$ and $\mathscr{B}_\alpha$ are countable, there is a function $f_\alpha : \mathscr{B}_\alpha \rightarrow X_\alpha$ such that $f_\alpha(B) \in B$ for all $B \in \mathscr{B}_\alpha$ and, if $S \in \mathscr{S}_\alpha$ and $S \cap B \cap X_\alpha \neq \emptyset$ for infinitely many $B \in \mathscr{B}_\alpha$, then $f_\alpha(B) \in S$ for infinitely many $B \in \mathscr{B}_\alpha$.

For $n \in \omega$, let $F(n) = \{B \in \mathscr{B}_\alpha|f_\alpha(B) = x_\gamma \in L_j \text{ for some } j \leqq i, \lambda_n < \gamma < \lambda, \text{ and } B \subset B_{\alpha,n}\}$. If $B \in F(0)$, then there is an n such that $B \in F(n) - F(n+1)$. If $B \in F(n) - F(n+1)$ and $f_\alpha(B) = x_\gamma$, then there is an integer m such that:

(a) $U_{\gamma,m} \subset B_{\alpha,n} \cap (\bigcup_{j\leqq i} L_j)$, and

(b) if $k < n$ and $\lambda^{k^*} < \lambda_n$, then $U_{\gamma,m} \cap Z_{\lambda^k} = \emptyset$.
(b) is possible since Z_{λ^k} is closed in (X_α, τ_α) and $Z_{\lambda^k} \subset X_{\lambda^{k^*}} \subset X_{\lambda^n}$ if $\lambda^{k^*} < \lambda_n$. Define $U_B = U_{\gamma,m}$ for $B \in F(0)$, and define $U_{\alpha,n} = \{x_\alpha\} \cup \bigcup\{U_B | B \in F(n)\}$.

Since a regular, Hausdorff, countable space is metrizable, the whole induction hypothesis will clearly hold for β if we can show that each $U_{\gamma,j}$ for $\gamma < \alpha$ and each $Z_{\lambda'}$ for $\lambda' \leqq \alpha$ is still closed in (X_β, τ_β).

If $\lambda' \in \Lambda$ and $\lambda' \leqq \alpha$, then $\lambda' = \lambda^k$ for some $k \in \omega$. Since $\lambda^{k^*} < \lambda^k \leqq \alpha = \lambda + a$ for some $a \in \omega$, there is an $n \in \omega$ such that $k < n$ and $\lambda^{k^*} < \lambda_n$. For all $m \geqq n$, $U_{\alpha,m} \cap Z_{\lambda'} = \emptyset$, so $x_\alpha \notin \mathrm{cl}_{\tau_\beta}(Z_{\lambda'})$. Thus $Z_{\lambda'}$ is closed in (X_β, τ_β) since $Z_{\lambda'}$ is closed in (X_β, τ_β).

Similarly, if $\gamma < \alpha$ and $j \in \omega$, the only possible boundary point of $U_{\gamma,j}$ in (X_β, τ_β) is x_α. Suppose that γ is the smallest ordinal such that $x_\alpha \in \mathrm{cl}_{\tau_\beta}(U_{\gamma,j})$ for some $j \in \omega$. Choose $B \in \mathscr{B}$ such that $x_\gamma \in B$ but $x_\alpha \notin \mathrm{cl}_\rho(B)$. By (5) of the induction hypothesis, there is a finite $G \subset \{U_{\mathscr{S},k} | \mathscr{S} < \gamma; k \in \omega\}$ such that $U_{\gamma,j} - B = \bigcup G - B$. Thus, if $x_\alpha \in \mathrm{cl}_{\tau_\beta}(U_{\gamma,j})$, then $x_\alpha \in \mathrm{cl}_{\tau_\beta}(\bigcup G)$. Hence $x \in \mathrm{cl}_{\tau_\beta}(U_{\mathscr{S},k})$ for some $\mathscr{S} < \gamma$ and $k \in \omega$. But this contradicts the minimality of γ.

Having defined τ_β for all $\beta \in \omega_1$ inductively, let τ be the topology on X induced by using $\{U_{\alpha,n} | \alpha \in \omega_1, n \in \omega\}$ as a subbasis. Clearly $|X| = \omega_1$, (X, τ) is Hausdorff, 0-dimensional, locally countable, and first countable.

We want to prove that (X, τ) is hereditarily separable, normal, and not countably paracompact. To this end we prove:

(a) *If W is open in $(\mathbf{R}, \rho)$, $Y \subset L_j$ for some $j \in \omega$, and $|Y \cap B| = \omega_1$ for all $W \supset B \in \mathscr{B}$, then there is a $\gamma \in \omega$, such that* $\mathrm{cl}_\tau(Y \cap X_\gamma) \supset ([W \cap (\bigcup_{j \leqq i} L_i)] - X_\gamma)$.

Proof. Since $|Y \cap B| = \omega_1$ for all $W \supset B \in \mathscr{B}$, we can choose $\omega < \gamma(0) < \mathscr{S}(0) < \gamma(1) < \mathscr{S}(1) < \ldots$ in ω_1 such that $S_{\mathscr{S}(n),j} \subset X_{\gamma(n+1)} - X_{\gamma(n)}$, $S_{\mathscr{S}(n),j} \subset Y$, and $\mathrm{cl}_\rho(S_{\mathscr{S}(n),j}) \supset W$. Let $\gamma = \sup\{\gamma(n) | n \in \omega\}$ and suppose that, contrary to (a), there is a smallest $a \geqq \gamma$ such that $x_\alpha \in W \cap L_i$ for some $i \geqq j$ and $x_\alpha \notin \mathrm{cl}_\tau(Y \cap X_\gamma)$. Since $\alpha \geqq \gamma > \omega$, there is a $\lambda \in \Lambda$ such that $\alpha = \lambda + a$ for some $a \in \omega$.

We first show that $\gamma \neq \lambda$. Since $x_\alpha \in \mathrm{cl}_\rho S_{\delta(n),j}$ for each $n \in \omega$, by (8), $x_\alpha \in \mathrm{cl}_{\tau_{\alpha+1}} \bigcup_n (S_{\delta(n),j}) \subset \mathrm{cl}_\tau \bigcup_n (S_{\delta(n),j}) \subset \mathrm{cl}_\tau(Y \cap X_\gamma)$. So $\gamma < \lambda$.

Next we suppose that $\lambda = \lambda' + \omega$ for some $\lambda' \in \Lambda$. By (7) $x_\alpha \in \mathrm{cl}_{\tau_{\alpha+1}}(S_{\lambda,j}) \subset \mathrm{cl}_\tau(S_{\lambda,j})$. Since $x_\alpha \in W$ and $\tau \supset \rho$, $x_\alpha \in \mathrm{cl}_\tau(S_{\lambda,j} \cap W)$.
But by the minimality of α, $(S_{\lambda,j} \cap W) \subset \mathrm{cl}_\tau(Y \cap X_\gamma)$ so $x_\alpha \in \mathrm{cl}_\tau(Y \cap X_\gamma)$.

Similarly, if λ is a limit of $\gamma = \lambda(0) < \lambda(1) < \ldots$ in Λ, then by the minimality of α, $(S_{\lambda(n),j} \cap W) \subset \mathrm{cl}_\tau(Y \cap X_\gamma)$ for all $\lambda(n)$. But by (8), since $x_\alpha \in W$ and $\tau \supset \rho$, $x_\alpha \in \mathrm{cl}_\tau \bigcup_n (S_{\lambda(n),j} \cap W)$. So $x_\alpha \in \mathrm{cl}_\tau(Y \cap X_\gamma)$ in all cases.

We next prove:

(b) *If $S \subset X$, there is a $\sigma(S) \in \omega_1$, and for each $i \in \omega$, a $W_{S,i}$ which is open in*

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$(\mathbf{R}, \rho)$ *such that* $\mathrm{cl}_\tau(S \cap X_{\sigma(S)}) \supset ([W_{S,i} \cap (\bigcup_{i \leqq j} L_j)] - X_{\sigma(S)})$ *and* $[(S \cap L_i) - W_{S,i}] \subset X_{\sigma(S)}$.

Proof. Fix $i \in \omega$ and let $S_i = S \cap L_i$. Let $\mathscr{J} = \{B \in \mathscr{B} \mid |B \cap S_i| \leqq \omega\}$ and let $J = \mathrm{cl}_\rho(\bigcup \mathscr{J})$. Since L_i is Lusin and $|\bigcup \{B \cap S_i | B \in \mathscr{J}\}| \leqq \omega$, $(J \cap S_i) \subset X_\delta$ for some $\delta \in \omega_1$. Define $W_{S,i} = X - J$.

By (a) there is a $\gamma \in \omega$, such that $\mathrm{cl}_\tau(S_i \cap X_\gamma) \subset ([W_{S,i} \cap (\bigcup_{i \leqq j} L_j)] - X_\gamma)$. Define $\sigma^i = \gamma + \mathscr{S}$.

Finally, define $\sigma(S) = \sup\{\sigma^i | i \in \omega\}$; since all of the desired properties hold, (b) is proved.

Let us now prove that (X, τ) is hereditarily separable. If $S \subset X$, then choose $\sigma(S)$ and $W_{S,i}$ as in (*b*). Thus $\mathrm{cl}_\tau(S \cap X_{\sigma(S)}) \supset S$ and S is separable.

It is somewhat more difficult to prove that (X, τ) is normal, but assume that H and K are closed and disjoint in (X, τ). Choose $\sigma(H)$, $\sigma(K)$, $W_{H,i}$, and $W_{K,i}$ for all i as in (b). Let $\sigma = \sigma(H) + \sigma(K)$. Observe that $W_H = \bigcup_{i \in \omega} W_{H,i}$ and $W_K = \bigcup_{i \in \omega} W_{K,i}$ are disjoint. To see this suppose that $i \leqq j$. By (b), $\mathrm{cl}_\tau H \supset (W_{H,i} \cap L_j) - X_\sigma$ and $\mathrm{cl}_\tau K \supset (W_{K,j} \cap L_j) - X_\sigma$. Thus, since L_j intersects every nonempty open subset of $(\mathbf{R}, \rho)$ in an uncountable set, $W_{H,i} \cap W_{K,j} = \emptyset$.

Define $H' = \{x \in H | x \in \mathrm{cl}_\rho W_K\}$ and $K' = \{x \in K | x \in \mathrm{cl}_\rho W_H\}$. Since $H' \cap W_H = \emptyset$, $H' \subset X_{\sigma(H)}$ by (b). Since H' is countable, $H' = A_\lambda$ for some $\lambda \in \Lambda$. So $H' \subset Z_\lambda \subset X_\lambda$ and Z_λ is clopen in (X, τ). Similarly there is a $\lambda' \in \Lambda$ with $K' \subset Z_{\lambda'} \subset X_{\lambda'}$ and $Z_{\lambda'}$ is clopen in (X, τ). Define $\sigma' = \sigma + \lambda + \lambda'$. Since $(X_{\sigma'}, \tau_{\sigma'})$ is metric, there are disjoint open U and V in $X_{\sigma'}$ such that $(H \cap X_{\sigma'}) \subset U$ and $(K \cap X_{\sigma'}) \subset V$. Let $U^* = (W_H - Z_{\lambda'}) \cup (U - \mathrm{cl}_\rho W_K) \cup (Z_\lambda \cap U)$ and $V^* = (W_K - Z_\lambda) \cup (V - \mathrm{cl}_\rho W_H) \cup (Z_{\lambda'} \cap V)$. Then $H \subset U^*$, $K \subset V^*$, and U^* and V^* are open in (X, τ) and disjoint. Thus we have proved that (X, τ) is normal.

It remains to show that (X, τ) is *not* countably paracompact. For each $n \in \omega$, define $D_n = \bigcup_{i \geqq n} L_i$; then D_n is closed in (X, τ) and $\bigcap_{n \in \omega} D_n = \emptyset$. If (X, τ) is countably paracompact, there are $U_n \supset D_n$ with U_n open in (X, τ) and $\bigcap_{n \in \omega} U_n = \emptyset$. Suppose that $U_n \supset D_n$ is open. Since L_0 is Lusin in $(\mathbf{R}, \rho)$, if $|L_0 - U_n| = \omega_1$, there is an open in $(\mathbf{R}, \rho)$ subset W of $\mathbf{R}$ such that $|B \cap (L_0 - U_n)| = \omega_1$ for all $W \supset B \in \mathscr{B}$. Thus, by (a) $\mathrm{cl}(L_0 - U_n)$ contains all of $L_n \cap W$ except a countable set. But this contradicts $L_n \subset D_n \subset U_n$ and $|L_n \cap W| = \omega_1$, so $L_0 - U_n$ is countable. Since $L_0 - U_n$ is countable for all $n \in \omega$, $\bigcap_{n \in \omega} U_n \neq \emptyset$ and (X, τ) is not countably paracompact.

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