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A dimension formula relating to algebraic groups

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An upper bound is given of the dimension of certain spaces of cusp harmonic forms of arithmetic subgroups Γ of semisimple algebraic groups G in terms of the multiplicities of corresponding irreducible unitary representations of the group G_{R} of real rational points of G in the space ${}^{0}L^2(G_{\mathrm{R}}/\Gamma)$ of cusp forms.

1. Introduction

Our formula can be considered to be related to the duality of Gel'fand and Pyateckii-Shapiro of a discrete subgroup Γ of G_{R} such that G_{R}/Γ is compact [3]. An essential point of the duality in [3] is that $L^2(G_{\mathrm{R}}/\Gamma)$ is a countable direct sum of irreducible unitary representations of G_{R}. If G_{R}/Γ is not compact, then $L^2(G_{\mathrm{R}}/\Gamma)$ contains continuous and discrete spectrum in general. However, the closed invariant subspace ${}^{0}L^2(G_{\mathrm{R}}/\Gamma)$ of $L^2(G_{\mathrm{R}}/\Gamma)$ is still a countable direct sum of irreducible unitary representations of G_{R} for an arithmetic subgroup Γ of G_{R} [4]. Consequently, we can obtain an upper bound of the Garland space of cusp harmonic forms which is a closed invariant subspace of the space studied in [2] by Garland. For basic definitions and facts about algebraic groups and their arithmetic subgroups, we refer to [1].

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2. The formula

We assume that G is a connected semisimple linear algebraic group which is defined and simple over $\mathbb{Q}$. Moreover, we assume that G has $\mathbb{Q}$-rank 1, that is, $\dim S_{\mathbb{Q}} = 1$, where $S_{\mathbb{Q}}$ is a maximal $\mathbb{Q}$-split torus of G. Let $\underline{\underline{G}}_{\mathbb{R}}$ denote the Lie algebra of $G_{\mathbb{R}}$ and $\underline{\underline{G}}$ denote the complexification of $\underline{\underline{G}}_{\mathbb{R}}$. Denote by Δ_G the Casimir operator which is a unique element of the center of the universal enveloping algebra of $\underline{\underline{G}}$.

The space ${}^{0}L^2(G_{\mathbb{R}}/\Gamma)$ of cusp forms consists of elements of $L^2(G_{\mathbb{R}}/\Gamma)$ satisfying

$$\int_{U_{\mathbb{R}}/U_{\mathbb{R}}\cap\Gamma} f(xu)\,du = 0$$

for almost all $x \in G_{\mathbb{R}}$, where U is the unipotent radical of an arbitrary parabolic subgroup P of G.

We fix a certain maximal compact subgroup $K \subset G_{\mathbb{R}}$. Let V be a finite dimensional complex vector space with a positive definite hermitian inner product. Then let $\sigma : K \to \operatorname{Aut} V$ be a representation of K which is unitary with respect to the given inner product. We let d_σ denote the complex dimension of V and let ξ_σ denote the character of σ. Let dk denote the Haar measure on K normalized so that $\int_K dk = 1$.

For $\nu \in \mathbb{C}$, the Garland space $G(\sigma, \nu)$ of harmonic forms is defined by

$$G(\sigma, \nu) = \Big\{ f \in L^2(G_{\mathbb{R}}/\Gamma) \cap C^\infty(G_{\mathbb{R}}/\Gamma) \,\Big|\, \Delta_G f = \nu f , \quad d_\sigma \int_K \xi_\sigma(k) f(k^{-1}x)\,dk = f(x),\ x \in G_{\mathbb{R}}/\Gamma \Big\} .$$

If $G(\sigma, \nu) \neq 0$ and G has $\mathbb{Q}$-rank 1, then ν is real and $\dim G(\sigma, \nu) < \infty$ ([2]). The Garland space ${}^{0}G(\sigma, \nu)$ of cusp harmonic forms is defined by

$${}^{0}G(\sigma, \nu) = \Big\{ f \in G(\sigma, \nu) \cap {}^{0}L^2(G_{\mathbb{R}}/\Gamma) \Big\} .$$

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Let $\hat{G}_{\mathbb{R}}$ denote the set of irreducible unitary representations π of $G_{\mathbb{R}}$. Let H_π be the representation space of π, $m(\pi)$ be the multiplicity of π in ${}^0L^2(G_{\mathbb{R}}/\Gamma)$ and Δ_π be the Casimir operator of the representation π. Since π is irreducible, there exists a complex number ν_π such that $\Delta_\pi\varphi = \nu_\pi\varphi$ for φ in the domain of Δ_π, which is dense in H_π. Let $\hat{G}_{\mathbb{R}}(\nu)$ denote the set of irreducible unitary representations π of $G_{\mathbb{R}}$ such that $\Delta_\pi = \nu_\pi\cdot 1$ and $\nu_\pi = \nu$. Fix an irreducible unitary representation π and its representation space H_π. For any irreducible unitary representation σ of K, we define a linear transformation E_σ in H_π by

$$E_\sigma v = d_\sigma \int_K \xi_\sigma(k)\pi\left(k^{-1}\right)v\,dk ,$$

for $v \in H_\pi$. Then E_σ is a continuous projection. We let $H_{\pi,\sigma} = E_\sigma\left(H_\pi\right)$. The dimension of $H_{\pi,\sigma}$ is finite dimensional and is denoted by $d\left(H_{\pi,\sigma}\right)$.

We write ${}^0L^2\left(G_{\mathbb{R}}/\Gamma\right) = \sum_{i=1}^{\infty} \oplus H_i$, where H_i is the representation space of the irreducible unitary representation π_i. Note that, for $\varphi \in C^\infty\left(G_{\mathbb{R}}/\Gamma\right) \cap {}^0L^2\left(G_{\mathbb{R}}/\Gamma\right)$, the regular representation λ of $G_{\mathbb{R}}$ on ${}^0L^2\left(G_{\mathbb{R}}/\Gamma\right)$ satisfies $\Delta_\lambda\varphi = \Delta_G\varphi$. This follows from an easy computation.

For any $f \in {}^0G(\sigma, \nu)$, $f \in {}^0L^2\left(G_{\mathbb{R}}/\Gamma\right) \cap C^\infty\left(G_{\mathbb{R}}/\Gamma\right)$, $\Delta_G f = \nu f$ and $E_\sigma f = f$. Let P_i be the projection of ${}^0L^2\left(G_{\mathbb{R}}/\Gamma\right)$ onto H_i. Since f is differentiable, for each $X \in \underline{\underline{G}}_{\mathbb{R}}$,

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$$\begin{aligned} P_i\lambda(X)f &= \lim_{t\to 0} \frac{1}{t} P_i\big(\lambda(\exp tX)f-f\big) \\ &= \lim_{t\to 0} \frac{1}{t} \big(\lambda(\exp tX)P_if-P_if\big) \\ &= \lim_{t\to 0} \frac{1}{t} \big(\pi_i(\exp tX)P_if-P_if\big) \\ &= \pi_i(X)P_if \ . \end{aligned}$$

Hence P_if is in the domain of Δ_{π_i}, $P_i\Delta_\lambda f = \Delta_{\pi_i}P_if$ and $\Delta_{\pi_i}P_if = \lambda_{\pi_i}P_if$. If $\pi_i \notin \hat{G}_R(\nu)$, then $P_if = 0$. Hence $f = P_{i_1}f + \ldots + P_{i_t}f$. Since $E_\sigma \cdot P_i = P_i \cdot E_\sigma$, $P_if \in H_{i,\sigma}$ for $i \in \{i_1, \ldots, i_t\}$. Consequently, we get the following

THEOREM. *Let* G *be a connected semisimple linear algebraic group which is defined and simple over* Q . *We assume that* G *has* Q*-rank* 1 . *Then*

$$\dim^0 G(\sigma, \nu) \le \sum_{\pi\in\hat{G}_R(\nu)} m(\pi)d\big(H_{\pi,\sigma}\big) \ .$$

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